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A NOTE ON GENERALIZED WEAKLY δ-SUPPLEMENTED MODULES

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Abstract: In this study we give simple different properties of generalized weakly δ-supplemented modules that characterized in [7]. And we also define generalized δ-coclosed submodule of a module M as a generalization of δ-coclosed submodules that introduced in [1] and show some basic characterizations.

Key words: generalized weakly δ-supplemented module; generalized δ-coclosed submodule.

Mathematics Subject Classification: 16D10, 16D90.

1. Introduction

Throughout this article, all rings are associative with identity and all modules are unitary left R-modules. A submodule L of a module M is called small in M (denoted by $L \ll M$), if for every proper submodule K of M, $L + K \neq M$. $L \leq M$, is said to be essential in M, denoted by $L \trianglelefteq M$, if $L \cap K \neq 0$ for each nonzero submodule $K \leq M$. A module M is said to be singular if $M \cong {}^{N}/_{L}$ for some module N and a submodule $L \leq N$ with $L \trianglelefteq N$. For two submodules N and K of M, N is called a supplement of K in M if N is minimal with the property $M = K + N$; equivalently $M = K + N$ and $N \cap L \ll N$. A module M is called supplemented if every submodule of M has a supplement in M. If $N + K = M$ and $N \cap K \ll M$, then K is called a weak supplement of N in M. M is weakly supplemented module if every submodule of M has a weak supplement in M. The sum of small submodules of a module M is denoted by $Rad(M)$. Let M be an R-module and N, K be any submodules of M with $M = N + K$. If $N \cap K \leq Rad(K)$ ($N \cap K \leq Rad(M)$) then K is called a generalized (weak) supplement of N in M. And M is called generalized supplemented module if every submodule N of M has a generalized supplement K in M. In [8],

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an R-module M is called generalized weakly supplemented if every submodule K of M has a generalized weak supplement N in M. For characterization of these modules we refer to [6] and [8].

By Zhou [9], a submodule L of M is called δ-small in M (denoted by $L \ll_{\delta} M$) if for any submodule N of M with ${}^{M}/_{N}$ singular, $M = N + L$ implies that $M = N$. Let $\wp$ be the class of all singular simple R-modules. For a module M, as in [9], let $\delta(M) = \bigcap\{N \leq M \mid {}^{M}/_{N} \in \wp\}$. The sum of δ-small submodules of a module M is denoted by $\delta(M)$. It is easy to see that every small submodule of a module M is δ-small in M so $Rad(M) \subseteq \delta(M)$.

Let K, N be submodules of module M, then N is called a δ-supplement of K in M if $M = N + K$ and $N \cap K \ll_{\delta} K$. N is called a weak δ-supplement of K in M if $M = N + K$ and $N \cap K \ll_{\delta} M$. A module M is called δ-supplemented if every submodule of M has a δ-supplement in M. Also M is called weakly δ-supplemented if every submodule of M has a weak δ-supplement in M.

A module M is said to be δ-local if $\delta(M) \ll_{\delta} M$ and $\delta(M)$ is a maximal submodule of M. [1]

Let M be an R-module and $N \leq M$. We call N a δ-coclosed submodule of M if ${}^{N}/_{X}$ is singular and ${}^{N}/_{X} \ll_{\delta} {}^{M}/_{X}$ for some $X \leq N$, then $X = N$. [1]

In this paper we define generalized δ-coclosed submodule of a module and give some basic properties of generalized weakly δ-supplemented modules.

2. Preliminaries

We give basic properties of δ-small submodules in the following lemma which is contained in [9].

Lemma 2.1: Let M be a module. Then we have the following.

1) If N is δ-small in M and $M = X + N$, then $M = X \oplus Y$ for a projective semisimple submodule Y with $Y \subseteq N$.
2) If K is δ-small in M and $f: M \longrightarrow N$ is a homomorphism, then $f(K)$ is δ-small in N. In particular, if K is δ-small in $M \subseteq N$, then K is δ-small in N.
3) Let $K_1 \subseteq M_1 \subseteq M, K_2 \subseteq M_2 \subseteq M$ and $M = M_1 \oplus M_2$. Then $K_1 \oplus K_2$ is δ-small in $M_1 \oplus M_2$ if and only if K_1 is δ-small in M_1 and K_2 is δ-small in M_2.

4) Let N, K be submodules of M with K δ-small in M and $N \leq K$. Then N is also δ-small in M.

Definition 2.2: Let M be a module and U, V be submodules of M. V is called a generalized δ-supplement of U in M if $M = U + V$ and $U \cap V \leq \delta(V)$.

A module M is called generalized δ-supplemented if every submodule of M has a generalized δ-supplement in M.

We refer to [7], for more detailed discussion about these modules.

3. Main Results

Theorem 3.1: Let M be a module and U, V be submodules of M. V is a generalized δ-supplement of U if and only if $U + V = M$ and $Rm \ll_{\delta} V$ for all $m \in U \cap V$.

Proof: Let V be a generalized δ-supplement of U. Then, $U + V = M$ and $U \cap V \subseteq \delta(V)$. Since $\delta(V)$ is the sum of all δ-small submodules of V, there exists elements $m_i \in V$ for every $1 \leq i \leq k$ such that $m = m_{i_1} + m_{i_2} + \cdots + m_{i_k}$ and $Rm_i \ll_{\delta} V$ for some $k \in N$. Following this, $Rm \ll_{\delta} V$ is obtained since the sum is finitely and $Rm \subseteq Rm_1 + Rm_2 + \cdots + Rm_k$. Conversely, assume that $U + V = M$ and $Rm \ll_{\delta} V$ for all $m \in U \cap V$. Then $Rm \subseteq \delta(V)$ since $\delta(V) = \sum_{L \ll_{\delta} V} L$. Hence, $U \cap V \leq \delta(V)$.

Definition 3.2: Let M be a module and U, V be submodules of M. V is called a generalized weak δ-supplement of U in M if $M = U + V$ and $U \cap V \leq \delta(M)$.

A module M is called generalized weakly δ-supplemented if every submodule of M has a generalized weak δ-supplement in M.

By definition it is clear that any generalized δ-supplemented module and weakly δ-supplemented module is generalized weakly δ-supplemented.

Theorem 3.3: Let M be a generalized weakly δ-supplemented module and δ(M) be δ-small submodule of M. Then M is weakly δ-supplemented.

Proof: Let U be an arbitrary submodule of M. Since M is generalized weakly δ-supplemented then $U + V = M$ and $U \cap V \leq \delta(M)$ for $V \leq M$. By hypothesis, $U \cap V \ll_{\delta} M$ is obtained.

Theorem 3.4: Let M be a module and V be a generalized weak δ-supplement of U in M. Then U is a generalized weak δ-supplement of V in M.

Proof: Since V is a generalized weak δ-supplement of U in M then $V+U=M$ and $V\cap U\leq \delta(M)$. Therefore it is clear that U is also a generalized weak δ-supplement of V in M.

Theorem 3.5: Let M be a module and U, V be submodules of M. V is a generalized weak δ-supplement of U if and only if $U+V=M$ and $Rm \ll_{\delta} M$ for all $m\in U\cap V$.

Proof: Let V be a generalized weak δ-supplement of U. Then, $U+V=M$ and $U\cap V\subseteq \delta(M)$. Since $\delta(M)$ is the sum of all δ-small submodules of M, there exists elements $m_i \in M$ for every $1\leq i\leq k$ such that $m=m_{i_1}+m_{i_2}+\cdots+m_{i_k}$ and $Rm_i \ll_{\delta} M$ for some $k\in N$. Following this, $Rm \ll_{\delta} M$ is obtained since the sum is finitely and $Rm\subseteq Rm_1+Rm_2+\cdots+Rm_k$. Conversely, assume that $U+V=M$ and $Rm\ll_{\delta} M$ for all $m\in U\cap V$. Then $Rm\subseteq \delta(M)$ since $\delta(M)=\sum_{L\ll_{\delta}M} L$. Hence, $U\cap V\leq \delta(M)$.

Theorem 3.6: Let M be a δ-local module and V be a generalized weak δ-supplement of U. Then V is weak δ-supplement of U.

Proof: If V is a generalized δ-supplement of U, then $U+V=M$ and $U\cap V\leq \delta(M)$. Since $M$is δ-local, $\delta(M)\ll_{\delta} M$ and so $U\cap V\ll_{\delta} M$. Hence, V is weak δ-supplement of U.

Theorem 3.7: Let M be a module and $K\leq L\leq M$ for submodules K, L of M. Then $L\leq \delta(M)$ if and only if $K\leq \delta(M)$ and ${}^{L}/_{K}\leq \delta({}^{M}/_{K})$.

Proof: Assume that $L\leq \delta(M)$. Clearly $K\leq \delta(M)$. Now let take into account natural epimorphism $p: M\longrightarrow {}^{M}/_{K}$. Since $p(\delta(M))\subseteq \delta(p(M))=\delta({}^{M}/_{K})$, then ${}^{L}/_{K}=p(L)\subseteq p(\delta(M))\leq \delta({}^{M}/_{K})$ is obtained. For the converse assume that $K\leq \delta(M)$ and ${}^{L}/_{K}\leq \delta({}^{M}/_{K})$. Now we show that $L\leq \delta(M)$. For this suppose that $L\nleq \delta(M)$. Then there is a maximal submodule X of M such that ${}^{M}/_{X}$ singular and $L\nleq X$. Namely, there is an element m in L with $m\notin X$. Then $X+Rm=M$ since X is maximal in M and so ${}^{X}/_{K}+{}^{Rm+K}/_{K}={}^{M}/_{K}$. Since ${}^{L}/_{K}\leq \delta({}^{M}/_{K})$ and ${}^{M}/_{X}$ is singular then ${}^{Rm+K}/_{K}\ll_{\delta} {}^{M}/_{K}$. Therefore, ${}^{X}/_{K}={}^{M}/_{K}$ and so $X=M$ is obtained but this fact contradicts with the maximality of X. Hence, $L\leq \delta(M)$.

Theorem 3.8: Let M be a module and $N \leq \delta(M)$. If M/N is generalized weakly δ-supplemented then M is also generalized weakly δ-supplemented.

Proof: Let U be an arbitrary submodule of M. Since M/N is generalized weakly δ-supplemented module there is a generalized weak δ-supplement X/N of $U+N/N$ in M/N. So, $(N+U)/N + X/N = M/N$ and $(N+U)/N \cap X/N = N+(U\cap X)/N \subseteq \delta(M/N)$. Here it is not difficult to see that $U+X=M$. By the previous theorem $U \cap X \subseteq \delta(M)$. Hence X is a generalized weak δ-supplement of U in M.

Corollary: Any δ-small cover of a generalized weakly δ-supplemented module is generalized weakly δ-supplemeneted.

Definition 3.9: Let M be an R-module and $N \leq M$. We call N a generalized δ-coclosed submodule of M if N/X is singular and $N/X \subseteq \delta(M/X)$ for some $X \leq N$, then $X = N$.

It is clear that every δ-coclosed submodule is generalized δ-coclosed.

Theorem 3.10: Let M be a module and $K \leq L \leq M$. If L is generalized δ-coclosed and $K \leq \delta(M)$ then $K \leq \delta(L)$. Additionally, $\delta(L) = L \cap \delta(M)$.

Proof: Assume that $K \nleq \delta(L)$. Then there is a submodule X of L with L/X singular simple that is not containing K. So there is an element m in K such that $m \notin X$. Since L/X is simple X is also a maximal submodule of L. Hence it can be seen easily that $X + Rm = L$. Now by taking into account the natural homomorphism $p: M \longrightarrow M/X$, $p(Rm) = (Rm+X)/X = L/X \ll_\delta M/X$ is obtained since $Rm \ll_\delta M$. This contradicts with the fact that L is generalized δ-coclosed. Finally, $K \leq \delta(L)$ must be true.

Additionally it is clear that $\delta(L) \leq L \cap \delta(M)$. Suppose that m be an arbitrary element in $L \cap \delta(M)$. Then $m \in \delta(M)$ and $Rm \ll_\delta M$ therefore $Rm \subseteq \delta(M)$. And so $Rm \subseteq \delta(L)$ since L is generalized δ-coclosed. From this reason, $\delta(L) = L \cap \delta(M)$ is obtained.

Theorem 3.11: Let M be a module and $K \leq L \leq M$. If L is generalized δ-coclosed then L/K is also generalized δ-coclosed in M/K.

Proof: Let assume that there is a submodule of X/K of L/K with L/X singular and $\frac{L/K}{X/K} \subseteq \delta\left(\frac{M/K}{X/K}\right)$. Then $L/X \subseteq \delta(M/X)$ and this contradicts with the fact that L is generalized δ-coclosed submodule of M. So, L/K is a generalized δ-coclosed submodule of M/K.

Theorem 3.12: Let M be a generalized weakly δ-supplemented module. If a submodule L of M is generalized δ-coclosed then so is L.

Proof: Let U be an arbitrary submodule of L. Since $U \leq M$ and M is generalized weakly δ-supplemented then there is a submodule V of M such that $U + V = M$ and $U \cap V \leq \delta(M)$. Following this $(U + V) \cap L = U + (V \cap L) = L$ and $U \cap (V \cap L) = (U \cap V) \cap L \leq \delta(M) \cap L = \delta(L)$ by theorem 3.10. Hence, $V \cap L$ is a generalized weakly δ-supplement of U in L.

Theorem 3.13: Let $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ be a short exact sequence. If L and N are generalized weakly δ-supplemented modules and L is a generalized weakly δ-supplement in M then M is also generalized weakly δ-supplemented. Conversely, if L is generalized δ-coclosed and M is generalized weakly δ-supplemented then L and N are generalized weakly δ-supplemented modules.

Proof: Let L be a generalized δ-supplement of X in M. So, $M = L + X$ and $L \cap X \leq \delta(M)$. Following this, $M/L \cap X = L/L \cap X \oplus X/L \cap X$ can be written. Since L is generalized weakly δ-supplemented $L/L \cap X$ is weakly generalized δ-supplemented. By using the isomorphisms $X/L \cap X \cong X + L/L = M/L \cong N$, $X/L \cap X$ is generalized weakly δ-supplemented since N is generalized weakly δ-supplemented. So $M/L \cap X$ is generalized weakly δ-supplemented. Additionally M is also generalized weakly δ-supplemented since $L \cap X \leq \delta(M)$. Conversely, let assume that M is generalized weakly δ-supplemented. Then $M/L \cong N$ is generalized weakly δ-supplemented and L is also generalized weakly δ-supplemented since L is generalized δ-coclosed.

Conflict of Interests

The author declares that there is no conflict of interests.

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