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A GENERALIZATION OF COFINITELY WEAK RAD-SUPPLEMENTED MODULES

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Abstract. We say that a module M is a *closed cofinitely weak generalized supplemented module* or briefly *ccwgs-module* if for every $N \leq_{cc} M$, N has a weak Rad-supplement in M. In this article, the various properties of ccwgs-modules are given as a proper generalization of cofinitely weak Rad-supplemented modules. We prove that every cofinite direct sum of a ccwgs-module is a ccwgs-module. In particular, we also prove that every ccwgs-module over a left Bass ring is a ccws-module. Finally, we show that the notion of cofinitely weak Rad-supplemented modules and the notion of ccwgs-modules are equivalent under some special conditions.

Keywords: supplements; cofinite submodule; cofinitely weak Rad-supplemented module; closed weak Rad-supplemented modules.

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1. Introduction

Throughout this paper, it is assumed that R is an associative ring with identity and all modules are unital left R-modules. Let R be such a ring and let M be an R-module. The notation $K \leq M$

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$(K < M)$ means that K is a (proper) submodule of M. A submodule N of M is called *cofinite* in M if the factor module $\frac{M}{N}$ is finitely generated. A module M is called *extending* if every submodule is essential in a direct summand of M [4]. Here a submodule $K \leq M$ is said to be *essential* in M, denoted as $K \trianglelefteq M$, if $K \cap N \neq 0$ for every non-zero submodule $N \leq M$. A *closed submodule* N of M, denoted by $N \leq_c M$, is a submodule which has no proper essential extension in M. Every direct summand of a module M is a closed submodule of M. If $L \leq_c N$ and $N \leq_c M$, then $L \leq_c M$ by [8, Proposition 1.5]. If N is closed and cofinite submodule of M, we denote as $N \leq_{cc} M$. As a dual notion of an essential submodule, a proper submodule S of M is called *small (in M)*, denoted as $S << M$, if $M \neq S + L$ for every proper submodule L of M [20, 19.1]. The Jacobson radical of M will be denoted by $RadM$. It is known that $RadM$ is the sum of all small submodules of M [20, 21.5]

A non-zero module M is said to be *hollow* if every proper submodule of M is small in M, and it is said to be *local* if it is hollow and is finitely generated. A module M is local if and only if it is finitely generated and $RadM$ is maximal (see [4, 2.12 §2.15]). A ring R is said to be *local* if J is maximal, where J is the Jacobson radical of R.

An R-module M is called *supplemented* if every submodule of M has a supplement in M. Here a submodule $K \leq M$ is said to be a *supplement* of N in M if K is minimal with respect to $N + K = M$, or equivalently, if $N + K = M$ and $N \cap K \ll K$ [20, page 349]. Every direct summand of a module M is a supplement submodule of M, and supplemented modules are a proper generalization of semisimple modules. In addition, every factor module of a supplemented module is again supplemented. As a generalization of supplemented modules, a module M is called *weakly supplemented* if any submodule N of M has a weak supplement K, i.e. there exists a submodule K of M such that $M = N + K$ and $N \cap K \ll M$ as in [11].

Alizade et al. [1] have defined cofinitely supplemented modules as a proper generalization of supplemented modules. They call a module M *cofinitely supplemented* if every cofinite submodule N of M has a *supplement* in M, and give characterizations of these modules over any rings and commutative domains (see [1]). In particular, it is shown in [1, Theorem 2.8] that a module M is cofinitely supplemented if and only if every maximal submodule of M

has a supplement in M. A module M is called *cofinitely weak supplemented* if every cofinite submodule has a weak supplement in M [2].

A module M is called *lifting* (or D_1*-module*) if, for every submodule N of M, there exists a direct summand K of M such that $K \leq N$ and $\frac{N}{K} << \frac{M}{K}$ [4, 22.2]. Mohamed and Müller has generalized the concept of lifting modules to $\oplus$-supplemented modules. M is called $\oplus$-*supplemented* if every submodule N of M has a supplement that is a direct summand of M [12]. Clearly every $\oplus$-supplemented module is supplemented, but a supplemented module need not be $\oplus$-supplemented in general (see [12, Lemma A.4 (2)]). It is shown in [12, Proposition A.7 and Proposition A.8] that if R is a dedekind domain, every supplemented R-module is $\oplus$-supplemented. Hollow modules are $\oplus$-supplemented.

In [5], Çalışıcı and Pancar call a module M $\oplus$*-cofinitely supplemented* if every cofinite submodule of M has a supplement that is a direct summand of M. They gave in the same paper some properties of these modules. In addition, it is proven in [5, Theorem 2.9] that a ring R is semiperfect (that is, $_RR$ is supplemented) if and only if every free left R-module is $\oplus$-cofinitely supplemented.

Let M be a module and U,V be submodules of M. A submodule V of M is called *Rad-supplement* (according to [19], *generalized supplement*) of U in M if $U+V=M$ and $U\cap V\subseteq RadV$ (see [4, Theorem 10.14]). A module M is called *Rad-supplemented* (according to [19], *generalized supplemented*) if every submodule U of M has a Rad-supplement in M. Since Jacobson radical of a module M is the sum of all small submodules of M, every supplement is a Rad-supplement. Then, clearly every supplemented module is Rad-supplemented but a Rad-supplemented module need not to be supplemented. Note that radical modules are Rad-supplemented. Let R be a non-local dedekind domain with quotient field K. Then K is Rad-supplemented, but it is not supplemented.

In [3], a module M is called *cofinitely Rad-supplemented* if every cofinite submodule has a Rad-supplement in M, and the closure properties of cofinitely Rad-supplemented modules is given. Lomp [11] calls a module M *semilocal* if $\frac{M}{RadM}$ is semisimple. Equivalently, every submodule N of M has a weak Rad-supplement K in M, that is, $M=N+K$ and $N\cap K\subseteq RadM$. A ring R is called semilocal if the left (or right) R-module R is semilocal. He show [11, Theorem

3.5] that R is semilocal if and only if every left R-module is semilocal. A submodule V of M is called *weak Rad-supplement* of U in M if $U+V=M$ and $U\cap V\subseteq RadM$ ([7]). A module M is called *cofinitely weak Rad-supplemented* if every cofinite submodule U of M, there exists a submodule V of M such that $U+V=M$ and $U\cap V\subseteq RadM$.

Let M be an R-module. M is called *Rad-$\oplus$-supplemented*, or *generalized $\oplus$-supplemented*, if every submodule of M has a Rad-supplement that is a direct summand of M ([6]). Clearly, Rad-$\oplus$-supplemented modules are Rad-supplemented. A module M is called *$\oplus$-cofinitely Rad-supplemented* (according to [9], *generalized $\oplus$-cofinitely supplemented*) if every cofinite submodule of M has a Rad-supplement that is a direct summand of M. Instead of a $\oplus$-cofinitely radical supplemented module, we will use a $cgs^{\oplus}$-module like for [13].

In [14], the notion of closed weak supplemented modules is studied as a generalization of weak supplemented modules. A module M is called a *closed weak supplemented module* if every closed submodule has a weak supplement in M. Then, Türkmen et al. call a module M *closed cofinitely weak supplemented module* (or briefly, *ccws-module*) if for $N\leq_{cc} M$, N has a weak supplement in M ([18]). The various properties of ccws-modules are given in the same paper. A module M is called *closed weak generalized supplemented* (or, *closed weak Rad-supplemented*) if every closed submodule has a weak generalized supplement (weak Rad-supplement) in M.

In this paper, we introduce the notion of closed cofinitely weak Rad-supplemented modules, denoted by ccwgs, as a proper generalization of ccws-modules. We provide some properties of these modules. An example is given to separate ccwgs-modules and cofinitely weak Rad-supplemented modules. We prove that every cofinite direct summand of a ccwgs-module is a ccwgs-module. We obtain that every ccwgs-module over a left Bass ring is a ccws-module. We also prove that a cofinitely strong refinable module M is $cgs^{\oplus}$-module if and only if M is a cofinitely weak supplemented modules.

2. ccwgs-Modules

In this section, we define the concept of ccwgs-modules as a generalization of cofinitely weak Rad-supplemented modules, and give various properties of them.

Definition 2.1. *Let M be a module. M is called a* closed cofinitely weak generalized supplemented module *(or briefly a* ccwgs-*module) if, for every $N \leq_{cc} M$, there exists a submodule K of M such that $M = K + N$ and $K \cap N \subseteq RadM$.*

Under given definitions, we clearly have the following implications on modules:

$$\begin{array}{c} cofinitely\, Rad-supplemented\; modules \\ \downarrow \\ cofinitely\, weak\, Rad-supplemented\; modules \\ \downarrow \\ ccwgs-modules \end{array}$$

It follows from [4, 1.16] that a module M is extending if and only if every closed submodule is a direct summand of M. Applying this fact, we obtain that every extending module is a ccwgs-module.

Example 2.2. *Consider the $\mathbb{Z}$-module $\mathbb{Z}$, where $\mathbb{Z}$ is the ring of all integers. Let $n\mathbb{Z} = N$ and $m\mathbb{Z} = M$ be proper submodules of $\mathbb{Z}$ such that $0, \mp 1 \neq n, m \in \mathbb{Z}$. Note that $0 \neq nm \in N \cap M \neq 0$. So, there is not a submodule M of $\mathbb{Z}$ such that $\mathbb{Z} = N + M$ and $N \cap M \subseteq Rad\mathbb{Z} = 0$. Hence $\mathbb{Z}$ is not cofinitely weak Rad-supplemented module. Since $\mathbb{Z}$ is uniform as a $\mathbb{Z}$-module and the direct summands of $\mathbb{Z}$ are 0 and $\mathbb{Z}$ itself. It is easy to see that $\mathbb{Z}$ is a ccwgs-module because of all closed submodules are 0 and $\mathbb{Z}$.*

Proposition 2.3. *Let M be a ccwgs-module. Then any cofinite direct summand of M is a ccwgs-module.*

Proof. Let N be any cofinite direct summand of M and $L \leq_{cc} N$. Since $N \leq_c M$, we obtain that $L \leq_c M$. In addition, since $\frac{M}{N}$ and $\frac{N}{L}$ is finitely generated, L is a cofinite submodule of M. It follows that there exists a submodule K of M such that $M = L + K$ and $L \cap K \subseteq RadM$. We have $N = L + (N \cap K)$ and $L \cap (N \cap K) = L \cap K \subseteq N \cap RadM$. Since N is a direct summand of M. We obtain that $N \cap RadM = RadN$. Note that $L \cap (N \cap K) \subseteq RadN$ by [20, 41.1(5)]. Therefore N is a ccwgs-module.

Lemma 2.4. *(See [15, Lemma 3.5]) Let N and L be cofinite submodules of a module M such that $N+L$ has a weak Rad-supplement H in M and $N\cap(H+L)$ has a weak Rad-supplement G in N. Then $H+G$ is a weak Rad-supplement of L in M.*

Proposition 2.5. *Let $M=M_1\oplus M_2$ such that each $M_i\,(i=1,2)$ is a ccwgs-module. Suppose that $M_i\cap(M_j+L)\leq_{cc} M_i$ and $M_j\cap(L+K)\leq_{cc} M_j$, where K is a weak Rad-supplement of $M_i\cap(M_j+L)$ in M_i, $i\neq j$, for any $L\leq_c M$. Then M is a ccwgs-module.*

Proof. Let $L\leq_{cc} M$, then $M=M_1+M_2+L$ has a weak Rad-supplement 0 in M. Since $M_1\cap(M_2+L)\leq_{cc} M_1$ and M_1 is a ccwgs-module, then there exists a submodule K of M_1 such that $M_1=M_1\cap(M_2+L)+K$ and $M_1\cap(M_2+L)\cap K=K\cap(M_2+L)\subseteq RadM_1$. By Lemma 2.4, K is a weak Rad-supplement of M_2+L in M, i.e. $M=K+(M_2+L)$. Since $M_2\cap(K+L)\leq_{cc} M_2$ and M_2 is a ccwgs-module, then $M_2\cap(K+L)$ has a weak Rad-supplement J in M_2. Again by Lemma 2.4, $K+J$ is a weak Rad-supplement of L in M. Hence M is a ccwgs-module.

Proposition 2.6. *Let $M=M_1+M_2$, where M_1 is a ccwgs-module and M_2 is any R-module. Suppose that for any $N\leq_{cc} M$, $N\cap M_1\leq_{cc} M_1$. Then M is a ccwgs-module if and only if every $N\leq_{cc} M$ with M_2 not contained in N has a weak Rad-supplement.*

Proof. $(\Longrightarrow)$ It is clear.

$(\Longleftarrow)$ Let $N\leq_{cc} M$ with $M_2\leq N$. Then $M=M_1+M_2=M_1+N$ and M_1+N has a weak Rad-supplement 0 in M. Since $N\cap M_1\leq_{cc} M_1$ and M_1 is a ccwgs-module, $N\cap M_1$ has a weak Rad-supplement H in M_1. By Lemma 2.4, H is a weak Rad-supplement of N in M. By the hypothesis, M is a ccwgs-module.

Recall from [10, page 185] that a left R-module M is said to be *singular* (respectively, *non-singular*) if $Z(M)=M$ (respectively, $Z(M)=0$), where $Z(M)=\{m\in M\,|\,Ann(m)\trianglelefteq R\}$.

Let M be a non-singular module and $N\leq_{cc} M$, then $N\cap L\leq_{cc} L$ for any submodule L of M and $M=N+L$.

Corollary 2.7. *Let $M=M_1+M_2$ be a non-singular module with M_1 ccwgs and M_2 any R-module. Then M is a ccwgs-module if and only if every $N\leq_{cc} M$ with M_2 not contained in N has a weak Rad-supplement.*

Recall from [16, 1.11] that a module M is said to be *distributive* if $(X+Y)\cap Z = (X\cap Z)+(Y\cap Z)$ for any submodules X, Y, and Z of M. This means that the submodule lattice $Lat(M)$ is distributive.

Theorem 2.8. *Let $M = M_1 \oplus M_2$ be a distributive module. Then M is a ccwgs-module if and ony if, for each M_i of M, $i \in \{1,2\}$, M_i is a ccwgs-module.*

Proof. Let $L \leq_{cc} M$. By the isomorphisms $\frac{M}{L} \cong \frac{M_1}{M_1\cap L} + \frac{M_2}{M_2\cap L}$ and $\frac{\frac{M}{L}}{\frac{M_i}{M_i\cap L}} \cong \frac{M_j}{M_j\cap L}$ for each M_i, $i \in \{1,2\}$ and $i \neq j$, we have $M_i \cap L$ is a cofinite submodule of M_i. In addition, since L is a closed submodule of M, then for each i, $i \in \{1,2\}$, $M_i \cap L$ is closed in M_i. So $M_i \cap L \leq_{cc} M_i$. In fact, suppose that $M_1 \cap L$ is essential in K. Since $M_2 \cap L$ is essential in $M_2 \cap L$ and M is distributive, we have that $L = (M_1\cap L)\oplus(M_2\cap L) = K \oplus (M_2 \cap L)$, because L is closed in M. Since for each i, $i \in \{1,2\}$, M_i is a ccwgs-module, there exists a submodule K_i of M_i such that $M_i = (L\cap M_i)+K_i$, $(L\cap M_i)\cap K_i = L\cap K_i \subseteq RadM_i$. Hence $M = M_1 \oplus M_2 = [(L\cap M_1)\oplus(L\cap M_2)]+(K_1+K_2) = L+(K_1\oplus K_2)$ and $L\cap(K_1\oplus K_2) = (L\cap K_1)\oplus(L\cap K_2) \leq RadM_1 \oplus RadM_2 = Rad(M_1\oplus M_2) = RadM$. Thus M is a ccwgs-module. The converse holds by Proposition 2.3.

Corollary 2.9. *Let $M = \oplus_{i=1}^{n} M_i$ be a duo module. Then M is a ccwgs-module if and only if for each cofinite direct summand M_i, $i \in \{1,2,\ldots,n\}$, M_i is a ccwgs-module.*

Recall from [17] that a module M is called *cofinitely strong refinable* if, for every cofinite submodule U of M and any submodule V of M with $U+V = M$, there exists submodules U' and V' of M with $U' \subseteq U$, $V' \subseteq V$, $M = U'+V$ and $M = U'\oplus V'$.

Proposition 2.10. *Let M be a cofinitely strong refinable module. Then the following statements are equivalent.*

(1) M is a $cgs^{\oplus}$-module.

(2) M is a cofinitely Rad-supplemented module.

(3) M is a cofinitely weak Rad-supplemented module.

Proof. $(1) \Rightarrow (2) \Rightarrow (3)$ are obvious.

$(3) \Rightarrow (1)$ Suppose that M is a cofinitely weak Rad-supplemented module. Let N be any cofinite submodule of M. Then there exists a submodule K of M such that $M = N+K$ and $N\cap K \subseteq RadM$. Since M is a cofinitely strong refinable module, there exist submodules N'

and $K^{'}$ with $N^{'} \subseteq N, K^{'} \subseteq K, M = N + K^{'}$ and $M = N^{'} \oplus K^{'}$. It follows that $M = N + K^{'}$ and $N \cap K^{'} \subseteq RadK^{'}$. Therefore M is a $cgs^{\oplus}$-module.

Proposition 2.11. *Let M be an R-module with $Rad(M) = 0$. Then, M is a ccwgs-module if and only if every closed cofinite submodule is a direct summand of M.*

Proof. $(\Rightarrow)$ Let $N \leq_{cc} M$. By the hypothesis, there exists a submodule K of M such that $M = N + K$ and $N \cap K \subseteq RadM$. So $N \cap K = 0$. Thus $M = N \oplus K$. Therefore N is a direct summand of M.

$(\Leftarrow)$ The converse is clear.

Using Proposition 2.10 and [4, 1.16], we obtain the following fact.

Corollary 2.12. *Let M be a finitely generated R-module with $Rad(M) = 0$. Then the following are equivalent.*

(1) M is a ccwgs-module.

(2) M is extending.

Recall [20, page 192] that a ring R is called *a left V-ring* if every simple left R-module is injective. Equivalently, a ring R is a left V-ring if and only if $Rad(M) = 0$ for all left R-modules M.

Theorem 2.13. *Let R be a left nonsingular V-ring. Then the following statements are equivalent.*

(1) Every nonsingular left R-module M is a ccwgs-module.

(2) For any closed cofinite submodule N of every nonsingular left R-module M , N is a direct summand of M.

Proof. Clear by Proposition 2.11.

Any finite sum of ccwgs-modules need not to be a ccwgs-module, in general. The following Example shows this.

Example 2.14. *Let $R = \mathbb{Z}[x]$, where $\mathbb{Z}$ is the ring of all integers. It can be seen that the left R-module R is a ccwgs-module and $M = R \oplus R$ is not an extending R-module. As $Rad(M) = 0$, by Corollary 2.12, M is not a ccwgs-module.*

Lemma 2.15. *(See [15, Lemma 4.10]) Let U and K be submodules of M such that K is a weak generalized supplement of a maximal submodule N of M. If $K+U$ has a weak Rad-supplement X in M, then U has a weak Rad-supplement in M.*

Theorem 2.16. *Suppose that for any cofinite submodule U of M, there exists a submodule K of M, which is a weak Rad-supplement of some maximal submodule N of M, such that $K+U$ is closed in M. Then M is a ccwgs-module if and only if M is a cofinitely weak Rad-supplemented module.*

Proof. $(\Rightarrow)$ Let $U \leq_{cc} M$. By the hypothesis, there exists a submodule K of M such that $M = N + K$, $N \cap K \subseteq RadM$ and $K+U \leq_c M$ for a maximal submodule N of M. It follows from $\frac{\frac{M}{U}}{\frac{K+U}{U}} \cong \frac{M}{K+U}$ and U is a cofinite submodule of M that $K+U$ is a cofinite submodule of M. Since M is a ccwgs-module, there exists a submodule X of M such that X is a weak Rad-supplement of $K+U$. By Lemma 2.15, U has a weak Rad-supplement in M. So M is a cofinitely weak Rad-supplemented module.

$(\Leftarrow)$ Clear.

Lemma 2.17. *Let M be a ccwgs-module. Suppose that $RadM$ is small in M. Then M is a ccwgs-module if and only if M is a ccws-module.*

Proof. Let $N \leq_{cc} M$. Since M is a ccwgs-module, there exists a submodule K of M such that $M = N + K$ and $N \cap K \subseteq RadM$. Since $RadM \ll M$, $N \cap K \ll M$. Thus M is a ccws-module. The converse is clear.

A module M is called *coatomic* if every proper submodule of M is contained in a maximal submodule of M. Note that coatomic modules have a small radical.

Corollary 2.18. *Let M be a coatomic module. Then M is a ccwgs-module if and only if it is a ccws-module.*

Recall from [4] that a ring R is a left Bass ring if every non-zero left R-module has a maximal submodule. It is known that the ring R is left Bass if and only if $RadM$ is small in M for every non-zero left R-module M. By using Lemma 2.17, we obtain the following corollary.

Corollary 2.19. *Every ccwgs-module over a left Bass ring is a ccws-module.*

Conflict of Interests

The authors declare that there is no conflict of interests.

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