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Q-FUZZY DERIVATIONS KU-IDEALS ON KU-ALGEBRAS

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Abstract. In this manuscript, we introduce a new concept, which is called *Q*- fuzzy left (right) derivations KU-ideals in KU-algebras. We state and prove some theorems about fundamental properties of it. Moreover, we give the concepts of the image and the pre-image of *Q*- fuzzy left (right) derivations KU-ideals under homomorphism of KU-algebras and investigated some its properties. Further, we have proved that every the image and the pre-image of *Q*-fuzzy left (right) derivations KU-ideals under homomorphism of KU- algebras are *Q*- fuzzy left (right) derivations KU-ideals. Furthermore, we give the concept of the Cartesian product of *Q*- fuzzy left (right) derivations KU - ideals in Cartesian product of KU – algebras.

Keywords. *KU*-algebras; *Q*- fuzzy left (right) derivations of *KU*-ideals; the image and the per- image of *Q*- fuzzy left (right) derivations KU – ideals; the Cartesian product of Q- fuzzy left (right) derivations KU – ideals.
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1. Introduction

As it is well known, BCK and BCI-algebras are two classes of algebras of logic. They were introduced by Imai and Iseki [10,11,12] and have been extensively investigated by many researchers. It is known that the class of BCK-algebras is a proper sub class of the BCI-algebras. The class of all *BCK*-algebras is a quasivariety. Is ́*e*ki posed an interesting problem (solved by Wro ́*n*ski [27]) whether the class of *BCK*-algebras is a variety. In connection with this problem, Komori [16] introduced a notion of *BCC*-algebras, and Dudek [7] redefined the notion of *BCC*-algebras by using a dual form of the ordinary definition in the sense of Komori. Dudek and Zhang [8] introduced a new notion of ideals in *BCC*-algebras and described connections between such ideals and congruences . C.Prabpayak and U.Leerawat ([24], [25]) introduced a new

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algebraic structure which is called KU - algebra . They gave the concept of homomorphisms of KU- algebras and investigated some related properties. Several authors [2,3,5,6,9,15] have studied derivations in rings and near rings. Jun and Xin [13] applied the notion of derivations in ring and near-ring theory to *BCI*-algebras, and they also introduced a new concept called a regular derivation in *BCI* -algebras. They investigated some of its properties, defined a d -derivation ideal and gave conditions for an ideal to be d-derivation. Later, Hamza and Al-Shehri [1], defined a left derivation in *BCI*-algebras and investigated a regular left derivation. Zhan and Liu [30] studied f-derivations in BCI-algebras and proved some results. G. Muhiuddin etl [22,23] introduced the notion of (α,β)-derivation in a BCI-algebra and investigated related properties. They provided a condition for a (α,β)- derivation to be regular. They also introduced the concepts of a $d_{(\alpha,\beta)}$- invariant (α,β) -derivation and α-ideal, and then they investigated their relations. Furthermore, they obtained some results on regular (α,β)- derivations. Moreover, they studied the notion of t-derivations on BCI-algebras and obtained some of its related properties. Further, they characterized the notion of p-semi-simple BCI-algebra X by using the notion of t-derivation. Later, Mostafa et al [19,20], introduced the notions of $((\ell,r)$ -$((r,\ell))$ -derivation of a KU-algebra and some related properties are explored. The concept of fuzzy sets was introduced by Zadeh [29]. In 1991, Xi [28] applied the concept of fuzzy sets to BCI, BCK, MV-algebras .Since its inception, the theory of fuzzy sets ,ideal theory and its fuzzification has developed in many directions and is finding applications in a wide variety of fields. Mostafa et al, in 2011[18] introduced the notion of fuzzy KU-ideals of KU-algebras and then they investigated several basic properties which are related to fuzzy KU-ideals. In Mostafa , Abd-eldayem [21] introduced the notion of fuzzy (left and right) derivations KU- ideals in KU - algebras and investigated related properties. Jun [14], he introduced the notion of Q- fuzzy subalgebras of BCK/BCI-algebras, and provided some appropriate examples and described Q- fuzzy subalgebras. Morover, he construct fuzzy subalgebras by using Q- fuzzy subalgebras and how the homomorphic images and inverse images of Q- fuzzy subalgebras become Q- fuzzy sub-algebras. A. Rezaei et al, in 2014[26] ,show that a KU –algebra is equivalent to the commutative self– distributive BE–algebra. Also, they show that a self –distributiveKU –algebra is equivalent to the Hilbert algebra.

Modifying the idea of Jun [14],in this paper, we introduce the the concept of *Q*- fuzzy (left and right) derivations KU-ideals in KU–algebras and homomorphic image (preimage) of *Q*- fuzzy left (right)-derivations KU-ideals in KU-algebras under homomorprhism of a KU -algebras. Also we discussed how the homomorphic images and inverse images of *Q*- fuzzy (left and right) derivations KU- ideals become *Q*- fuzzy (left and right) derivations KU- ideals in KU - algebras. Furthermore, we give the concept of the Cartesian product of *Q*- fuzzy left (right) derivations KU - ideals in Cartesian product of KU – algebras. Many related results have been derived.

2. Preliminaries

In this section, we recall some basic definitions and results that are needed for our work.

Definition 2.1 [24,25] Let X be a set with a binary operation $*$ and a constant 0.
($X, *, 0$) is called KU-algebra if the following axioms hold : $\forall x, y, z \in X$:

(KU_1) $(x*y)*[(y*z)*(x*z)] = 0$

(KU_2) $x*0 = 0$

(KU_3) $0*x = x$

(KU_4) $if\ x*y = 0 = y*x\ implies\ \ x = y$.

Define a binary relation $\leq$ by: $x \leq y \Leftrightarrow y*x = 0$,

Lemma 2.2 On KU-algebra (*X; *;* 0). We define a binary relation $\leq$ on X by putting $x \leq y$ if and only if $y*x$ = 0. Then (X; $\leq$) is a partially ordered set and 0 is its smallest element.

Proof. Let X be KU-algebra $\forall a,b,c \in X$, we have

1. $\leq$ is reflexive as $a \leq a$.
2. if $a \leq b,\ b \leq a$,then $a = b$. Hence $\leq$ is anti-symmetric.
3. if $a \leq b,\ b \leq c$,then we want to prove that $a \leq c$.

   Since $c*a = 0*(c*a) = (c*b)*(c*a) \leq b*a = 0$,we have $c*a = 0 \Rightarrow a \leq c$, then $\leq$ is transitive. Hence $(X, \leq)$ is partial order set.

Throughout this article, X will denote a KU-algebra unless otherwise mentioned

Corollary 2.3 [18,24] In KU-algebra the following identities are true for all $x, y, z \in X$:

(i) $z*z = 0$

(ii) $z*(x*z) = 0$

(iii) If $x \leq y$ *impliesthat* $y * z \leq x * z$

(v) $z * (y * x) = y * (z * x)$

(vi) $y * [(y * x) * x] = 0$

Definition 2.4 [24,25] A subset S of KU-algebra X is called sub algebra of X if $x * y \in S$, whenever $x, y \in S$

Definition 2.5 [24,25] Anon empty subset A of KU-algebra X is called ideal of X if it is satisfied the following conditions:

(i) $0 \in A$

(ii) $y * z \in A$, $y \in A$ *implies* $z \in A$ $\forall y, z \in X$.

Definition 2.6 [18] A non - empty subset A of a KU-algebra X is called KU- ideal of X if it satisfies the following conditions :

(1) $0 \in A$,

(2) $x * (y * z) \in A$, $y \in A$ implies x $*$ z $\in$ A , for all $x, y, z \in X$

Definition 2.7[18] Let X be a KU - algebra, a fuzzy set μ in X is called fuzzy subalgebra if it satisfies:

(S_1) $\mu(0) \geq \mu(x)$,

(S_2) $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$ for all $x, y \in X$.

Definition 2.8 [18] Let X be a KU-algebra, a fuzzy set μ in X is called a fuzzy KU-ideal of X if it satisfies the following conditions:

(F_1) $\mu(0) \geq \mu(x)$,

(F_2) $\mu(x * z) \geq \min\{\mu(x * (y * z)), \mu(y)\}$.

Definition 2.9 For elements x and y of KU-algebra ($X, *, 0$), we denote $x \wedge y = (x * y) * y$.

Definition 2.10[19] Let X be a KU-algebra. A self map $d : X \to X$ is a left –right derivation (briefly, (ℓ, r) -derivation) of X if it satisfies the identity

$$d(x * y) = (d(x) * y) \wedge (x * d(y)) \ \forall x, y \in X$$

If d satisfies the identity

$$d(x*y) = (x*d(y)) \wedge (d(x)*y) \ \forall x, y \in X$$

d is called right-left derivation (briefly, (r,ℓ) -derivation) of X . Moreover, if d is both (ℓ, r) *and* $(r,\ell)-$*derivation then* d is called a derivation of X .

Definition 2.11[19] A derivation of KU-algebra is said to be regular if $d(0) = 0$.

Lemma 2.12[19] A derivation d of KU-algebra X is regular.

Example 2.13 [19] Let $X = \{0,1,2,3,4\}$ be a set in which the operation $*$ is defined as follows:

*	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	2	4
2	0	0	0	1	4
3	0	0	0	0	4
4	0	1	1	1	0

Using the algorithms in Appendix A, we can prove that (X, *, 0) is a KU-algebra. Define a map $d: X \to X$ by

$$d(x) = \begin{cases} 0 & if \ x = 0,1,2,3 \\ 4 & if \ x = 4 \end{cases}$$

Then it is easy to show that d is both a (ℓ, r) and (r,ℓ) -derivation of X .

Example 2.14. Let $X = \mathrm{N} \cup \{0\}$ and * binary operation on X defined by

$$x*y = \begin{cases} 0 & if \ x \geq y \\ y-x & if \ y > x \end{cases}$$

Then $X = (\mathrm{N} \cup \{0\}, *, 0)$ is a KU-algebra. If the map $d: X \to X$ is defined by d (x) =x −1 for all $x \in \mathrm{N}$ .Then for all $x, y \in X$, we have

$d(x*y) = d(y-x) = y-x-1$ ……………………………………………….(I),

$d(x)*y = y-d(x) = y-(x-1) = 1+y-x$ and $x*d(y) = d(y)-x = y-x-1$ but

$d(x)*y \wedge \ x*d(y)=((1+y-x)*\ (y-x-1))*\ (y-x-1)=$

$$=(y-x-1)-[(y-x-1)-(1+y-x)] \ = \ y+1-x\ldots\ldots\ldots \text{(II)}$$

From (I) and (II), d is not (ℓ, r) derivation of X.

On other hand

$d(x*y)=d(y-x)=y-x-1$……………………………………………..(I)

$x*d(y)= \ d(y)-x \ =y-x-1, \ \ d(x)*y= \ y-d(x)=y-(x-1) \ =y+1-x$, but

$x*d(y)\wedge \ d(x)*y=[\ (x*d(y))*\ (d(x)*y)]*\ (d(x)*y)$

$$=(y-d(x))-[(y-d(x))-(d(y)-x)] \ = \text{y}-\text{x}-1\ldots\ldots \text{(III)}$$

From (I) and (III), d is (r,ℓ) derivation of X .Hence (r,ℓ) -derivation and (ℓ, r) derivation are not coincide.

Proposition 2.15[19] Let X be a KU-algebra with partial order $\leq$, and let d be a derivation of X . Then the following hold $\forall\, x, y \in X$:

(i) $d(x) \leq x$.

(ii) $d(x*y) \leq d(x)*y$.

(iii) $d(x*y) \leq x*d(y)$.

(v) $d(x*d(x))=0$.

(vi) $d^{-1}(0)=\{x \in X \mid d(x)=0\}$ is a sub algebra of X .

Definition 2.16 [19] Let X be a KU-algebra and d be a derivation of X .

Denote $Fix_d(X)=\{x \in X : d(x)=x\}$.

Proposition 2.17[19] Let X be a KU-algebra and d be a derivation of X .Then $Fix_d(X)$ is a sub algebra of X .

3. Q-Fuzzy derivations KU- ideals of KU-algebras

In this section, we will discuss and investigate a new notion called $Q-$fuzzy (left and right) derivations KU - ideals of KU - algebras and study several basic properties which are related to fuzzy left derivations KU - ideals.

Definition 3.1 Let X be a KU-algebra and $d : X \to X$ be self map. A non - empty subset A of a KU-algebra X is called left derivations KU- ideal of X if it satisfies the following conditions:

(1) $0 \in A$,

(2) $d(x)*(y*z)) \in A$, $d(y) \in A$ implies $d(x*z) \in A$, for all $x, y, z \in X$

Definition 3.2 Let X be a KU-algebra and $d : X \to X$ be self map. A non - empty subset A of a KU-algebra X is called right derivations KU ideal of X if it satisfies the following conditions:

(1) $0 \in A$,

(2) $x*d(y*z)) \in A$, $d(y) \in A$ implies $d(x*z) \in A$, for all $x, y, z \in X$

Definition 3.3 Let X be a KU-algebra and $d : X \to X$ be self map .A non - empty subset A of a KU-algebra X is called derivations KU -ideal of X if it satisfies the following conditions:

(1) $0 \in A$,

(2) $d(x*(y*z)) \in A$, $d(y) \in A$ implies d(x * z) $\in A$, for all $x, y, z \in X$

Definition 3.4 Let X be a KU-algebra and $d : X \to X$ be self map. A fuzzy set $\mu : X \times Q \to [0,1]$ in X is called Q- fuzzy left derivations KU-ideal (briefly,(Q- F, ℓ) ,d) of X if it satisfies the following conditions :

(F_1) $\mu(0, q) \geq \mu(x, q)$

(F_2) $\mu(d(x*z), q) \geq \min\{\mu((d(x)*(y*z)), q), \mu(d(y), q)\}$ $\forall\, x, y, z \in X$ *and* $q \in Q$.

Definition 3.5 Let X be a KU-algebra and $d : X \to X$ be self map. A fuzzy set $\mu : X \times Q \to [0,1]$ in X is called Q- fuzzy right derivations KU-ideal(briefly, (Q- F, r) - derivation) of X if it satisfies the following conditions:

(F_1) $\mu(0, q) \geq \mu(x, q)$.

(F_2) $\mu(d(x*z), q) \geq \min\{\mu((x*d(y*z)), q), \mu(d(y), q)\}$

$\forall\, x, y, z \in X$ and $q \in Q$.

Definition 3.6 Let X be a KU-algebra and $d : X \to X$ be self map. A fuzzy set $\mu : X \times Q \to [0,1]$ in X is called a Q- fuzzy derivations KU-ideal of X ,if it satisfies the following conditions ***:***

(F_1) $\mu(0, q) \geq \mu(x, q)$.

(F_2) $\mu(d(x*z), q) \geq \min\{\mu(d(x*(y*z)), q), \mu(d(y), q)\}$

$\forall\ x, y, z \in X$ and $q \in Q$.

Example 3.7 Let $X = \{0,1,2,3,4\}$ be a set as in example 2.13: Using the algorithms in Appendix A, we can prove that (X, *, 0) is a KU-algebra.

Define a self map $d: X \to X$ by

$$d(x) = \begin{cases} 0 & if\ x = 0,1,2,3 \\ 4 & if\ x = 4 \end{cases}$$, and

a fuzzy set $\mu : X \times Q \to [0,1]$,by $\mu(d(0), q) = t_0$, $\mu(d(1), q) = \mu(d(2), q) = t_1$, $\mu(d(3), q) = \mu(d(4), q) = t_2$, where t_0 , t_1 , $t_2 \in [0,1]$ with $t_0 > t_1 > t_2$ .Routine calculations give that μ is a Q- fuzzy (left and right)- derivations KU- ideal of KU- algebra X .

Lemma 3.8 Let μ be a Q- fuzzy left derivations KU - ideal of KU - algebra X , if the inequality $x * y \le d(z)$ holds in X , then $\mu(d(y), q) \ge \min\{\mu(d(x), q), \mu(z, q)\}$, $\forall\ x, y, z \in X$ and $q \in Q$.

Proof. Assume that the inequality $x * y \le d(z)$ holds in X , then $d(z) * (x * y) = 0$, $z * (x * y) = 0$, since $d(z) \le z$ from (Proposition 2.15(i)) and definition 3.4 (F_2) we have

$\mu(d(z * y), q) \ge \min\{\mu((d(z) * (x * y), q), \mu(d(x), q)\} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$(A)

$= \min\{\mu(0, q), \mu(d(x), q)\} = \mu(d(x), q)$

Put z=0 in (A), we have

$\mu(d(0 * y), q) = \mu(d(y), q) \ge \min\{\mu((x * y), q), \mu(d(x), q)\}\ldots\ldots$ ..(a), but

$\mu((x * y), q) \ge \min\{\mu((x * (z * y), q), \mu(z, q)\} = \min\{\mu((z * (x * y), q), \mu(z, q)\}$

$= \min\{\mu(0, q), \mu(z, q)\} = \mu(z, q)\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots$(b)

From (a) , (b) , we get $\mu(d(y), q) \ge \min\{\mu(z, q), \mu(d(x), q)\}$.

This completes the proof.

Lemma 3.9 If μ is a Q- fuzzy left derivations KU - ideal of KU - algebra X and if $x \le d(y)$, then $\mu(d(x), q) \ge \mu(d(y), q)$.

Proof. Straight forward.

Proposition 3.10 The intersection of any set of a Q- fuzzy left derivations KU - ideals of KU – algebra X is also Q- fuzzy left derivations KU - ideal.

Proof. let $\{\mu_i\}$ be a family of a Q-fuzzy left derivations KU - ideals of KU- algebra X, then for any $x, y, z \in X$ and $q \in Q$.

$(\bigcap \mu_i)(0,q) = \inf(\mu_i(0,q)) \geq \inf(\mu_i(d(x),q)) = (\bigcap \mu_i)(d(x),q)$ and

$(\bigcap \mu_i)(d(x*z),q) = \inf(\mu_i(d(x*z),q) \geq \inf(\min\{\mu_i((d(x)*(y*z)),q), \mu_i(d(y),q)\} =$
$\min\{\inf(\mu_i((d(x)*(y*z)),q), \inf(\mu_i(d(y),q)\} = \min\{(\bigcap \mu_i)((d(x)*(y*z)),q), (\bigcap \mu_i)(d(y),q)\}.$

This completes the proof .

Lemma 3.11 The intersection of any set of a Q- fuzzy right derivations KU - ideals of KU – algebra X is also a Q- fuzzy right derivations KU - ideal.

proof. Straight forward.

Theorem3.12 Let μ be a Q- fuzzy set in X then μ is a Q- fuzzy left derivations KU- ideal of X if and only if it satisfies : For all $\alpha \in [0,1]$,

(A_1) $U(\mu, \alpha) = \{x \in X \,/\, \mu(d(x), q) \geq \alpha\} \neq \varphi$ implies $U(\mu, \alpha)$ is KU- ideal of X .

Proof . Assume that μ is a Q- fuzzy left derivations KU- ideal of X , let $\alpha \in [0,1]$ be such that $U(\mu, \alpha) \neq \varphi$, and $x, y \in X$ such that $x \in U(\mu, \alpha)$, then $\mu(d(x), q) \geq \alpha$ and so by (definition 3.4 (F_2)) we have ,

$\mu(d(0), q) = \mu(d(y*0), q) \geq \min\{\mu(d(y)*(x*0), q), \mu(d(x), q)\} =$

$\min\{\mu(d(y)*0), q), \mu(d(x), q)\} = \min\{\mu(0), \mu(d(x), q)\} \geq \alpha$, hence

$0 \in U(\mu, \alpha)$. Let $d(x)*(y*z) \in U(\mu, \alpha)$, $d(y) \in U(\mu, \alpha)$,

it follows from(definition 3.4 (F_2)) that

$\mu(d(x*z), q) \geq \min\{\mu(d(x)*(y*z), q), \mu(d(y), q)\} \geq \alpha$, hence

$x*z \in U(\mu, \alpha)$ and so $U(\mu, \alpha)$ is KU - ideal of X .

Conversely, suppose that μ satisfies (A_1) , let $x, y, z \in X$ and $q \in Q$, be such that

$\mu(d(x*z), q) < \min\{\mu(d(x)*(y*z)), q), \mu(d(y), q)\}$,taking

$\beta_0 = 1/2\,\{\mu(d(x*z), q) + \min\{\mu(d(x)*(y*z), q), \mu(d(y), q)\}$, we have

$\beta_0 \in [0,1]$ and $\mu(d(x*z), q) < \beta_0 < \min\{\mu(d(x)*(y*z)), q), \mu(d(y), q)\}$,it follows that

$d(x)*(y*z) \in$ U (μ, β_0) and $d(x*z) \notin$ U (μ, β_0) , this is a contradiction and therefore μ is a Q- fuzzy left derivations KU - ideal of X .

Theorem3.13 Let μ be a Q- fuzzy set in X then μ is a $Q-$fuzzy right derivations KU- ideal of X if and only if it satisfies : For all $\alpha \in [0,1]$),U (μ , α) ≠ φ implies U(μ ,α) is KU- ideal of X .

proof. Straight forward.

Proposition 3.14 If μ is a Q- fuzzy left derivations KU - ideal of X , then

$$\mu(\,(d(x)*(x*y)),q) \geq \mu(d(y),q).$$

proof . Taking $z = x*y$ in (definition 3.4 (F_2)), we get

$$\mu(d(x)*(x*y),q) \geq \min\{\mu(d(x)*(y*(x*y)),q), \mu(d(y),q)\}$$
$$=\min\{\mu(d(x)*(x*(y*y),q), \mu(d(y),q)\}$$
$$=\min\{\mu(d(x)*(x*0)),q), \mu(d(y),q)\}$$
$$=\min\{\mu(0,q), \mu(d(y),q)\} = \mu(d(y),q).$$

Definition3.15 Let μ be a Q- fuzzy left derivations KU - ideal of KU - algebra X ,the KU - ideals $\mu_t := \{x \in X \mid \mu(x,q) \geq t\}$, $t \in [0,1]$ are called level KU - ideal of X .

Corollary3.16 Let I be an KU - ideal of KU - algebra X , then for any fixed number t in an open interval (0,1) , there exist a $Q-$ fuzzy left derivations KU – ideal μ of X such that μ_t = I .

proof. The proof is similar the corollary 4.4 [17] .

4. Image (Pre-image) of a Q- fuzzy derivations KU-ideals under homomorphism

In this section, we introduce the concepts of the image and the pre-image of a $Q-$fuzzy (left – right) derivations KU-ideals in KU-algebras under homomorphism.

Definition 4.1 Let f be a mapping from the set X to a set Y . If μ is a Q- fuzzy subset of X , then the Q- fuzzy subset β of Y is defined by

$$f(\mu)(y) = \beta(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \mu(x,q), & \text{if } f^{-1}(y) = \{x \in X, f(x) = y\} \neq \phi \\ 0 & \text{otherwise} \end{cases}$$

is said to be the image of μ under *f*.

Similarly if β is a *Q*-fuzzy subset of Y, then the fuzzy subset $\mu = \beta \circ f$ in X (i.e the fuzzy subset defined by $\mu(x,q) = \beta(f(x),q)$ for all $x \in X$) is called the preimage of β under f .

Theorem 4.2 An onto homomorphic preimage of a *Q*- fuzzy left derivations KU - ideal is also a *Q*- fuzzy left derivations KU - ideal .

Proof. Let $f : X \to X`$ be an onto homomorphism of KU - algebras , $\beta$ a *Q*- fuzzy left derivations KU - ideal of $X`$ and μ the preimage of β under f , then

$\beta(f(d(x),q) = \mu(d(x),q)$, for all $x \in X$.

Let $x \in X$, then $\mu(d(0),q) = \beta(f(d(0),q)) \geq \beta(f(d(x)),q)) = \mu(d(x),q)$.

Now let $x, y, z \in X$, then

$$\begin{aligned} \mu(d(x*z),q) &= \beta(f(d(x*z),q)) \\ &\geq \min\{\beta(f(d(x)*`(f(y)*` f(z))),q),\beta(f(d(y),q))\} \\ &= \min\{\beta(f(d(x)*(y*z)),q),\beta(f(d(y)),q)\} \\ &= \min\{\mu(d(x)*(y*z)),q), \mu(d(y),q)\} . \end{aligned}$$

The proof is completed.

Theorem 4.3 An onto homomorphic preimage of a *Q*- fuzzy right derivations KU - ideal is also a *Q*- fuzzy right derivations KU – ideal

Proof . *Straightforward.*

Definition 4.4 **[4]** A *Q*- fuzzy subset μ of X has sup property if for any subset T of X ,there exist $t_0 \in T$ such that , $\mu(t_0,q) = \sup_{t \in T} \mu(t,q)$.

Theorem 4.5 Let $f : X \to Y$ be a homomorphism between KU - algebras X and Y .

For every *Q*- fuzzy left derivations KU - ideal μ in X , f (μ) is a *Q*- fuzzy left derivations KU - ideal of Y .

Proof. By definition $\beta(d(y'),q) = f(\mu)(d(y'),q) = \sup_{d(x) \in f^{-1}((d(y'))} \mu(d(x),q)$ for all $y' \in Y$

and $\sup\phi = 0$. We have to prove that

$\beta(d(x' * z'),q) \geq \min\{\beta((d(x') * (y' * z')),q), \beta(d(y'),q)\}, \forall\ x`,\ y`,\ z` \in Y$.

Let $f : X \to Y$ be an onto a homomorphism of KU - algebras , μ a Q- fuzzy left derivations KU - ideal of X with sup property and β the image of μ under f , since μ is a Q- fuzzy left derivations KU - ideal of X , we have μ($d(0)$, q) ≥ μ($d(x)$, q) for all $x \in X$. Note that $0 \in f^{-1}(0`)$ , where 0 , 0` are the zeros of X and Y respectively

Thus, $\beta(d(0'),q) = \sup\limits_{d(t)\in f^{-1}(d(0'))} \mu(d(t),q) = \mu(d(0),q) = \mu(0,q) \geq \mu(d(x),q),$ for all $x \in X$,

which implies that $\beta(d(0'),q) \geq \sup\limits_{d(t)\in f^{-1}(d(x'))} \mu(d(t),q) = \beta(d(x'),q),$ for any $x' \in Y$.

For any $x', y', z' \in Y$,let $d(x_0) \in f^{-1}(d(x'))$, $d(y_0) \in f^{-1}(d(y'))$, $d(z_0) \in f^{-1}(d(z'))$

be such that $\mu(d(x_0 * z_0),q) = \sup\limits_{d(t)\in f^{-1}(d(x' * z`))} \mu(d(t),q)$, $\mu(y_0,q) = \sup\limits_{d(t)\in f^{-1}(d(y'))} \mu(d(t),q)$

and

$$\mu((d(x_0) * (y_0 * z_0)),q) = \beta\{f(d(x_0) * (y_0 * z_0)),q\} = \beta(d(x') * (y' * z')),q) =$$
$$\sup_{(d(x_0)*(y_0 * z_0))\in f^{-1}(d(x')*(y'*z'))} \mu(d(x_0) * (y_0 * z_0)),q) = \sup_{d(t)\in f^{-1}(d(x')*(y'*z'))} \mu(d(t),q) \ .$$

Then

$$\beta(d(x' * z'),q) = \sup_{d(t)\in f^{-1}(d(x' * z'))} \mu(d(t),q) = \mu(d(x_0 * z_0),q)$$
$$\geq \min\{\mu((d(x_0) * (y_0 * z_0)),q), \mu(d(y_0),q)\} =$$
$$\min\left\{\sup_{d(t)\in f^{-1}(d(x')*(y'*z'))} \mu(d(t),\mathrm{q}),\ \sup_{d(t)\in f^{-1}(d(y`))} \mu(d(t),q)\right\} = \min\{\beta((d(x') * (y' * z')),q)\ ,\ \beta(d(y'),q)\}$$

.

Hence β is a Q- fuzzy left derivations KU-ideal of Y.

Theorem 4.6 Let $f : X \to Y$ be a homomorphism between KU - algebras X and Y .

For every $Q-$ fuzzy right derivations KU - ideal μ in X , f (μ) is a Q-fuzzy right derivations KU - ideal of Y .

proof. Straight forward.

5. Cartesian product of a Q- fuzzy derivations KU-ideals

Definition 5.1 A fuzzy μ is called a Q- fuzzy relation on any set S , if μ is a fuzzy subset $\mu : (S \times S)\times Q \to [0,1]$.

Definition 5.2 If μ is a fuzzy relation on a set S and β is a fuzzy subset of S ,
then μ is a Q- fuzzy relation on β if

$\mu((x, y),q) \leq \min \{\beta(x,q), \beta(y,q)\}, \forall\ x, y \in S, q\in Q.$

Definition 5.3 Let μ and β be a Q-fuzzy subset of a set S , the Cartesian product of μ and β is defined by $(\mu \times \beta)(x, y) = \min \{\mu(x,q), \beta(y,q)\}$, for all $x, y \in S$, $q\in Q$

Lemma 5.4[4] Let μ and β be a fuzzy subset of a set S ,then

(i) $\mu\times\beta$ is a fuzzy relation on S .

(ii) $(\mu\times\beta)_t = \mu_t \times \beta_t$ for all t $\in[0,1]$.

Definition 5.5 If μ is a Q- fuzzy derivations relation on a set S and β is a Q- fuzzy derivations subset of S , then μ is a Q- fuzzy derivations relation on β if

$$\mu(d(x,y),q) \leq \min \{\beta(d(x),q), \beta(d(y),q)\},\ \forall\ x, y \in S \text{ and } q\in Q.$$

Definition 5.6 Let μ and β be Q- fuzzy derivations subset of a set S , the Cartesian product of μ and β is defined by $(\mu \times \beta)(d(x,y),q) = \min \{\mu(d(x),q), \beta(d(y),q)\}$, $\forall\ x, y \in S$ and $q\in Q$.

Definition 5.7 If β is a Q- fuzzy derivations subset of a set S, the strongest fuzzy relation on S , that is a Q- fuzzy derivations relation on β is μ_β given by

$\mu_\beta(d(x,y),q) = \min \{\beta(d(x),q), \beta(d(y),q)\},\ \forall\ x, y \in S$ and $q\in Q$.

Analogous to [17] , we have a similar result for Q-fuzzy derivations KU-ideal, which can be proved in similar manner ,we state the result without proof.

Lemma 5.8 For a given a Q- fuzzy derivations subset S, let μ_β be the strongest fuzzy derivations relation on S ,then for t $\in$ [0,1] , we have $(\mu_\beta)_t = \beta_t \times \beta_t$.

Theorem 5.9 Let μ and β be a Q- fuzzy derivations subset of KU-algebra X,
Such that $\mu\times\beta$ is a Q- fuzzy derivations KU-ideal of $X\times X$, then

(i) either $\mu(d(x),q) \leq \mu(d(0),q)$ or $\beta(d(x),q) \leq \beta(d(0),q)$ $\forall\ x\in X, q\in Q$.

(ii) if $\mu(d(x),q) \leq \mu(d(0),q)$ $\forall\ x\in X, q\in Q$, then

either $\mu(d(x),q) \leq \beta(d(0),q)$ or $\beta(d(x),q) \leq \beta(d(0),q)$,

(iii) if $\beta(d(x),q) \le \beta(d(0),q) \quad \forall\ x \in X$, then either $\mu(d(x),q) \le \mu(d(0),q)$ or $\beta(d(x),q) \le \mu(d(0),q)$,

(iv) either μ or β is Q- fuzzy derivations KU-ideal of X.

Remark5.10 Let X and Y be KU- algebras, we define $*$ on $X \times Y$ by :
For every $(x, y), (u, v) \in X \times Y$, $(x, y) * (u, v) = (x * u, y * v)$, then clearly $(X \times Y, *, (0, 0))$ is a KU- algebra .

Theorem 5.11 Let μ and β be a Q- fuzzy derivations KU- ideals of KU - algebra X, then $\mu \times \beta$ is a Q- fuzzy derivations KU-ideal of $X \times X$.
Proof : for any $(x, y) \in X \times X$, we have

$$(\mu \times \beta)(d(0,0),q) = \min\{\mu(d(0),q), \beta(d(0),q)\}$$
$$= \min\{\mu(0,q), \beta(0,q)\}$$
$$\ge \min\{\mu(d(x),q), \beta(d(y),q)\} = (\mu \times \beta)(d(x,y),q).$$

Now let $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in X \times X$, then

$(\mu \times \beta)(d((x_1 * z_1), (x_2 * z_2)), q) = \min\{\mu(d(x_1 * z_1), q), \beta(d(x_2 * z_2), q)\} \ge \min$
$\{\min\{\mu(d(x_1 * (y_1 * z_1)), q), \mu(d(y_1), q)\}, \min\{\beta(d(x_2 * (y_2 * z_2)), q), \beta(d(y_2), q)\}\}$
$= \min\{\min\{\mu(d(x_1 * (y_1 * z_1)), q), \beta(d(x_2 * (y_2 * z_2)), q)\}, \min\{\mu(d(y_1), q), \beta(d(y_2), q)\}\}$
$= \min\{(\mu \times \beta)((d(x_1 * (y_1 * z_1)), q), (d(x_2 * (y_2 * z_2)), q), (\mu \times \beta)((d(y_1), q), (d(y_2), q)\}$.
Hence $\mu \times \beta$ is a fuzzy Q- derivations KU- ideal of $X \times X$.

Theorem 5.13 Let β be a Q- fuzzy derivations subset of KU- algebra X and let μ_β be the strongest Q- fuzzy derivations relation on X, then β is a Q- fuzzy derivations KU - ideal of X if and only if μ_β is a Q- fuzzy derivations KU- ideal of $X \times X$.
proof : Assume that β is a fuzzy derivations KU- ideal of X, we note from (F_1) that

$\mu_\beta((0,0),q) = \min\{\beta(d(0),q), \beta(d(0),q)\} = \min\{\beta(0,q), \beta(0,q)\}$
$\ge \min\{\beta(d(x),q), \beta(d(y),q)\} = \mu_\beta((d(x), d(y)),q)$.

Now, for any $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in X \times X$, we have from ($F_2$) :

$\mu_\beta(d((x_1 * z_1, x_2 * z_2)), q) = \min\{\beta(d(x_1 * z_1)), q), \beta(d(x_2 * z_2), q)\} \ge \min$
$\{\min\{\beta(d((x_1 * (y_1 * z_1)), q), \beta(d(y_1), q)\}, \min\{\beta(d(x_2 * (y_2 * z_2)), q), \beta(y_2), q\}\} =$

$\min\{\min\{\beta(d(x_1*(y_1*z_1)),q),\beta(d(x_2*(y_2*z_2)),q)\},\min\{\beta(d(y_1),q),\beta(d(y_2),q)\}\}$

$= \min\{\mu_\beta(d(x_1*(y_1*z_1)), d(x_2*(y_2*z_2)),q), \mu_\beta(d(y_1), d(y_2)),q)\}$.

Hence μ_β is a fuzzy derivations KU - ideal of $X\times X$.

Conversely. For all $(x, y)\in X\times X$, we have

Min $\{\beta(0,q), \beta(0,q)\} = \mu_\beta((0,0),q) \geq \mu_\beta((x,y),q) = \min\{\beta(x,q), \beta(y,q)\}$.

It follows that $\beta(0,q)\geq\beta(x,q)$ for all $x\in X$, which prove (F_1).

Now, let $(x_1, x_2), (y_1, y_2), (z_1, z_2)\in X\times X$, then

$\min\{\beta(d(x_1*z_1),q), \beta(d(x_2*z_2),q)\} = \mu_\beta(d(x_1*z_1), d(x_2*z_2)),q) \geq$

$\min\{\mu_\beta(d((x_1, x_2)*((y_1, y_2)*(z_1, z_2))),q), \mu_\beta((d(y_1),d(y_2)),q)\}=$

$\min\{\mu_\beta(d(x_1*(y_1*z_1)),q), d(x_2*(y_2*z_2)),q), \mu_\beta(d(y_1), d(y_2),q)\}= \min$

$\{\min\{\beta(d(x_1*(y_1*z_1)),q), \beta(d(x_2*(y_2*z_2)),q)\}, \min\{\beta(d(y_1),q), \beta(d(y_2),q)\}\}=$

$\min\{\min\{\beta(d(x_1*(y_1*z_1)),q),\beta(d(y_1),q)\},\min\{\beta((d(x_2*(y_2*z_2)),q), \beta(d(y_2),q)\}\}$

In particular, if we take $x_2 = y_2 = z_2 = 0$, then ,

$\beta(d(x_1*z_1),q)\geq\min\{\beta(d(x_1*(y_1*z_1)),q), \beta(d(y_1),q)\}$ This prove (F_2) and completes the proof.

Conclusion

Derivation is a very interesting and important area of research in the theory of algebraic structures in mathematics. In the present paper, the notion of *Q*- fuzzy left derivations KU - ideal in KU-algebra are introduced and investigated the useful properties of *Q*- fuzzy left derivations KU - ideals in KU-algebras.

In our opinion, these definitions and main results can be similarly extended to some other algebraic systems such as BCI-algebra, BCH-algebra, Hilbert algebra, BF-algebra -J-algebra, WS-algebra, CI-algebra, SU-algebra, BCL-algebra, BP-algebra, Coxeter algebra, BO-algebra, PU- algebras and so forth.

The main purpose of our future work is to investigate:

(1) The interval value, bipolar and intuitionistic *Q*- fuzzy left derivations KU - ideal in KU-algebra.

(2) To consider the cubic structure of left derivations KU - ideal in KU-algebra.

We hope the fuzzy left derivations KU - ideals in KU-algebras, have applications in different branches of theoretical physics and computer science.

Algorithm for KU-algebras

Input (X :set, $*$:binary operation)

Output (“ X is a KU-algebra or not”)

Begin

If $X = \phi$ then go to (1.);

End If

If $0 \notin X$ then go to (1.);

End If

Stop: =false;

$i := 1$;

While $i \le |X|$ and not (Stop) do

If $x_i * x_i \ne 0$ then

Stop: = true;

End If

$j := 1$

While $j \le |X|$ and not (Stop) do

If $x_i * (y_j * x_i) \ne 0$ then

Stop: = true;

End If

End If

$k := 1$

While $k \le |X|$ and not (Stop) do

If $(x_i * y_j) * ((y_j * z_k) * (x_i * z_k)) \ne 0$ then

Stop: = true;

End If

End While

End While

End While

If Stop then

(1.) Output (“ X is not a KU-algebra”)

Else

Output (“ X is a KU-algebra”)

End If

End.

Conflict of Interests

The author declares that there is no conflict of interests.

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