Available online at http://scik.org
Algebra Letters, 2015, 2015:4
ISSN: 2051-5502

$\tilde{\alpha}$ - INTERVAL VALUED FUZZY NEW IDEAL OF PU-ALGEBRA

SAMY M. MOSTAFA, MOKHTAR A. ABDEL NABY AND ALAA ELDIN I. ELKABANY*
Department of mathematics, Ain Shams University, Roxy, Cairo

Abstract: Our aim in this paper is to introduce and study, the notion $\tilde{\alpha}$ -interval valued fuzzy *new*-ideal of a *PU*-algebra. The homomorphic images (pre images) of $\tilde{\alpha}$ -interval valued fuzzy *new*-ideal under homomorphism of a *PU*-algebras have been obtained. And some related results have been derived. Finally, we give the properties of the concept of Cartesian product of an $\tilde{\alpha}$ -interval valued fuzzy *new*-ideal of a *PU*-algebra.

Keywords: ***PU***-algebra; $\tilde{\alpha}$ **-**interval valued fuzzy *new*-ideal.

2010 Mathematics Subject Classification: 03F55, 08A72.

1. Introduction

In 1966, Imai and Iseki [1,2,3] introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [4,5], Hu and Li introduced a wide class of abstract algebras: BCH-algebras. They are shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. In [13], Neggers and Kim introduced the notion of d-algebras, which is a generalization of BCK-algebras and investigated a relation between d-algebras and BCK-algebras. Neggers et al.[14] introduced the notion of Q-algebras , which is a generalization of BCH/BCI/BCK-algebras. Megalai and Tamilarasi [9] introduced the notion of a TM-algebra which is a generalization of BCK/BCI/BCH-algebras and several results are presented .Mostafa et al.[12] introduced a new algebraic structure called ***PU***-algebra, which is a dual for TM-algebra and investigated several basic properties. Moreover they derived new view of several ideals on ***PU***-algebra and studied

*Corresponding author
Received March 29, 2015

some properties of them .The concept of fuzzy sets was introduced by Zadeh [17]. In 1991, Xi [16] applied the concept of fuzzy sets to BCI, BCK, MV-algebras. Since its inception, the theory of fuzzy sets, ideal theory and its fuzzification has developed in many directions and is finding applications in a wide variety of fields [6,7,8,10,11,15]. Here in this paper, we modify the ideas of Xi [16], to introduce the notion, $\tilde{\alpha}$ -interval valued fuzzy ***new***-ideal of a ***PU***-algebra. The homomorphic image (pre image) of $\tilde{\alpha}$ -interval valued fuzzy ***new***-ideal of a ***PU***-algebra under homomorprhism of a ***PU***-algebras are discussed. Many related results have been derived.

2. Preliminaries

Now, we will recall some known concepts related to PU-algebra from the literature, which will be helpful in further study of this article

Definition 2.1 [12] A ***PU***-algebra is a non-empty set X with a constant $0 \in X$ and a binary operation $*$ satisfying the following conditions:

(I) $0 * x = x$,

(II) $(x * z) * (y * z) = y * x$ for any $x, y, z \in X$.

On X we can define a binary relation " $\leq$ " by: $x \leq y$ if and only if $y * x = 0$.

Example 2.2 [12] Let X = {0, 1, 2, 3, 4} in which $*$ is defined by

$*$	0	1	2	3	4
0	0	1	2	3	4
1	4	0	1	2	3
2	3	4	0	1	2
3	2	3	4	0	1
4	1	2	3	4	0

Then (X, $*$, 0) is a ***PU***-algebra.

Proposition 2.3 [12] In a ***PU***-algebra (X, $*$, 0) the following hold for all x, y, z $\in$ X:

(a) $x * x = 0$.

(b) $(x * z) * z = x$.

(c) $x * (y * z) = y * (x * z)$.

(d) $x * (y * x) = y * 0$.

(e) $(x * y) * 0 = y * x$.

(f) If $x \leq y$, then $x * 0 = y * 0$.

(g) $(x * y) * 0 = (x * z) * (y * z)$.

(h) $x * y \leq z$ if and only if $z * y \leq x$.

(i) $x \leq y$ if and only if $y * z \leq x * z$.

(j) In a ***PU***-algebra (X, $*$, 0) , the following are equivalent:

(1) $x = y$, (2) $x * z = y * z$, (3) $z * x = z * y$.

(k) The right and the left cancellation laws hold in X.

(l) $(z * x) * (z * y) = x * y$,

(m) $(x * y) * z = (z * y) * x$.

(n) $(x * y) * (z * u) = (x * z) * (y * u)$ for all x, y, z and u $\in$ X.

Definition 2.4 [12] A non-empty subset *I* of a ***PU***-algebra (X, $*$, 0) is called a sub-algebra of X if $x * y \in I$ whenever $x, y \in I$.

Definition 2.5 [12] A non-empty subset *I* of a ***PU***-algebra (X, $*$, 0) is called a ***new***-ideal of X if,

(i) $0 \in I$,

(ii) $(a * (b * x)) * x \in I$, for all a, b $\in I$ and x $\in X$.

Example 2.6 [12] Let $X = \{0, a, b, c\}$ in which $*$ is defined by the following table:

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Then (X,∗, 0) is a ***PU***-algebra. It is easy to show that I_1 = {0, a}, I_2 = {0, b}, I_3 = {0, c} are ***new***-ideals of *X*.

Lemma 2.7 [12] If (*X*, ∗, 0) is a ***PU***-algebra, then (x ∗ (y ∗ z)) ∗ z = (y ∗ 0) ∗ x for all x, y, z ∈ *X*.

Theorem 2.8 Any sub-algebra *S* of a ***PU***-algebra *X* is a ***new***-ideal of *X*.

Proof: Let *S* be a sub-algebra of a ***PU***-algebra *X*. Let x ∈ *S*, it follows by the definition of sub-algebra and properties of ***PU***-algebra that x ∗ x = 0 ∈ *S*. Let a, b ∈ *S* and x ∈ *X*. Since 0 ∈ *S*, then b ∗ 0 ∈ *S*. Hence (b ∗ 0) ∗ a ∈ *S*. It follows (by Lemma 2.7), that (a ∗ (b ∗ x)) ∗ x = (b ∗ 0) ∗ a. Then we have (a ∗ (b ∗ x)) ∗ x ∈ *S*. Therefore *S* is a ***new***-ideal of *X*.

Definition 2.9 [12] Let (X, ∗, 0) and (X\, ∗\, 0\) be ***PU***-algebras. A map *f*: X → X\ is called a homomorphism if *f*(x ∗ y) = *f*(x) ∗\ *f*(y) for all x, y ∈ X.

Proposition 2.10 Let $(X,*,0)$ and $(X^{\backslash},*^{\backslash},0^{\backslash})$ be ***PU***-algebras and $f: X \to X^{\backslash}$ be a homomorphism, then $\ker f$ is a ***new***-ideal of *X*.

Proof: By the definition of ***PU***-algebra and its properties, we have that $f(0) = f(0*0) = f(0)*^{\backslash} f(0) = 0^{\backslash}$, then $0 \in \ker f$. Let $a,b \in \ker f$ and $x \in X$, it follows (by Lemma 2.7), that $(a*(b*x))*x = (b*0)*a$. Then we have that $f((a*(b*x))*x) = f((b*0)*a) = f(b*0)*^{\backslash} f(a) = (f(b)*^{\backslash} f(0))*^{\backslash} f(a) =$ $(0^{\backslash} *^{\backslash} 0^{\backslash})*^{\backslash} 0^{\backslash} = 0^{\backslash}$. Hence $((a*(b*x))*x) \in \ker f$. Therefore $\ker f$ is a ***new***-ideal of *X*.

3. $\tilde{\alpha}$ -Interval valued fuzzy new-ideal of PU-algebra

In this section, we will discuss and investigate a new notion called $\tilde{\alpha}$ -interval valued fuzzy ***new***-ideal of a ***PU***-algebra and study several basic properties which related to $\tilde{\alpha}$ -interval valued fuzzy ***new***-ideal.

Definition 3.1 [17] Let *X* be a non-empty set, a fuzzy subset μ in *X* is a function $\mu : X \to [0,1]$.

Remark 3.2 [16] An interval-valued fuzzy subset (briefly i-v fuzzy subset) A defined in the set X is given by A = $\{(x,\ [\mu_A^L(x), \mu_A^U(x)])\}$, for all $x \in X$. (briefly, it is denoted by A = $[\mu_A^L(x), \mu_A^U(x)]$ where $\mu_A^L(x)$ and $\mu_A^U(x)$ are any two fuzzy subsets in X such that $\mu_A^L(x) \leq \mu_A^U(x)$ for all $x \in X$. Let $\tilde{\mu}_A(x) = [\mu_A^L(x), \mu_A^U(x)]$, for all $x \in X$ and let D[0,1] be denotes the family of all closed sub-intervals of [0,1]. It is clear that if $\mu_A^L(x) = \mu_A^U(x) = c$, where $0 \leq c \leq 1$, then $\tilde{\mu}_A(x) = [c, c]$ in D[0,1] , then $\tilde{\mu}_A(x) \in$ D[0,1] , for all $x \in X$. Therefore the i-v fuzzy subset A is given by: $A = \{(x, \tilde{\mu}_A(x))\}$, for all $x \in X$, where $\tilde{\mu}_A : X \to$ D[0,1]. Now we define the refined minimum (briefly r min) and order " $\leq$ " on elements $D_1 = [a_1, b_1]$ and $D_2 = [a_2, b_2]$ of D[0, 1] as follows:

r min $(D_1, D_2) = [\min\{a_1, a_2\}, \min\{b_1, b_2\}]$, $D_1 \leq D_2 \Leftrightarrow a_1 \leq a_2$ and $b_1 \leq b_2$. Similarly we can define ($\geq$) and (=). Also we can define $D_1 + D_2 = [a_1 + a_2, b_1 + b_2]$, and if $c \in [0,1]$, then $cD_1 = [ca_1, cb_1]$. Also if $D_i = [a_i, b_i]$, $i \in I$ then we define rsup $(D_i) = [\sup a_i, \sup b_i]$ and rinf $(D_i) = [\inf a_i, \inf b_i]$. We will consider that $\tilde{1} = [1,1]$ and $\tilde{0} = [0,0]$. In what follows, let X denotes a ***PU***-algebra unless otherwise specified, we begin with the following definition.

Definition 3.3 Let X be a ***PU***-algebra. An interval valued fuzzy subset $\tilde{\mu}$ in X is called an interval valued fuzzy sub-algebra of X if $\tilde{\mu}(x * y) \geq r\min\{\tilde{\mu}(x), \tilde{\mu}(y)\}$, for all $x, y \in X$.

Definition 3.4 Let $\tilde{\mu}$ be an interval valued fuzzy subset of a ***PU***-algebra X. Let $\tilde{\alpha} \in$ D[0,1]. Then the interval valued fuzzy set $\tilde{\mu}^{\tilde{\alpha}}$ of X is called the $\tilde{\alpha}$-interval valued fuzzy subset of X (w.r.t. interval valued fuzzy set $\tilde{\mu}$) and is defined by

$$\tilde{\mu}^{\tilde{\alpha}}(x) = r\min\{\tilde{\mu}(x), \tilde{\alpha}\} \quad , \quad \text{for all } x \in X.$$

Remark 3.5 Clearly, $\tilde{\mu}^{\tilde{1}} = \tilde{\mu}$ and $\tilde{\mu}^{\tilde{0}} = \tilde{0}$.

Lemma 3.6 If $\tilde{\mu}$ is an interval valued fuzzy sub-algebra of a ***PU***-algebra X and $\tilde{\alpha} \in$ D[0,1], then $\tilde{\mu}^{\tilde{\alpha}}(x * y) \geq r\min\{\tilde{\mu}^{\tilde{\alpha}}(x), \tilde{\mu}^{\tilde{\alpha}}(y)\}$, for all $x, y \in X$.

Proof: Let X be a ***PU***-algebra and $\tilde{\alpha} \in$ D[0,1]. Then by Definition 3.4, we have that

$$\tilde{\mu}^{\tilde{\alpha}}(x * y) = r\min\{\tilde{\mu}(x * y), \tilde{\alpha}\} \geq r\min\{r\min\{\tilde{\mu}(x), \tilde{\mu}(y)\}, \tilde{\alpha}\}$$
$$= r\min\{r\min\{\tilde{\mu}(x), \tilde{\alpha}\}, r\min\{\tilde{\mu}(y), \tilde{\alpha}\}\}$$
$$= r\min\{\tilde{\mu}^{\tilde{\alpha}}(x), \tilde{\mu}^{\tilde{\alpha}}(y)\}, \text{for all } x, y \in X.$$

Definition 3.7 Let X be a ***PU***-algebra. An interval valued fuzzy subset $\tilde{\mu}^{\tilde{\alpha}}$ in X is called an $\tilde{\alpha}$ -interval valued fuzzy sub-algebra of X if $\tilde{\mu}^{\tilde{\alpha}}(x * y) \geq r\min\{\tilde{\mu}^{\tilde{\alpha}}(x), \tilde{\mu}^{\tilde{\alpha}}(y)\},$ for all $x, y \in X.$

It is clear that an $\tilde{\alpha}$ **-**interval valued fuzzy sub-algebra of a ***PU***-algebra X is a generalization of an interval valued fuzzy sub-algebra of X and an interval valued fuzzy sub-algebra of X is an $\tilde{\alpha}$ **-** interval valued fuzzy sub-algebra of X in case of $\alpha = 1$.

Definition 3.8 Let (X, $*$, 0) be a ***PU***-algebra, an interval valued fuzzy subset $\tilde{\mu}$ in X is called an interval valued fuzzy ***new***-ideal of X if it satisfies the following conditions:

$(\tilde{F}_1)$ $\tilde{\mu}(0) \geq \tilde{\mu}(x),$

$(\tilde{F}_2)$ $\tilde{\mu}((x * (y * z)) * z) \geq r\min\{\tilde{\mu}(x), \tilde{\mu}(y)\},$ for all $x, y, z \in X$.

Lemma 3.9 If $\tilde{\mu}$ is an interval valued fuzzy ***new***-ideal of a ***PU***-algebra X and $\tilde{\alpha} \in$ D[0,1] , then

$(\tilde{F}_1^{\tilde{\alpha}})$ $\tilde{\mu}^{\tilde{\alpha}}(0) \geq \tilde{\mu}^{\tilde{\alpha}}(x),$

$(\tilde{F}_2^{\tilde{\alpha}})$ $\tilde{\mu}^{\tilde{\alpha}}((x * (y * z)) * z) \geq r\min\{\tilde{\mu}^{\tilde{\alpha}}(x), \tilde{\mu}^{\tilde{\alpha}}(y)\},$ for all $x, y, z \in X$.

Proof: Let X be a ***PU***-algebra and $\tilde{\alpha} \in$D[0,1]. Then by Definition 3.4 and Definition 3.8, we have that: $\tilde{\mu}^{\tilde{\alpha}}(0) = r\min\{\tilde{\mu}(0), \tilde{\alpha}\} \geq r\min\{\tilde{\mu}(x), \tilde{\alpha}\} = \tilde{\mu}^{\tilde{\alpha}}(x),$ for all $x \in X$.

$$\tilde{\mu}^{\tilde{\alpha}}((x * (y * z)) * z) = r\min\{\tilde{\mu}((x * (y * z)) * z), \tilde{\alpha}\}$$
$$\geq r\min\{r\min\{\tilde{\mu}(x), \tilde{\mu}(y)\}, \tilde{\alpha}\}$$
$$= r\min\{r\min\{\tilde{\mu}(x), \tilde{\alpha}\}, r\min\{\tilde{\mu}(y), \tilde{\alpha}\}\}$$
$$= r\min\{\tilde{\mu}^{\tilde{\alpha}}(x), \tilde{\mu}^{\tilde{\alpha}}(y)\}, \text{ for all } x, y, z \in X.$$

Definition 3.10 Let (X, $*$, 0) be a ***PU***-algebra, an $\tilde{\alpha}$ **-**interval valued fuzzy subset $\tilde{\mu}^{\tilde{\alpha}}$ in X is called an $\tilde{\alpha}$ **-**interval valued fuzzy ***new***-ideal of X if it satisfies the following conditions:

$(\tilde{F}_1^{\tilde{\alpha}})$ $\tilde{\mu}^{\tilde{\alpha}}(0) \geq \tilde{\mu}^{\tilde{\alpha}}(x)$,

$(\tilde{F}_2^{\tilde{\alpha}})$ $\tilde{\mu}^{\tilde{\alpha}}((x*(y*z))*z) \geq r\min\{\tilde{\mu}^{\tilde{\alpha}}(x), \tilde{\mu}^{\tilde{\alpha}}(y)\}$, for all $x, y, z \in X$.

It is clear that an $\tilde{\alpha}$ **-**interval valued fuzzy ***new***-ideal of a ***PU***-algebra X is a generalization of an interval valued fuzzy ***new***-ideal of X and an interval valued fuzzy ***new***-ideal of X is special case, when $\alpha = 1$.

Example 3.11 Let X = {0, 1, 2, 3} in which $*$ is defined by the following table:

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then (X, $*$, 0) is a ***PU***-algebra.

Define an $\tilde{\alpha}$ **-**interval valued fuzzy subset $\tilde{\mu}^{\tilde{\alpha}} : X \to D[0,1]$ by

$$\tilde{\mu}^{\tilde{\alpha}}(x) = \begin{cases} r\min\{\tilde{\alpha}, [0.3, 0.9]\} & if \quad x \in \{0,1\} \\ r\min\{\tilde{\alpha}, [0.1, 0.6]\} & otherwise \end{cases}$$

Routine calculation gives that $\tilde{\mu}^{\tilde{\alpha}}$ is an $\tilde{\alpha}$ **-**interval valued fuzzy ***new***-ideal of X.

Lemma 3.12 Let $\tilde{\mu}^{\tilde{\alpha}}$ be an $\tilde{\alpha}$ **-**interval valued fuzzy ***new***-ideal of a ***PU***-algebra X. If the inequality $x*y \leq z$ holds in X, then $\tilde{\mu}^{\tilde{\alpha}}(y) \geq r\min\{\tilde{\mu}^{\tilde{\alpha}}(x), \tilde{\mu}^{\tilde{\alpha}}(z)\}$.

Proof: Assume that the inequality $x*y \leq z$ holds in X, then $z*(x*y) = 0$ and by $(\tilde{F}_2^{\tilde{\alpha}})$ $\tilde{\mu}^{\tilde{\alpha}}(\overbrace{(z*(x*y))}^{0}*y) \geq r\min\{\tilde{\mu}^{\tilde{\alpha}}(x), \tilde{\mu}^{\tilde{\alpha}}(z)\}$. Since $\tilde{\mu}^{\tilde{\alpha}}(y) = \tilde{\mu}^{\tilde{\alpha}}(0*y)$, then we have that $\tilde{\mu}^{\tilde{\alpha}}(y) \geq r\min\{\tilde{\mu}^{\tilde{\alpha}}(x), \tilde{\mu}^{\tilde{\alpha}}(z)\}$.

Corollary 3.13 Let $\tilde{\mu}$ be an interval valued fuzzy ***new***-ideal of a ***PU***-algebra *X*. If the inequality $x * y \le z$ holds in *X*, then $\tilde{\mu}(y) \ge r\min\{\tilde{\mu}(x), \tilde{\mu}(z)\}$.

Lemma 3.14 If $\tilde{\mu}^{\tilde{\alpha}}$ is an $\tilde{\alpha}$ **-**interval valued fuzzy subset of a ***PU***-algebra *X* and if $x \le y$, then $\tilde{\mu}^{\tilde{\alpha}}(x) = \tilde{\mu}^{\tilde{\alpha}}(y)$.

Proof: If $x \le y$, then $y * x = 0$. Hence by the definition of ***PU***-algebra and its properties we have that $\tilde{\mu}^{\tilde{\alpha}}(x) = r\min\{\tilde{\mu}(x), \tilde{\alpha}\}$ $= r\min\{\tilde{\mu}(0 * x), \tilde{\alpha}\}$ $= r\min\{\tilde{\mu}((y * x) * x), \tilde{\alpha}\} = r\min\{\tilde{\mu}(y), \tilde{\alpha}\} = \tilde{\mu}^{\tilde{\alpha}}(y)$.

Corollary 3.15 If $\tilde{\mu}$ is an interval valued fuzzy subset of a ***PU***-algebra *X* and if $x \le y$, then $\tilde{\mu}(x) = \tilde{\mu}(y)$.

Definition 3.16 Let $\tilde{\mu}^{\tilde{\alpha}}$ be an $\tilde{\alpha}$ -interval valued fuzzy ***new***-ideal of a ***PU***-algebra *X* and let *x* be an element of *X*. We define $(\bigcap_{i\in I} \tilde{\mu}_i^{\tilde{\alpha}})(x) = r\inf(\tilde{\mu}_i^{\tilde{\alpha}}(x))_{i\in I}$.

Proposition 3.17 The intersection of any set of $\tilde{\alpha}$ **-**interval valued fuzzy ***new***-ideals of a ***PU***-algebra *X* is also an $\tilde{\alpha}$ -interval valued fuzzy ***new***-ideal of *X*.

Proof: Let $\{\tilde{\mu}_i^{\tilde{\alpha}}\}_{i\in I}$ be a family of $\tilde{\alpha}$ **-**interval valued fuzzy ***new***-ideals of a ***PU***-algebra *X*, then for any $x, y, z \in X$, $(\bigcap_{i\in I} \tilde{\mu}_i^{\tilde{\alpha}})(0) = r\inf(\tilde{\mu}_i^{\tilde{\alpha}}(0))_{i\in I} \ge r\inf(\tilde{\mu}_i^{\tilde{\alpha}}(x))_{i\in I} = (\bigcap_{i\in I} \tilde{\mu}_i^{\tilde{\alpha}})(x)$ and

$$(\bigcap_{i\in I} \tilde{\mu}_i^{\tilde{\alpha}})((x * (y * z)) * z) = r\inf(\tilde{\mu}_i^{\tilde{\alpha}}((x * (y * z)) * z)_{i\in I}$$

$$\ge r\inf(r\min\{\tilde{\mu}_i^{\tilde{\alpha}}(x), \tilde{\mu}_i^{\tilde{\alpha}}(y)\})_{i\in I}$$

$$= r\min\{r\inf(\tilde{\mu}_i^{\tilde{\alpha}}(x))_{i\in I}, r\inf(\tilde{\mu}_i^{\tilde{\alpha}}(y))_{i\in I}\}$$

$$= r\min\{(\bigcap_{i\in I} \tilde{\mu}_i^{\tilde{\alpha}})(x), (\bigcap_{i\in I} \tilde{\mu}_i^{\tilde{\alpha}})(y)\}.$$ This completes the proof.

Theorem 3.18 Let $\tilde{\mu}^{\tilde{\alpha}}$ be an $\tilde{\alpha}$ **-**interval valued fuzzy subset of a ***PU***-algebra *X*. Then $\tilde{\mu}^{\tilde{\alpha}}$ is an $\tilde{\alpha}$ **-**interval valued fuzzy ***new***-ideal of *X* if and only if it satisfies:

$(\forall \tilde{\varepsilon} \in D[0,1])$ $(U(\tilde{\mu}^{\tilde{\alpha}}; \tilde{\varepsilon}) \neq \phi \Rightarrow U(\tilde{\mu}^{\tilde{\alpha}}; \tilde{\varepsilon})$ is a ***new***-ideal of *X*), where

$U(\tilde{\mu}^{\tilde{\alpha}}; \tilde{\varepsilon}) = \{x \in X : \tilde{\mu}^{\tilde{\alpha}}(x) \geq \tilde{\varepsilon}\}.$

Proof: Assume that $\tilde{\mu}^{\tilde{\alpha}}$ is an $\tilde{\alpha}$ -interval valued fuzzy ***new***-ideal of *X*. Let $\tilde{\varepsilon} \in D[0,1]$ be such that $U(\tilde{\mu}^{\tilde{\alpha}}; \tilde{\varepsilon}) \neq \phi$. Let $x \in U(\tilde{\mu}^{\tilde{\alpha}}; \tilde{\varepsilon})$, then $\tilde{\mu}^{\tilde{\alpha}}(x) \geq \tilde{\varepsilon}$. Since $\tilde{\mu}^{\tilde{\alpha}}(0) \geq \tilde{\mu}^{\tilde{\alpha}}(x)$ for all $x \in X$, then $\tilde{\mu}^{\tilde{\alpha}}(0) \geq \tilde{\varepsilon}$. Thus $0 \in U(\tilde{\mu}^{\tilde{\alpha}}; \tilde{\varepsilon})$. Let $x \in X$ and $a, b \in U(\tilde{\mu}^{\tilde{\alpha}}; \tilde{\varepsilon})$, then $\tilde{\mu}^{\tilde{\alpha}}(a) \geq \tilde{\varepsilon}$ and $\tilde{\mu}^{\tilde{\alpha}}(b) \geq \tilde{\varepsilon}$. It follows by the definition of $\tilde{\alpha}$ -interval valued fuzzy ***new***-ideal that $\tilde{\mu}^{\tilde{\alpha}}((a*(b*x))*x) \geq r\min\{\tilde{\mu}^{\tilde{\alpha}}(a), \tilde{\mu}^{\tilde{\alpha}}(b)\} \geq \tilde{\varepsilon}$, so that $(a*(b*x))*x \in U(\tilde{\mu}^{\tilde{\alpha}}; \tilde{\varepsilon})$. Hence $U(\tilde{\mu}^{\tilde{\alpha}}; \tilde{\varepsilon})$ is a ***new***-ideal of *X*.

Conversely, suppose that $(\forall \tilde{\varepsilon} \in D[0,1])(U(\tilde{\mu}^{\tilde{\alpha}}; \tilde{\varepsilon}) \neq \phi \Rightarrow U(\tilde{\mu}^{\tilde{\alpha}}; \tilde{\varepsilon})$ is a ***new***-ideal of *X*), where $U(\tilde{\mu}^{\tilde{\alpha}}; \tilde{\varepsilon}) = \{x \in X : \tilde{\mu}^{\tilde{\alpha}}(x) \geq \tilde{\varepsilon}\}$. If $\tilde{\mu}^{\tilde{\alpha}}(0) < \tilde{\mu}^{\tilde{\alpha}}(x)$ for some $x \in X$, then $\tilde{\mu}^{\tilde{\alpha}}(0) < \tilde{\varepsilon}_0 < \tilde{\mu}^{\tilde{\alpha}}(x)$ by taking $\tilde{\varepsilon}_0 = (\tilde{\mu}^{\tilde{\alpha}}(0) + \tilde{\mu}^{\tilde{\alpha}}(x))/2$. Hence $0 \notin U(\tilde{\mu}^{\tilde{\alpha}}; \tilde{\varepsilon}_0)$, which is a contradiction.

Let $a, b, c \in X$ be such that $\tilde{\mu}^{\tilde{\alpha}}((a*(b*c))*c) < r\min\{\tilde{\mu}^{\tilde{\alpha}}(a), \tilde{\mu}^{\tilde{\alpha}}(b)\}$. Taking $\tilde{\varepsilon}_1 = (\tilde{\mu}^{\tilde{\alpha}}(a*(b*c)*c) + r\min\{\tilde{\mu}^{\tilde{\alpha}}(a), \tilde{\mu}^{\tilde{\alpha}}(b)\})/2$, we have $\tilde{\varepsilon}_1 \in D[0,1]$ and $\tilde{\mu}^{\tilde{\alpha}}((a*(b*c))*c) < \tilde{\varepsilon}_1 < r\min\{\tilde{\mu}^{\tilde{\alpha}}(a), \tilde{\mu}^{\tilde{\alpha}}(b)\}$. It follows that $a, b \in U(\tilde{\mu}^{\tilde{\alpha}}; \tilde{\varepsilon}_1)$ and $(a*(b*c))*c \notin U(\tilde{\mu}^{\tilde{\alpha}}; \tilde{\varepsilon}_1)$. This is a contradiction, and therefore $\tilde{\mu}^{\tilde{\alpha}}$ is an $\tilde{\alpha}$ -interval valued fuzzy ***new***-ideal of *X*.

Corollary 3.19 Let $\tilde{\mu}$ be an interval valued fuzzy subset of a ***PU***-algebra *X*. Then $\tilde{\mu}$ is an interval valued fuzzy ***new***-ideal of *X* if and only if it satisfies:

$(\forall \tilde{\varepsilon} \in D[0,1])$ $(U(\tilde{\mu}; \tilde{\varepsilon}) \neq \phi \Rightarrow U(\tilde{\mu}; \tilde{\varepsilon})$ is a ***new***-ideal of *X*),

where $U(\tilde{\mu}; \tilde{\varepsilon}) = \{x \in X : \tilde{\mu}(x) \geq \tilde{\varepsilon}\}$.

Definition 3.20 Let *f* be a mapping from *X* to *Y*. If $\tilde{\mu}^{\tilde{\alpha}}$ is an $\tilde{\alpha}$ -interval valued fuzzy subset of *X*, then the $\tilde{\alpha}$ -interval valued fuzzy subset $\tilde{\beta}^{\tilde{\alpha}}$ of *Y* defined by

$$f(\tilde{\mu}^{\tilde{\alpha}})(y) = \tilde{\beta}^{\tilde{\alpha}}(y) = \begin{cases} r\sup\limits_{x \in f^{-1}(y)} \tilde{\mu}^{\tilde{\alpha}}(x) & if \quad f^{-1}(y) \neq \phi \\ \tilde{0} & otherwise \end{cases}$$

is said to be the image of $\tilde{\mu}^{\tilde{\alpha}}$ under *f*.

Similarly if $\tilde{\beta}^{\tilde{\alpha}}$ is an $\tilde{\alpha}$ **-**interval valued fuzzy subset of *Y*, then the $\tilde{\alpha}$ **-**interval valued fuzzy subset $\tilde{\mu}^{\tilde{\alpha}} = (\tilde{\beta}^{\tilde{\alpha}} \circ f)$ of *X* (i.e. the $\tilde{\alpha}$ -interval valued fuzzy subset defined by $\tilde{\mu}^{\tilde{\alpha}}(x) = \tilde{\beta}^{\tilde{\alpha}}(f(x))$ for all $x \in X$) is called the pre-image of $\tilde{\beta}^{\tilde{\alpha}}$ under *f*.

Theorem 3.21 Let $(X,*,0)$ and $(X^{\backslash},*^{\backslash},0^{\backslash})$ be ***PU***-algebras and $f : X \to X^{\backslash}$ be a homomorphism. If $\tilde{\beta}^{\tilde{\alpha}}$ is an $\tilde{\alpha}$ -interval valued fuzzy ***new***-ideal of $X^{\backslash}$ and $\tilde{\mu}^{\tilde{\alpha}}$ is the pre-image of $\tilde{\beta}^{\tilde{\alpha}}$ under *f*, then $\tilde{\mu}^{\tilde{\alpha}}$ is an $\tilde{\alpha}$ -interval valued fuzzy ***new***-ideal of *X*.

Proof: Since $\tilde{\mu}^{\tilde{\alpha}}$ is the pre-image of $\tilde{\beta}^{\tilde{\alpha}}$ under *f*, then $\tilde{\mu}^{\tilde{\alpha}}(x) = \tilde{\beta}^{\tilde{\alpha}}(f(x))$ for all $x \in X$. Let $x \in X$, then $\tilde{\mu}^{\tilde{\alpha}}(0) = \tilde{\beta}^{\tilde{\alpha}}(f(0)) \geq \tilde{\beta}^{\tilde{\alpha}}(f(x)) = \tilde{\mu}^{\tilde{\alpha}}(x).$ Now let $x, y, z \in X$, then

$$
\begin{aligned}
\tilde{\mu}^{\tilde{\alpha}}((x*(y*z))*z) &= \tilde{\beta}^{\tilde{\alpha}}(f((x*(y*z))*z)) \\
&= \tilde{\beta}^{\tilde{\alpha}}(f(x*(y*z))*^{\backslash} f(z)) \\
&= \tilde{\beta}^{\tilde{\alpha}}((f(x)*^{\backslash} f(y*z))*^{\backslash} f(z)) \\
&= \tilde{\beta}^{\tilde{\alpha}}((f(x)*^{\backslash}(f(y)*^{\backslash} f(z)))*^{\backslash} f(z)) \\
&\geq r\min\{\tilde{\beta}^{\tilde{\alpha}}(f(x)), \tilde{\beta}^{\tilde{\alpha}}(f(y))\} \\
&= r\min\{\tilde{\mu}^{\tilde{\alpha}}(x), \tilde{\mu}^{\tilde{\alpha}}(y)\},
\end{aligned}
$$

and the proof is completed.

Theorem 3.22 Let $(X,*,0)$ and $(Y,*^{\backslash},0^{\backslash})$ be ***PU***-algebras. Let $f : X \to Y$ be a homomorphism, $\tilde{\mu}^{\tilde{\alpha}}$ be an $\tilde{\alpha}$ **-**interval valued fuzzy subset of X and $\tilde{\beta}^{\tilde{\alpha}}$ be the image of $\tilde{\mu}^{\tilde{\alpha}}$ under *f*. If $\tilde{\mu}^{\tilde{\alpha}}$ is an $\tilde{\alpha}$ **-**interval valued fuzzy ***new***-ideal of *X*, then $\tilde{\beta}^{\tilde{\alpha}}$ is an $\tilde{\alpha}$ **-**interval valued fuzzy ***new***-ideal of *Y*.

Proof: Since $0 \in f^{-1}(0^{\backslash})$, then $f^{-1}(0^{\backslash}) \neq \phi.$ It follows that $\tilde{\beta}^{\tilde{\alpha}}(0^{\backslash}) = r\sup_{t \in f^{-1}(0^{\backslash})} \tilde{\mu}^{\tilde{\alpha}}(t) = \tilde{\mu}^{\tilde{\alpha}}(0) \geq \tilde{\mu}^{\tilde{\alpha}}(x),$ for all $x \in X$. Thus $\tilde{\beta}^{\tilde{\alpha}}(0^{\backslash}) = r\sup_{t \in f^{-1}(x^{\backslash})} \tilde{\mu}^{\tilde{\alpha}}(t)$ for all $x^{\backslash} \in Y$. Hence $\tilde{\beta}^{\tilde{\alpha}}(0^{\backslash}) \geq \tilde{\beta}^{\tilde{\alpha}}(x^{\backslash})$ for all $x^{\backslash} \in Y$.

For any $x^{\backslash}, y^{\backslash}, z^{\backslash} \in Y$, If $f^{-1}(x^{\backslash}) = \phi$ or $f^{-1}(y^{\backslash}) = \phi$, then $\tilde{\beta}^{\tilde{\alpha}}(x^{\backslash}) = \tilde{0}$ or $\tilde{\beta}^{\tilde{\alpha}}(y^{\backslash}) = \tilde{0}$, it follows that $r\min\{\tilde{\beta}^{\tilde{\alpha}}(x^{\backslash}), \tilde{\beta}^{\tilde{\alpha}}(y^{\backslash})\} = \tilde{0}$ and hence

$\tilde{\beta}^{\tilde{\alpha}}((x^{\backslash} *^{\backslash} (y^{\backslash} *^{\backslash} z^{\backslash})) *^{\backslash} z^{\backslash}) \geq r\min\{\tilde{\beta}^{\tilde{\alpha}}(x^{\backslash}), \tilde{\beta}^{\tilde{\alpha}}(y^{\backslash})\}$.

If $f^{-1}(x^{\backslash}) \neq \phi$ and $f^{-1}(y^{\backslash}) \neq \phi$, let $x_0 \in f^{-1}(x^{\backslash})$, $y_0 \in f^{-1}(y^{\backslash})$ be such that $\tilde{\mu}^{\tilde{\alpha}}(x_0) = r\sup_{t \in f^{-1}(x^{\backslash})} \tilde{\mu}^{\tilde{\alpha}}(t)$ and $\tilde{\mu}^{\tilde{\alpha}}(y_0) = r\sup_{t \in f^{-1}(y^{\backslash})} \tilde{\mu}^{\tilde{\alpha}}(t)$. It follows by given and properties of ***PU***-algebra that

$\tilde{\beta}^{\tilde{\alpha}}((x^{\backslash} *^{\backslash} (y^{\backslash} *^{\backslash} z^{\backslash})) *^{\backslash} z^{\backslash}) = \tilde{\beta}^{\tilde{\alpha}}((z^{\backslash} *^{\backslash} (y^{\backslash} *^{\backslash} z^{\backslash})) *^{\backslash} x^{\backslash}) = \tilde{\beta}^{\tilde{\alpha}}((y^{\backslash} *^{\backslash} (z^{\backslash} *^{\backslash} z^{\backslash})) *^{\backslash} x^{\backslash}) =$

$\tilde{\beta}^{\tilde{\alpha}}((y^{\backslash} *^{\backslash} 0^{\backslash}) *^{\backslash} x^{\backslash}) = \tilde{\beta}^{\tilde{\alpha}}((f(y_0) *^{\backslash} f(0)) *^{\backslash} f(x_0)) = \tilde{\beta}^{\tilde{\alpha}}(f((y_0 * 0) * x_0)) = \tilde{\mu}^{\tilde{\alpha}}((y_0 * 0) * x_0)$

$= \tilde{\mu}^{\tilde{\alpha}}((y_0 * (z_0 * z_0)) * x_0) = \tilde{\mu}^{\tilde{\alpha}}((z_0 * (y_0 * z_0)) * x_0) = \tilde{\mu}^{\tilde{\alpha}}((x_0 * (y_0 * z_0)) * z_0) \geq$

$r\min\{\tilde{\mu}^{\tilde{\alpha}}(x_0), \tilde{\mu}^{\tilde{\alpha}}(y_0)\} = r\min\{r\sup_{t \in f^{-1}(x^{\backslash})} \tilde{\mu}^{\tilde{\alpha}}(t), r\sup_{t \in f^{-1}(y^{\backslash})} \tilde{\mu}^{\tilde{\alpha}}(t)\} = r\min\{\tilde{\beta}^{\tilde{\alpha}}(x^{\backslash}), \tilde{\beta}^{\tilde{\alpha}}(y^{\backslash})\}$.

Hence $\tilde{\beta}^{\tilde{\alpha}}$ is an $\tilde{\alpha}$ -interval valued fuzzy ***new***-ideal of *Y*.

Corollary 3.23 Let $(X,*,0)$ and $(Y,*^{\backslash},0^{\backslash})$ be ***PU***-algebras, $f : X \to Y$ be a homomorphism, $\tilde{\mu}$ be an interval valued fuzzy subset of *X*, $\tilde{\beta}$ be the image of $\tilde{\mu}$ under *f*. If $\tilde{\mu}$ is an interval valued fuzzy ***new***-ideal of *X*, then $\tilde{\beta}$ is an interval valued fuzzy ***new***-ideal of *Y*.

4. Cartesian product of $\tilde{\alpha}$ -interval valued fuzzy new-ideals of PU-algebras

In this section, we introduce the concept of Cartesian product of an $\tilde{\alpha}$ -interval valued fuzzy ***new***-ideal of a ***PU***-algebra.

Definition 4.1 An $\tilde{\alpha}$ -interval valued fuzzy relation on any set *S* is an $\tilde{\alpha}$ -interval valued fuzzy subset $\tilde{\mu}^{\tilde{\alpha}} : S \times S \to D[0,1]$.

Definition 4.2 If $\tilde{\mu}^{\tilde{\alpha}}$ is an $\tilde{\alpha}$ -interval valued fuzzy relation on a set *S* and $\tilde{\beta}^{\tilde{\alpha}}$ is an $\tilde{\alpha}$ -interval valued fuzzy subset of *S*, then $\tilde{\mu}^{\tilde{\alpha}}$ is an $\tilde{\alpha}$ -interval valued fuzzy relation on $\tilde{\beta}^{\tilde{\alpha}}$ if $\tilde{\mu}^{\tilde{\alpha}}(x,y) \leq r\min\{\tilde{\beta}^{\tilde{\alpha}}(x), \tilde{\beta}^{\tilde{\alpha}}(y)\}$, for all $x, y \in S$.

Definition 4.3 If $\widetilde{\beta}^{\tilde{\alpha}}$ is an $\tilde{\alpha}$ **-**interval valued fuzzy subset of a set *S*, the strongest $\tilde{\alpha}$ **-**interval valued fuzzy relation on *S* that is an $\tilde{\alpha}$ **-**interval valued fuzzy relation on $\widetilde{\beta}^{\tilde{\alpha}}$ is $\widetilde{\mu}^{\tilde{\alpha}}_{\widetilde{\beta}^{\tilde{\alpha}}}$ given by $\widetilde{\mu}^{\tilde{\alpha}}_{\widetilde{\beta}^{\tilde{\alpha}}}(x,y) = r\min\{\widetilde{\beta}^{\tilde{\alpha}}(x), \widetilde{\beta}^{\tilde{\alpha}}(y)\}$ for all $x, y \in S$.

Definition 4.4 We define the binary operation * on the Cartesian product $X \times X$ as follows: $(x_1, x_2) * (y_1, y_2) = (x_1 * y_1, x_2 * y_2)$ for all $(x_1, x_2), (y_1, y_2) \in X \times X$.

Lemma 4.5 If (*X*, ∗, 0) is a ***PU***-algebra, then $(X \times X, *, (0,0))$ is a ***PU***-algebra, where $(x_1, x_2) * (y_1, y_2) = (x_1 * y_1, x_2 * y_2)$ for all $(x_1, x_2), (y_1, y_2) \in X \times X$.

Proof: Clear.

Theorem 4.6 Let $\widetilde{\beta}^{\tilde{\alpha}}$ be an $\tilde{\alpha}$ **-**interval valued fuzzy subset of a ***PU***-algebra *X* and $\widetilde{\mu}^{\tilde{\alpha}}_{\widetilde{\beta}^{\tilde{\alpha}}}$ be the strongest $\tilde{\alpha}$ **-**interval valued fuzzy relation on *X*, then $\widetilde{\beta}^{\tilde{\alpha}}$ is an $\tilde{\alpha}$ **-**interval valued fuzzy ***new***-ideal of *X* if and only if $\widetilde{\mu}^{\tilde{\alpha}}_{\widetilde{\beta}^{\tilde{\alpha}}}$ is an $\tilde{\alpha}$ **-**interval valued fuzzy ***new***-ideal of $X \times X$.

Proof: $(\Rightarrow)$: Assume that $\widetilde{\beta}^{\tilde{\alpha}}$ is an $\tilde{\alpha}$ **-**interval valued fuzzy ***new***-ideal of *X*, we note from $(\widetilde{F}_1^{\tilde{\alpha}})$ that:

$$\widetilde{\mu}^{\tilde{\alpha}}_{\widetilde{\beta}^{\tilde{\alpha}}}(0,0) = r\min\{\widetilde{\beta}^{\tilde{\alpha}}(0), \widetilde{\beta}^{\tilde{\alpha}}(0)\} \geq r\min\{\widetilde{\beta}^{\tilde{\alpha}}(x), \widetilde{\beta}^{\tilde{\alpha}}(y)\} = \widetilde{\mu}^{\tilde{\alpha}}_{\widetilde{\beta}^{\tilde{\alpha}}}(x,y) \text{ for all } x, y \in X.$$

Now, for any $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in X \times X$, we have from $(\widetilde{F}_2^{\tilde{\alpha}})$:

$$\widetilde{\mu}^{\tilde{\alpha}}_{\widetilde{\beta}^{\tilde{\alpha}}}(((x_1, x_2) * ((y_1, y_2) * (z_1, z_2))) * (z_1, z_2)) = \widetilde{\mu}^{\tilde{\alpha}}_{\widetilde{\beta}^{\tilde{\alpha}}}(((x_1, x_2) * (y_1 * z_1, y_2 * z_2)) * (z_1, z_2)) =$$
$$\widetilde{\mu}^{\tilde{\alpha}}_{\widetilde{\beta}^{\tilde{\alpha}}}((x_1 * (y_1 * z_1), x_2 * (y_2 * z_2)) * (z_1, z_2)) =$$
$$\widetilde{\mu}^{\tilde{\alpha}}_{\widetilde{\beta}^{\tilde{\alpha}}}((x_1 * (y_1 * z_1)) * z_1, (x_2 * (y_2 * z_2)) * z_2) =$$
$$r\min\{\widetilde{\beta}^{\tilde{\alpha}}((x_1 * (y_1 * z_1)) * z_1), \widetilde{\beta}^{\tilde{\alpha}}((x_2 * (y_2 * z_2)) * z_2)\} \geq$$
$$r\min\{r\min\{\widetilde{\beta}^{\tilde{\alpha}}(x_1), \widetilde{\beta}^{\tilde{\alpha}}(y_1)\}, r\min\{\widetilde{\beta}^{\tilde{\alpha}}(x_2), \widetilde{\beta}^{\tilde{\alpha}}(y_2)\}\} =$$
$$r\min\{r\min\{\widetilde{\beta}^{\tilde{\alpha}}(x_1), \widetilde{\beta}^{\tilde{\alpha}}(x_2)\}, r\min\{\widetilde{\beta}^{\tilde{\alpha}}(y_1), \widetilde{\beta}^{\tilde{\alpha}}(y_2)\}\} = r\min\{\widetilde{\mu}^{\tilde{\alpha}}_{\widetilde{\beta}^{\tilde{\alpha}}}(x_1, x_2), \widetilde{\mu}^{\tilde{\alpha}}_{\widetilde{\beta}^{\tilde{\alpha}}}(y_1, y_2)\}.$$

Hence $\widetilde{\mu}^{\tilde{\alpha}}_{\widetilde{\beta}^{\tilde{\alpha}}}$ is an $\tilde{\alpha}$ **-**interval valued fuzzy ***new***-ideal of $X \times X$.

$(\Leftarrow)$: For all $(x,x)\in X\times X$, we have $\tilde{\mu}^{\tilde{\alpha}}_{\tilde{\beta}^{\tilde{\alpha}}}(0,0) = r\min\{\tilde{\beta}^{\tilde{\alpha}}(0),\tilde{\beta}^{\tilde{\alpha}}(0)\} \geq \tilde{\mu}^{\tilde{\alpha}}_{\tilde{\beta}^{\tilde{\alpha}}}(x,x)$.

Then $\tilde{\beta}^{\tilde{\alpha}}(0) = r\min\{\tilde{\beta}^{\tilde{\alpha}}(0),\tilde{\beta}^{\tilde{\alpha}}(0)\} \geq r\min\{\tilde{\beta}^{\tilde{\alpha}}(x),\tilde{\beta}^{\tilde{\alpha}}(x)\} = \tilde{\beta}^{\tilde{\alpha}}(x)$ for all $x\in X$.

Now, for all $x,y,z\in X$, we have

$$\begin{aligned}
\tilde{\beta}^{\tilde{\alpha}}((x*(y*z))*z) &= r\min\{\tilde{\beta}^{\tilde{\alpha}}((x*(y*z))*z),\tilde{\beta}^{\tilde{\alpha}}((x*(y*z))*z)\} \\
&= \tilde{\mu}^{\tilde{\alpha}}_{\tilde{\beta}^{\tilde{\alpha}}}((x*(y*z))*z,(x*(y*z))*z) \\
&= \tilde{\mu}^{\tilde{\alpha}}_{\tilde{\beta}^{\tilde{\alpha}}}((x*(y*z),x*(y*z))*(z,z)) \\
&= \tilde{\mu}^{\tilde{\alpha}}_{\tilde{\beta}^{\tilde{\alpha}}}(((x,x)*((y*z),(y*z)))*(z,z)) \\
&= \tilde{\mu}^{\tilde{\alpha}}_{\tilde{\beta}^{\tilde{\alpha}}}(((x,x)*((y,y)*(z,z)))*(z,z)) \\
&\geq r\min\{\tilde{\mu}^{\tilde{\alpha}}_{\tilde{\beta}^{\tilde{\alpha}}}(x,x),\tilde{\mu}^{\tilde{\alpha}}_{\tilde{\beta}^{\tilde{\alpha}}}(y,y)\} \\
&= r\min\{r\min\{\tilde{\beta}^{\tilde{\alpha}}(x),\tilde{\beta}^{\tilde{\alpha}}(x)\},r\min\{\tilde{\beta}^{\tilde{\alpha}}(y),\tilde{\beta}^{\tilde{\alpha}}(y)\}\} \\
&= r\min\{\tilde{\beta}^{\tilde{\alpha}}(x),\tilde{\beta}^{\tilde{\alpha}}(y)\}.
\end{aligned}$$

Hence $\tilde{\beta}^{\tilde{\alpha}}$ is an $\tilde{\alpha}$ -interval valued fuzzy ***new***-ideal of X.

Definition 4.7 Let $\tilde{\mu}$ and $\tilde{\delta}$ be the interval valued fuzzy subsets in *X*. The Cartesian product $\tilde{\mu}\times\tilde{\delta}: X\times X\to D[0,1]$ is defined by $(\tilde{\mu}\times\tilde{\delta})(x,y) = r\min\{\tilde{\mu}(x),\tilde{\delta}(y)\}$, for all $x,y\in X$.

Definition 4.8 Let $\tilde{\mu}^{\tilde{\alpha}}$ and $\tilde{\delta}^{\tilde{\alpha}}$ be the $\tilde{\alpha}$ -interval valued fuzzy subsets in *X*. The Cartesian product $\tilde{\mu}^{\tilde{\alpha}}\times\tilde{\delta}^{\tilde{\alpha}}: X\times X\to D[0,1]$ is defined by

$$(\tilde{\mu}^{\tilde{\alpha}}\times\tilde{\delta}^{\tilde{\alpha}})(x,y) = r\min\{\tilde{\mu}^{\tilde{\alpha}}(x),\tilde{\delta}^{\tilde{\alpha}}(y)\},\ \text{for all } x,y\in X.$$

Theorem 4.9 If $\tilde{\mu}^{\tilde{\alpha}}$ and $\tilde{\delta}^{\tilde{\alpha}}$ are $\tilde{\alpha}$ -interval valued fuzzy ***new***-ideals in a ***PU***-algebra *X*, then $\tilde{\mu}^{\tilde{\alpha}}\times\tilde{\delta}^{\tilde{\alpha}}$ is an $\tilde{\alpha}$ -interval valued fuzzy ***new***-ideal in $X\times X$.

Proof:

$(\tilde{\mu}^{\tilde{\alpha}} \times \tilde{\delta}^{\tilde{\alpha}})(0,0) = r\min\{\tilde{\mu}^{\tilde{\alpha}}(0), \tilde{\delta}^{\tilde{\alpha}}(0)\} \geq r\min\{\tilde{\mu}^{\tilde{\alpha}}(x_1), \tilde{\delta}^{\tilde{\alpha}}(x_2)\} = (\tilde{\mu}^{\tilde{\alpha}} \times \tilde{\delta}^{\tilde{\alpha}})(x_1, x_2)$ for all $(x_1, x_2) \in X \times X$. Let (x_1, x_2), (y_1, y_2), $(z_1, z_2) \in X \times X$. Then we have that

$(\tilde{\mu}^{\tilde{\alpha}} \times \tilde{\delta}^{\tilde{\alpha}})(((x_1, x_2) * ((y_1, y_2) * (z_1, z_2))) * (z_1, z_2)) =$

$(\tilde{\mu}^{\tilde{\alpha}} \times \tilde{\delta}^{\tilde{\alpha}})(((x_1, x_2) * (y_1 * z_1, y_2 * z_2)) * (z_1, z_2)) =$

$(\tilde{\mu}^{\tilde{\alpha}} \times \tilde{\delta}^{\tilde{\alpha}})((x_1 * (y_1 * z_1), x_2 * (y_2 * z_2)) * (z_1, z_2)) =$

$(\tilde{\mu}^{\tilde{\alpha}} \times \tilde{\delta}^{\tilde{\alpha}})((x_1 * (y_1 * z_1)) * z_1, (x_2 * (y_2 * z_2)) * z_2) =$

$r\min\{\tilde{\mu}^{\tilde{\alpha}}(x_1 * (y_1 * z_1)) * z_1), \tilde{\delta}^{\tilde{\alpha}}(x_2 * (y_2 * z_2)) * z_2)\} \geq$

$r\min\{r\min\{\tilde{\mu}^{\tilde{\alpha}}(x_1), \tilde{\mu}^{\tilde{\alpha}}(y_1)\}, r\min\{\tilde{\delta}^{\tilde{\alpha}}(x_2), \tilde{\delta}^{\tilde{\alpha}}(y_2)\}\} =$

$r\min\{(\tilde{\mu}^{\tilde{\alpha}} \times \tilde{\delta}^{\tilde{\alpha}})(x_1, x_2), (\tilde{\mu}^{\tilde{\alpha}} \times \tilde{\delta}^{\tilde{\alpha}})(y_1, y_2)\}$. Therefore $\tilde{\mu}^{\tilde{\alpha}} \times \tilde{\delta}^{\tilde{\alpha}}$ is an $\tilde{\alpha}$ -interval valued fuzzy ***new***-ideal in $X \times X$.

5. Conclusions

In the present paper, we have introduced the concept of $\tilde{\alpha}$ -interval valued fuzzy ***new***-ideal of PU -algebras and investigated some of their useful properties. We believe that these results are very useful in developing algebraic structures also these definitions and main results can be similarly extended to some other algebraic structure such as PS -algebras , Q- algebras, SU- algebras , IS-algebras, β algebras and semirings. It is our hope that this work would other foundations for further study of the theory of BCI-algebras. In our future study of fuzzy structure of PU-algebras, may be the following topics should be considered:

(1) To establish the interval value, bipolar and intuitionistic α -fuzzy ***new***-ideal in PU-algebras.

(2) To consider the structure of $(\tilde{\tau}, \tilde{\rho})$ - interval–valued α -fuzzy ***new***-ideal of PU-algebras.

(3) To get more results in $\tilde{\tau}$ - cubic α -fuzzy ***new***-ideal of PU -algebras and its application.

Algorithms for PU-algebra

Input (X: set with 0 element, *: Binary operation)

Output ("X is a PU-algebra or not")

If X = ϕ then;

Go to (1.)

End if

If $0 \notin X$ then go to (1.);

End If

Stop: = false

i = 1;

While $i \leq |X|$ and not (Stop) do

If 0 * $x_i \neq x_i$, then

Stop: = true

End if

j = 1;

While $j \leq |X|$, and not (Stop) do

k = 1;

While $k \leq |X|$ and not (stop) do

If $(x_i * x_k)*$ $(x_j * x_k) \neq x_j * x_i$, then

Stop: = true

End if

End while

End if

End while

If stop then

Output ("X is a PU-algebra")

Else

(1.) Output ("X is not a PU-algebra")

End if

End.

Algorithms for PU-ideal in PU-algebra

Input (X: PU-algebra, I: subset of X)

Output ("I is a PU-ideal of X or not")

If I = ϕ then

Go to (1.);

End if

If $0 \notin$ I then

Go to (1.);

End if

Stop: = false

i = 1;

While $i \leq |X|$ and not (stop) do

j = 1

While $j \leq |X|$ and not (stop) do

k = 1

While $k \leq |X|$ and not (stop) do

If $x_j * x_i \in$ I, and $x_i * x_k \in$ I then

If $x_{j*}\ x_k \notin$ I then

Stop: = false

End if

End while

End while

End while

If stop then

Output ("I is a PU-ideal of X")

Else

(1.) Output ("I is not ("I is a PU-ideal of X")

End if

End.

Algorithm for fuzzy subsets

Input (X : PU-algebra, $A: X \rightarrow [0,1]$);

Output (“ A is a fuzzy subset of X or not”)

Begin

Stop: =false;

$i := 1$;

While $i \leq |X|$ and not (Stop) do

If ($A(x_i) < 0$) or ($A(x_i) > 1$) then

Stop: = true;

End If

End While

If Stop then

Output (“ A is a fuzzy subset of X ”)

Else

Output (“ A is not a fuzzy subset of X ”)

End If

End

Algorithm for fuzzy new-ideal

Input (X : PU-algebra, I : subset of X);

Output (“ I is an ***new***-ideal of X or not”);

Begin

If $I = \phi$ then go to (1.);

End If

If $0 \notin I$ then go to (1.);

End If

Stop: =false;

$i := 1$;

While $i \leq |X|$ and not (Stop) do

$j := 1$

While $j \leq |X|$ and not (Stop) do

$k := 1$

While $k \leq |X|$ and not (Stop) do

If $x_i, x_j \in I$ and $x_k \in X$,then

If $(x_i * (x_j * x_k)) * x_k \notin I$ then

Stop: = true;

End If

End If

End While

End While

End While

If Stop then

Output (“ I is is ***new***-ideal of X ”)

Else (1.) Output (“ I is not is ***new***-ideal of X ”)

End If

End

Conflict of Interests

The author declares that there is no conflict of interests.

Acknowledgements.

The authors would like to express sincere appreciation to the referees for their valuable suggestions and comments helpful in improving this paper.

REFERENCES

[1] K. Iseki, On BCI-algebras, Mathematics Seminar Notes, 8(1980), no. 1, 125–130.

[2] K. Iseki and S. Tanaka, An introduction to the theory of BCKalgebras, Mathematica Japonica, 23(1978), no. 1, 1–26.

[3] K. Iseki and S. Tanaka, Ideal theory of BCK-algebras, Mathematica Japonica, 21(1976), no. 4, 351–366.

[4] Q.P. Hu and X. Li, On proper BCH-algebras, Math. Japonicae, 30 (1985), 659-661.

[5] Q.P. Hu and X. Li, On BCH-algebras, Math. Sem. Notes Kobe Univ., 11(1983), no. 2, 313-320.

[6] J. Meng, On ideals in BCK-algebras, Math. Japon., 40 (1994), 143–154.

[7] J.Meng, X.L.Xin, Implicative BCI-algebras, Pure and Appl. Math. 8(1992), 99 – 103.

[8] J. Meng, Y. B. Jun, and H. S. Kim, Fuzzy implicative ideals of BCK-algebras, Fuzzy Sets and Systems 89 (1997), 243-248.

[9] K.Megalai and A.Tamilarasi, Classification of TM-algebra, IJCA Special Issue on “Computer Aided Soft Computing Techniques for Imaging and Biomedical Applications” CASCT, (2010).

[10] Y. B. Jun, K. J. Lee, and M. S. Kang, "Cubic structures applied to ideals of BCI-algebras," Computers & Mathematics with Applications, 62(2011), no. 9, 3334–3342.

[11] Samy M.Mostafa, Fuzzy implicative ideals in BCK-algebras, Fuzzy sets and systems, 87(1997), 361 – 368.

[12] S. M. Mostafa, M. A. Abdel Naby, A. I. Elkabany, New View Of Ideals On PU-Algebra. International Journal of Computer Applications, 111(2015), No. 4, 1-18

[13] J. Neggers, Y.B. Jun and H.S. Kim, On d-ideals in d-algebras, Math. Slovaca, 49 (1999) , 243-251.

[14] J.Neggers , S.S. Ahn and H.S. Kim, On Q-algebras , IJMMS, 27 (2001), 749-757.

[15] Rosenfeld, Fuzzy groups, J. Math. Anal. Appl., 35 (1971), 512–517.

[16] G. Xi, Fuzzy BCK-algebras, Math. Japon. 36(5) (1991), 935-942.

[17] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338-353.