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THE MAXIMAL SUBGROUPS OF THE UNITARY GROUP PSU(6, q), WHERE $q = 2^K$

RAUHI I. ELKHATIB*

Department of Mathematics, Faculty of Applied Science, Thamar University, Yemen

Abstract. The purpose of this paper is to study maximal subgroups of the Unitary group PSU(6, q), where $q = 2^k$. The main result is a list of maximal subgroups called "the main theorem". The main theorem has been proved by using Aschbacher's Theorem {see [1]}. Thus, this work is divided into two main parts:

Part (1): In this part, we will find the maximal subgroups in the classes $C_1 - C_8$ of Aschbacher's Theorem {see [1]}.

Part (2): In this part, we will find the maximal subgroups in the class C_9 of Aschbacher's Theorem {see [1]}, so, we will find the maximal primitive subgroups H of G which have the property that the minimal normal subgroup M of H is not abelian group and simple, thus, we divided this part into two cases:

Case (1): M is generated by transvections: In this case, we will use result of Kantor {see [9]}.

Case (2): M is a finite primitive subgroup of rank three: In this case, we will use the classification of Kantor and Liebler {see [8]}.

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1. Introduction: The main theorem of this paper is the following theorem:

Theorem (1.1): Let G = PSU(6, q), $q = 2^k$. If H is a maximal subgroup of G, then H isomorphic to one of the following subgroups:

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*Corresponding author

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1. $H_1 = q^9{:}(PGL(1, q^2) \times PSU(4, q))$.
2. $H_2 = q^{12}{:}(PGL(2, q^2) \times PSU(2, q))$.
3. $H_3 = q^9{:}PGL(3, q^2)$.
4. $H_4 = PSU(1, q) \times PSU(5, q)$.
5. $H_5 = PSU(2, q) \times PSU(4, q)$.
6. $H_6 = PSU(2, q){:}S_3$.
7. $H_7 = PSU(3, q){:}S_2$.
8. $H_8 = PSU(2, q^3).3$.
9. $H_9 = PSU(3, q^2).2$.
10. $H_{10} = PSU(2, q) \circ PSU(3, q)$.
11. $H_{11} = PSU(6, q')$, where $q' = 2^{k'}$ and k′ is a prime number divides k.
12. $H_{12} = PSp(6, q)$;
13. $3.P\Omega^-(6, 3)$;

We will prove this theorem by Aschbacher's theorem (Result 2.9) {see [1]}:

If $x \in GF(q^2)$ we set $\bar{x} = x^q$ and assume that X is a matrix, then the matrix $\bar{X}^t$ is obtained from X by replacing each entry x with $\bar{x}$ and then taking the transpose. Thus, in a matrix form, ***The unitary group*** U(n, q) consists of all matrices X ∈ GL(n, q) such that $\bar{X}^t J X = J$, where

$$J = \begin{pmatrix} & & & 1 \\ & & \cdot & \\ & \cdot & & \\ 1 & & & \end{pmatrix}$$

The special unitary group SU(n, q) = {X∈U(n, q) : det X = 1}, and *the projective special unitary group* PSU(n, q) is the factor group SU(n, q)/(SU(n, q) ∩ Z), where the center Z = {diag(λ, λ, …, λ): λ∈F, $\lambda^n = 1$, $\lambda\bar{\lambda} = 1$ }. PSU(n, q) is simple except for PSU(2, 2), PSU(2, 3) and PSU(3, 2). The group U(6, q), q even can generate by the two elements:

$$\left\{ \begin{pmatrix} \alpha & . & . & . & . & . \\ . & 1 & . & . & . & . \\ . & . & 1 & . & . & . \\ . & . & . & 1 & . & . \\ . & . & . & . & 1 & . \\ . & . & . & . & . & \bar{\alpha}^{-1} \end{pmatrix} \text{ and } \begin{pmatrix} 1 & . & . & 1 & . & . \\ 1 & . & . & . & . & . \\ . & 1 & . & . & . & . \\ . & . & . & . & 1 & . \\ . & . & 1 & . & . & 1 \\ . & . & -1 & . & . & . \end{pmatrix} \right\}$$

, where α is a generator element of the multiplicative group of GF(q^2) {see [15]}. Throughout this article, G will denote PSU(6, q), $q = 2^k$, unless otherwise stated. G is a simple group of order $\frac{1}{d} q^{15}(q^6-1)(q^5+1)(q^4-1)(q^3+1)(q^2-1)$, where d = 3 or 1 depending on whether k is odd or even respectively. G acts as a doubly transitive permutation group on the points of the projective space PG(5, q).{see [2]}.

2. Aschbacher's theorem:

A classification of the maximal subgroups of GL(n, q) by Aschbacher's theorem {see [1]}, is a very strong tool in the finite groups for finding the maximal subgroups of finite linear groups. There are many good works in finite groups which simplify this theorem, see for example [12]. But before giving a brief description of this theorem, we will give the following definitions:

Definition 2.1: *A split extension* (*a semidirect product*) A**:**B is a group G with a normal subgroup A and a subgroup B such that G = AB and A∩B = 1. *A non-split extension* A**.**B is a group G with a normal subgroup A and G/A ≅ B, but with no subgroup B satisfying G = AB and A∩B = 1. A group G = A∘B is *a central product* of its subgroups A and B if G = AB and [A, B], the commutator of A and B = {1}, in this case A and B are normal subgroups of G and A∩B ≤ Z(G). If A∩B = {1}, then A∘B = AB.

Definition 2.2: Let V be a vector space of dimensional n over a finite field q, a subgroup H of GL(n, q) is called *reducible* if it stabilizes a proper nontrivial subspace of V. If H is not reducible, then it is called *irreducible*. If H is irreducible for all field extension F of F_q, then H is *absolutely irreducible*. An irreducible subgroup H of GL(n, q) is called *imprimitive* if there are subspaces $V_1, V_2, \ldots, V_k$, $k \geq 2$, of V such that $V = V_1 \oplus \ldots \oplus V_k$ and H permutes the elements of the set $\{V_1, V_2, \ldots, V_k\}$ among themselves. When H is not imprimitive then it is called *primitive*.

Definition 2.3: A group $H \leq GL(n, q)$ is a *superfield group* of degree s if for some s divides n with $s > 1$, the group H may be embedded in $GL(n/s, q^s)$.

Definition 2.4: If the group $H \leq GL(n, q)$ preserves a decomposition $V = V_1 \otimes V_2$ with $\dim(V_1) \neq \dim(V_2)$, then H is a *tensor product group.*

Definition 2.5: Suppose that $n = r^m$ and $m > 1$. If the group $H \leq GL(n, q)$ preserves a decomposition $V = V_1 \otimes \ldots \otimes V_m$ with $\dim(V_i) = r$ for $1 \leq i \leq m$, then H is a *tensor induced* group.

Definition 2.6: A group $H \leq GL(n, q)$ is a *subfield group* if there exists a subfield $F_{q_o} \subset F_q$ such that H can be embedded in $GL(n, q_o)$**.**Z, where Z is the centre group of H.

Definition 2.7: A p-group H is called a *special group* if $Z(H) = H'$ and is called *an extraspecial group* if also $|Z(H)| = p$.

Definition 2.8: Let Z denote the centre group of H. Then H is *almost simple modulo scalars* if there is a non-abelian simple group T such that $T \leq H/Z \leq Aut(T)$, the automorphism group of T.

A classification of the maximal subgroups of GL(n, q) by Aschbacher's theorem {see [1]}, can be summarized as follows:

Result 2.9. (Aschbacher's theorem):

Let H be a subgroup of GL(n, q), $q = p^e$ with the centre Z and let V be the underlying n-dimensional vector space over a field q. If H is a maximal subgroup of GL(n, q), then one of the following holds:

C_1:- H is a reducible group.

C_2:- H is an imprimitive group.

C_3:- H is a superfield group.

C_4:- H is a tensor product group.

C_5:- H is a subfield group.

C_6:- H normalizes an irreducible extraspecial or symplectic-type group.

C_7:- H is a tensor induced group.

C_8:- H normalizes a classical group in its natural representation.

C_9:- H is absolutely irreducible and $H/(H\cap Z)$ is almost simple.

3. Classes $C_1 – C_8$ of Result 2.9:

In this section, we will find the maximal subgroups in the classes $C_1 – C_8$ of Result 2.9:

Lemma 3.1: There are five reducible maximal subgroups of C_1 in G which are:

1. $H_1 = q^9{:}(PGL(1, q^2) \times PSU(4, q))$.
2. $H_2 = q^{12}{:}(PGL(2, q^2) \times PSU(2, q))$.
3. $H_3 = q^9{:}PGL(3, q^2)$.
4. $H_4 = PSU(1, q) \times PSU(5, q)$.
5. $H_5 = PSU(2, q) \times PSU(4, q)$.

Proof:

Let H be a reducible subgroup of the unitary group SU(n, q) and W be an invariant subspace of H. Let r = dim (W), $1 \leq r \leq n/2$ and let $G_r = G_{(W)}$ denote the subgroup of SU(n, q) containing all elements fixing W as a whole and $H \subseteq G_{(W)}$. with a suitable choice of a basis, $G_{(W)}$ consists of all matrices of the form $\begin{pmatrix} A & D & E \\ & B & F \\ & & C \end{pmatrix}$ where D and F are two elementary abelian groups of order $q^{r(n-2r)}$, A is a p- group of upper triangular matrix of identity diagonal of order $q^{\frac{r(r-1)}{2}}$, C is a p- group of upper triangular matrix of order $q^{\frac{r(r+1)}{2}}$, $E \in GL(r, q^2)$, $B \in SU(n-2r, q)$ such that $\bar{A}^t JC = J$, and J is anti-diagonal identity matrix. Thus the maximal parabolic subgroups are the stabilizers of totally isotopic subspaces $\langle e_1, e_2, ..., e_r \rangle$ is isomorphic to a group of the form $q^{r(2n-3r)}$**:**(GL(r, q^2) ×SU(n-2r, q)). Thus, we have the following reducible maximal subgroups of PSU(6, q):

1. If r = 1, then we get a group stabilizing a point is isomorphic to a group of the form H_1 = q^9**:**(PGL(1, q^2) ×PSU(4, q)).
2. If r = 2, then we get a group stabilizing a line is isomorphic to a group of the form H_2 = q^{12}**:**(PGL(2, q^2) ×PSU(2, q)).
3. If r = 3, then we get a group $G_{(2-\pi)}$, stabilizing a plane is isomorphic to a group of the form H_3 = q^9**:**PGL(3, q^2).

Also, H is a maximal reducible subgroup of the unitary group SU(n, q) which stabilizers of non-singular subspaces of dimension d have the shape H = SU(d, q) ×SU(b, q) where n = d + b and $1 \leq d < b$. Thus, we have the following reducible maximal subgroups of PSU(6, q):

4. If d = 1 and b = 5, then we get a group H_4 = PSU(1, q) ×PSU(5, q).
5. If d = 2 and b = 4, then we get a group H_5 = PSU(2, q) ×PSU(4, q).

Which prove the points (1), (2), (3), (4) and (5) of the main theorem 1.1.

Lemma 3.2: There are two imprimitive group of C_2 in G which are

1. $H_6 = PSU(2, q){:}S_3$.
2. $H_7 = PSU(3, q){:}S_2$.

Proof:

If H is imprimitive of the unitary group SU(n, q), then H preserves a decomposition of V as orthogonal direct sum $V = V_1 \oplus_{\dots} \oplus V_t$ with $t \geq 2$, of t subspaces of V each of dimension m = n/t, which are permuted transitively by H, thus H are isomorphic to $SU(m, q){:}S_t$ with $0 < m < n = mt$, $t \geq 2$.

Thus, we have the following two imprimitive maximal subgroups of PSU(6, q):

1. If m = 2 and t = 3, then we get a group $H_6 = PSU(2, q){:}S_3$.
2. If m = 3 and t = 2, then we get a group $H_7 = PSU(3, q){:}S_2$.

Which prove the points (6) and (7) of the main theorem 1.1.

Lemma 3.3: There are two semilinear groups of C_3 in G which are

1. $H_8 = PSU(2, q^3).3$.
2. $H_9 = PSU(3, q^2).2$.

Proof:

Let H is (superfield group) a semilinear groups of PSU(n, q) over extension field F_r of GF(q) of prime degree $r > 1$ where r prime number divide n. Thus V is an F_r-vector space in a natural way, so there is an F-vector space isomorphism between n-dimensional vector space over F and the m-dimensional vector space over F_r, where m = n/r, thus H embeds in $PSU(m, q^r).r$.

Thus, we have the following two semilinear maximal subgroups of PSU(6, q):

1. If m = 2 and r = 3, then we get a group $H_8 = PSU(2, q^3).3$.
2. If m = 3 and r = 2, then we get a group $H_9 = PSU(3, q^2).2$.

Which prove the points (8) and (9) of the main theorem 1.1.

Lemma 3.4: There is one tensor product group H_{10} = PSU(2, q)∘PSU(3, q) of C_4 in G.

Proof:

If H is a tensor product group of SU(n, q), then H preserves a decomposition of V as a tensor product $V_1 \otimes V_2$, where $\dim(V_1) \neq \dim(V_2)$ of spaces of dimensions k and m over GF(q) and n = km, k ≠ m. So, H stabilizes the tensor product decomposition $F^k \otimes F^m$. Thus, H is a subgroup of the central product of SU(k, q)∘SU(m, q). Consequently, there is one C_4 group H_{10} = PSU(2, q)∘PSU(3, q) in PSU(6, q). This proves the point (10) of the main theorem 1.1.

Lemma 3.5: There are subfield groups of C_5 in G which are H_{11} = PSU(6, q′), where $q' = 2^{k'}$ and k′ is a prime number divides k.

Proof:

If H is a subfield group of the unitary group SU(n, q) and $q = p^k$, then H is the unitary group over subfield of GF(q) of prime index. Thus H can be embedded in SU(n, p^f), where f is prime number divides k. Consequently, since $q = 2^k$, then there are subfield groups in PSU(6, 2^k) which are H_{11} = PSU(6, q′), where $q' = 2^{k'}$ and k′ is a prime number divides k. This proves the point (11) of the main theorem 1.1.

Lemma 3.6: There are no C_6 groups in G.

Proof:

For the dimension $n = r^m$, if r = 2 and 4 divides q-1, then H = 2^{2m+1}**.**$O^-(2m, 2)$ normalizes a 2-group of symplectic type of order 2^{2m+2} {see [12]}, consequently, there are no C_6 groups in PSU(6, q) since 6 is not prime power.

Lemma 3.7: There is no tensor induced group of C_7 in G.

Proof:

If H is a tensor induced of the unitary group SU(n, q), then H preserves a decomposition of V as $V_1 \otimes V_2 \otimes \ldots \otimes V_r$, where V_i are isomorphic, each V_i has dimension m, dim V = n = m^r , and the set of V_i is permuted by H, so H stabilize the tensor product decomposition $F^m \otimes F^m \otimes \ldots \otimes F^m$, where $F = F_q$. Thus, $H/Z \leq PSU(m, q){:}S_r$. Consequently, there are no C_7 groups in PSU(6, q) since 6 is not a proper power.

Lemma 3.8: There one maximal C_8 group in G which is H_{12} = PSp(6, q).

Proof:

The groups in this class are stabilizers of forms, this means H is the normalizers of one classical groups PSL(n, q), $PO^{\epsilon}(n, q)$ or PSp(n, q) as a subgroup of PSU(n, q). Thus, from [4], if n is even, then the normalize of PSp(n, q) is a maximal subgroup of PSU(n, q) and from [5] the normalizer of PSp(6, q) in PSU(6, q) is PSp(6, q).

Also, from [6], if n is even and q is even, then the normalizers of $PO^+(n, q)$ and $PO^-(n, q)$ are maximal subgroups of PSp(n, q) except when n = 4 and $\in$ = -. Consequently, $PSGO^+(6, q)$ and $PSGO^-(6, q)$ are two irreducible maximal subgroups of PSU(6, q) but not maximal in PSU(6, q) since they also are subgroups of PSp(6, q). This proves the point (12) of theorem 1.1.

In the following, we will find the maximal subgroups of class C_9 of Result 2.9:

4. The maximal subgroups of C_9:

In Corollary 4.1, we will find the primitive non abelian simple subgroups of G. In Theorem 4.2, we will find the maximal primitive subgroups H of G which have the property that the minimal normal subgroup M of H is not abelian group and simple. We will prove this Theorem 4.2 by finding the normalizers of the groups of Corollary 4.1 and determine which of them are maximal.

Corollary 4.1: If M is a non abelian simple group of a primitive subgroup H of G, then M is isomorphic to one of the following groups:

(i) PSU(6, q′) where $q' = 2^{k'}$ and k′ is a prime number divides k;

(ii) PSp(6, q);

(iii) $PSO^-(6, q)$;

(iv) $PSO^+(6, q)$.

(v) $3.P\Omega^-(6, 3)$.

(vi) $P\Omega^-(6, q) \cong PSU(4, q)$, where $q = 2^k$;

(vii) PSU(6, 2);

(viii) PSU(3, 3);

(ix) PSp(6, 2);

Proof:

Let H be a primitive subgroup of G with a minimal normal subgroup M of H which is not abelian and simple. So, we will discuss the possibilities of M of H according to:

(I) M contains transvections, {section 4.1}.

(II) M is a finite primitive subgroup of rank three, {section 4.2}.

4.1 Primitive subgroups H of G which have the property that a minimal normal subgroup of H is not abelian is generated by transvections:

Definition 4.1.1: An element $T \in GL(n, q)$ is called *a transvection* if T satisfies $rank(T - I_n) = 1$

and $(T - I_n)^2 = 0$. The collineation of projective space induced by a transvection is called *elation*. *The axis* of the transvection is the hyperplane $Ker(T - I_n)$; this subspace is fixed elementwise by T, Dually, *the centre* of T is the image of $(T - I_n)$.

To find the primitive subgroups H of G which have the property that a minimal normal subgroup of H is not abelian and is generated by transvections, we will use the following result of Kantor [9]:

Result 4.1.2: Let H be a proper irreducible subgroup of $SU(n, q^i)$ generated by transvections. Then H is one of:

1. $SU(n, q)$;
2. $Sp(n, q)$ for n even;
3. $O^{\pm}(n, q)$ for n and q even;
4. S_n for n even;
5. S_{n+1} for n even;
6. $SL(2, 5) < Sp(2, 9^i)$;
7. $3.P\Omega^{-}(6, 3) < SU(6, 2^i)$;
8. $A{:}S_n < SU(n, s)$, $|A| = a^{n-1}$, $a | s+1$, s even;
9. Dihedral subgroups of $Sp(2, 2^i)$.

In the following corollary, we will find the primitive subgroups of PSU(6, q) which is generated by transvections:

Corollary 4.1.3: If M is a non abelian simple group and contains some transvections, then M is isomorphic to one of the groups:

(i) PSU(6, q′) where $q' = 2^{k'}$ and k′ is a prime number divides k;

(ii) PSp(6, q);

(iii) $PSO^-(6, q)$;

(iv) $PSO^+(6, q)$.

(v) $3.P\Omega^-(6, 3)$.

Proof:

We will discuss the different possibilities of Result (4.1.2), so, M is isomorphic to one of the following groups:

1. From Lemma 3.5, PSU(6, q′) is an irreducible subgroup of G, where $q' = 2^{k'}$ and k′ is a prime number divides k.
2. From Lemma 3.8, PSp(6, q) is an irreducible subgroup of G.
3. From Lemma 3.8, $PSO^-(6, q)$ and $PSO^+(6, q)$ are irreducible subgroups of G.
4. $S_6 \not\subset G$, since, the irreducible 2-modular characters for S_6 by GAP are:

   [[1, 1], [4, 2], [16, 1]]

   (gap> CharacterDegrees(CharacterTable("S6")mod 2);)

   And non of these characters of degree 6.
5. $S_7 \subset G$, since the irreducible 2-modular characters for S_7 by GAP are:
   [[1, 1], [6, 1], [8, 1], [14, 1], [20, 1]]

   (gap> CharacterDegrees(CharacterTable("S7")mod 2);)

   Thus there exist one irreducible character of degree 6 but the symmetric group S_7 is not a simple group.

6. $SL(2, 5) \not\subset G$, since the irreducible 2-modular characters for SL(2, 5) by GAP are:

[[1, 1], [2, 2], [4, 1]]

(gap> CharacterDegrees(CharacterTable("L2(5)") mod 2);)

And non of these characters of degree 6.

7. From the statements of result 4.1.2, it is clear that $3.P\Omega^{-}(6, 3) \subset G$,

8. If a|q+1, then $a^5:S_6$ is a subgroup of G but not simple.

9. If M is a Dihedral subgroups of $Sp(2, 2^i)$, then $M \not\subset G$, since M is not a simple group.

4.2 Primitive subgroups H of G which have the property that a minimal normal subgroup M of H which is not abelian is a finite primitive subgroup of rank three:

A group G has rank 3 in its permutation representation on the cosets of a subgroup K if there are exactly 3 (K, K)-double cosets and the rank of a transitive permutation group is the number of orbits of the stabilizer of a point, thus we may consider PSU(n, q) as group of permutations of the absolute points of the corresponding projective space, then PSU(n, q) is a transitive group of rank 3.

In this section, we will consider the minimal normal subgroup M of H is not abelian and a finite primitive subgroup of rank three, so will use the classification of Kantor and Liebler {Result 4.2.2} for the primitive groups of rank three {see [8]}. The following Corollary is the main result of this section:

Corollary 4.2.1: If M is a non abelian simple group which is a finite primitive subgroup of rank three group of H, then M is isomorphic to one of the following groups:

1. $P\Omega^{-}(6, q) \cong PSU(4, q)$, where $q = 2^k$;

2. PSU(6, 2);

3. PSU(3, 3);

4. PSp(6, 2);

Proof:

Let M is not an abelian finite primitive subgroup of rank three of H, and will use the classification of Kantor and Liebler {Result 4.2.2} for the primitive groups of rank three {see [8]}. So, we will prove Corollary 4.2.1 by series of Lemmas 4.2.3 through Lemmas 4.2.17 and Result 4.2.2.

Result 4.2.2: If Y acts as a primitive rank 3 permutation group on the set X of cosets of a subgroup K of Sp(2n-2, q), $\Omega^{\pm}(2n, q)$, $\Omega(2n-1, q)$ or SU(n, q). Then for $n \geq 3$, Y has a simple normal subgroup M^*, and $M^* \subseteq Y \subseteq Aut(M^*)$, where M^* as follows:

(i) M = Sp(4, q), SU(4, q), SU(5, q), $\Omega^-(6, q)$, $\Omega^+(8, q)$ or $\Omega^+(10, q)$.

(ii) M = SU(n, 2), $\Omega^{\pm}(2n, 2)$, $\Omega^{\pm}(2n, 3)$ or $\Omega(2n-1, 3)$.

(iii) M = $\Omega(2n-1, 4)$ or $\Omega(2n-1, 8)$;

(iv) M = SU(3, 3);

(v) SU(3, 5);

(vi) SU(4, 3);

(vii) Sp(6, 2);

(viii) $\Omega(7, 3)$;

(ix) SU(6, 2);

In the following, we will discuss the different possibilities of Result 4.2.2;

Lemma 4.2.3: If M = PSp(4, q), then $M \not\subset G$.

Proof:

Since PSp(2n, q), $n \geq 2$, has no projective representation in G of degree less than $\frac{1}{2}(q^n-1)$, if q is odd, and $\frac{1}{2}(q^{n-1})(q^{n-1}-1)(q-1)$ if q is even, {see [13] and [14]}, thus PSp(4, q) has no projective representation in G for all $n \geq 2$, thus $M \not\subset G$.

Lemma 4.2.4: $PSU(4, q) \cong P\Omega^{-}(6, q) \subset G$, where $q = 2^k$.

Proof:

$PSU(4, q) \cong P\Omega^{-}(6, q)$, but, $P\Omega^{-}(6, q) \subset G$ (see Lemma 3.8), where $q = 2^k$. Which prove the point (1) of Corollary 4.2.1.

Lemma 4.2.5: $PSU(5, q) \not\subset G$.

Proof:

$SU(n, q)$, $n \geq 3$, has no projective representation in G of degree less than $q(q^{n-1}-1)/(q+1)$, if n is odd, and $(q^{n}-1)/(q+1)$, if n is even, {see [13] and [14]}, thus $PSU(5, q)$ has no projective representation in G for all $q \geq 2$, thus $PSU(5, q) \not\subset G$.

Lemma 4.2.6: $P\Omega^{+}(8, q) \not\subset G$, $P\Omega^{+}(10, q) \not\subset G$.

Proof:

$P\Omega^{+}(2n, q)$, $n \geq 4$, $q \neq 2, 3, 5$, has no projective representation in G of degree less than $(q^{n-1}-1)(q^{n-2}+1)$, and $P\Omega^{+}(2n, q)$, $n \geq 4$, $q = 2, 3$ or 5, has no projective representation in G of degree less than $q^{n-2}(q^{n-1}-1)$, { see [13] and [14]}, but these bounds are greater than 6 for all $n \geq 4$, thus $P\Omega^{+}(8, q) \not\subset G$ and $P\Omega^{+}(10, q) \not\subset G$.

Lemma 4.2.7: $PSU(6, 2) \subset G$.

Proof:

See Corollary 4.1.3. Which prove the point (2) of Corollary 4.2.1.

Lemma 4.2.8: If M = $P\Omega^{\pm}(2n, 2)$, then M $\not\subset$ G.

Proof:

In our case n = 6, thus we need to consider $P\Omega^{\pm}(12, 2)$:

- $P\Omega^{+}(2n, q)$, n ≥ 4, q = 2 has no projective representation in G of degree less than $q^{n-2}(q^{n-1}-1)$, { see [13] and [14]}, but this bound is greater than 6 for all n ≥ 4 and q = 2, thus $P\Omega^{+}(12, 2) \not\subset$G.

- $P\Omega^{-}(2n, q)$, n ≥ 4, has no projective representation in G of degree less than $(q^{n-1} + 1)(q^{n-2} - 1)$, {see [13] and [14]}, but this bound is greater than 6 for all n ≥ 4 and q = 2, thus $P\Omega^{-}(12, 2) \not\subset$G.

Lemma 4.2.9: If M = $P\Omega^{\pm}(2n, 3)$, then M $\not\subset$ G.

Proof:

In our case n = 6, thus we need to consider $P\Omega^{\pm}(12, 3)$:

- $P\Omega^{+}(2n, q)$, n ≥ 4, q = 2 has no projective representation in G of degree less than $q^{n-2}(q^{n-1}-1)$, {[13] and [14]}, but this bound is greater than 6 for all n ≥ 4 and q = 3, thus $P\Omega^{+}(12, 3) \not\subset$G.

- $P\Omega^{-}(2n, q)$, n ≥ 4, has no projective representation in G of degree less than $(q^{n-1} + 1)(q^{n-2} - 1)$, {see [13] and [14]}, but this bound is greater than 6 for all n ≥ 4 and q = 3, thus $P\Omega^{-}(12, 3) \not\subset$G.

Lemma 4.2.10: If M = $P\Omega(2n-1, 3)$, then M $\not\subset$ G.

Proof:

In our case n = 6, thus, we have PΩ(11, 3) ⊄ G, since PΩ(2n+1, q), n ≥ 3, q = 3, has no projective representation in G of degree less than $q^{n-1}(q^{n-1}- 1)$, {see [13] and [14]}, which is greater than 6 for all n ≥ 3 and q = 3.

Lemma 4.2.11: If M = PΩ(2n-1, 4), then M ⊄ G.

Proof:

In our case n = 6, thus we have PΩ(11, 4) ⊄ G. since, PΩ(2n+1, q) ≅ PSp(2n, q) for q even, then PΩ(11, 4) ≅ PSp(10, 4), and PSp(2n, q), n ≥ 2, has no projective representation in G of degree less than $½(q^{n-1})(q^{n-1}-1)(q-1)$ if q is even {see [13] and [14]}, which is greater than 6 for all n ≥ 2 and q = 4.

Lemma 4.2.12: If M = PΩ(2n-1, 8), then M ⊄ G.

Proof:

In our case n = 6, thus we have PΩ(11, 8) ⊄ G. since, PΩ(2n+1, q) ≅ PSp(2n, q) for q even, then PΩ(11, 8) ≅ PSp(10, 8), and PSp(2n, q), n ≥ 2, has no projective representation in G of degree less than $½(q^{n-1})(q^{n-1}-1)(q-1)$ if q is even {see [13] and [14]}, which is greater than 6 for all n ≥ 2 and q = 8.

Lemma 4.2.13: PSU(3, 3) ⊂ G.

Proof:

Since the irreducible 2-modular characters for PSU(3, 3) by GAP are:

[[1, 1], [6, 1], [14, 1], [32, 2]], {gap> CharacterDegrees(CharacterTable("U3(3)")mod 2);}

Which prove the point (3) of Corollary 4.2.1.

Lemma 4.2.14: PSU(3, 5) $\not\subset$ G.

Proof:

Since the irreducible 2-modular characters for PSU(3, 5) by GAP are:

[[1, 1], [20, 1], [28, 3], [104, 1], [144, 2]]
(gap> CharacterDegrees(CharacterTable("U3(5)")mod 2);)
And non of these of degree 6.

Lemma 4.2.15: PSU(4, 3) $\not\subset$ G.

Proof:

Since the irreducible 2-modular characters for PSU(3, 5) by GAP are:

[[1, 1], [20, 1], [34, 2], [70, 4], [120, 1], [640, 2],[896, 1]]
(gap> CharacterDegrees(CharacterTable("U4(3)")mod 2);)
and non of these of degree 6.

Lemma 4.2.16: PSp(6, 2) $\subset$ G.

Proof:

See Corollary 4.1.3. Which prove the point (4) of Corollary 4.2.1.

Lemma 4.2.17: PΩ(7, 3) $\not\subset$ G.

Proof:

$P\Omega(7, 3) \not\subset G$, since $P\Omega(2n+1, q)$, $n \geq 3$, $q = 3$, has no projective representation in G of degree less than $q^{n-1}(q^{n-1}- 1)$, {see [13] and [14]}, which is greater than 6 for all $n \geq 3$ and $q = 3$.

Now, we will determine the maximal primitive group of C_9:

Theorem 4.2: If H is a maximal primitive subgroup of G which has the property that a minimal normal subgroup M of H is not abelian group, then H is isomorphic to one of the following subgroups of G:

(i) PSU(6, q′) where $q' = 2^{k'}$ and k′ is a prime number divides k;

(ii) PSp(6, q);

(iii) $3.P\Omega^{-}(6, 3)$;

Proof:

We will prove this theorem by finding the normalizers N of the groups of **Corollary 4.1** and determine which of them are maximal:

From [5], the normalizer of Sp(2n, k) in SL(2n, k) is SGSp(2n, k) = GSp(2n, k) ∩ SL(2n, k). From [4], the normalizer of SU(n, k) in SL(n, k) is SGU(n, k) = GU(n, k) ∩ SL(n, k). From [10], the normalizer of SO(n, k) in SL(n, k) is SGO(n, k) = GO(n, k) ∩ SL(n, k). Thus,

- If Y = PSU(6, q′) where $q' = 2^{k'}$ and k′ is a prime number divides k, then N = PSGU(6, q) but in PSU(6, q), PSGU(6, q′) = PSU(6, q′), which prove the point (i) of this theorem.
- If Y = PSp(6, q), then N = PSGSp(6, q) but in PSU(6, q), PSGSp(6, q) = PSp(6, q), which prove the point (ii) of this theorem.
- If $Y = PSO^{-}(6, q)$, then $N = PSGO^{-}(6, q)$, but in PSU(6, q), $PSGO^{-}(6, q) = PSO^{-}(6, q)$, in this case Y is a subgroup of PSp(6, q), thus Y is not a maximal subgroup of G.

- If Y = $PSO^+(6, 2)$, then N = $PSGO^+(6, 2)$, but in PSU(6, q), $PSGO^+(6, 2) = PSO^+(6, 2)$, in this case Y is a subgroup of PSp(6, q), thus Y is not a maximal subgroup of G.

- If Y = $3.P\Omega^-(6, 3)$, then the normalizer in G is $3.P\Omega^-(6, 3)$, which prove the point (iii) of theorem 4.2.

- If Y = $P\Omega^-(6, q) \cong PSU(4, q)$, where $q = 2^k$, then the normalizer in G is N = $P\Omega^-(6, q) \cong PSU(4, q)$, in this case Y is a subgroup of PSp(6, q), thus Y is not a maximal subgroup of G.

- If Y = PSU(6, 2), then N = PSGU(6, 2) but in PSU(6, q), PSGU(6, 2) = PSU(6, 2), in this case Y is a subgroup of PSU(6, q′) where $q' = 2^{k'}$ and k′ is a prime number divides k, thus Y is not a maximal subgroup of G.

- If Y = PSU(3, 3), then the normalizer in G is N = PSU(3, 3). in this case Y is a subgroup of PSp(6, q), thus Y is not a maximal subgroup of G.

- If Y = PSp(6, 2), then N = PSGSp(6, 2) but in PSU(6, q), PSGSp(6, 2) = PSp(6, 2), in this case Y is a subgroup of PSp(6, q), thus Y is not a maximal subgroup of G.

This completes the proof of theorem 1.1.

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