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ON FORMAL LOCAL COHOMOLOGY MODULES

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Abstract. Let I, $\mathfrak{a}$ be two ideals of a Noetherian ring R. Let M be an R-module. There exists a systematic study of the formal cohomology modules $\varprojlim_{n\in\mathbb{N}} \mathrm{H}^i_I(M/\mathfrak{a}^n M)$, $0 \leq i \in \mathbb{Z}$. It is what will be done in this paper.

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1. Introduction

Throughout this paper, R is a commutative ring with non-zero identity. The theory of local cohomology has developed for six decades after its introduction by Grothendieck. There exists a relation between local cohomology and formal local cohomology. We study here this latter module.

2. Preliminaries

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Let I be an ideal of R, and let M be an R-module. In [1], the ith local cohomology module $\mathrm{H}_I^i(M)$ of M with respect to I is defined by

$$\mathrm{H}_I^i(M) = \varinjlim_{t\in\mathbb{N}} \mathrm{Ext}_R^i\left(R/I^t, M\right),$$

for all $0 \leq i \in \mathbb{Z}$. Now, for $\mathfrak{a}$ other ideal of R, consider the family of local cohomology modules given by $\left\{\mathrm{H}_I^i(M/\mathfrak{a}^n M)\right\}_{n\in\mathbb{N}}$. According to [4], for every $n \in \mathbb{N}$, we have that there exists a natural homomorphism

$$\phi_{n+1,n} : \mathrm{H}_I^i\left(M/\mathfrak{a}^{n+1}M\right) \to \mathrm{H}_I^i(M/\mathfrak{a}^n M).$$

These families form an inverse system. Their inverse limit that is given by $\varprojlim_{n\in\mathbb{N}} \mathrm{H}_I^i(M/\mathfrak{a}^n M)$ is called, according to [4], the ith formal local cohomology module of M with respect to $\mathfrak{a}$, and will be denoted by $\mathfrak{F}_{\mathfrak{a},I}^i(M)$. Moreover, for a Noetherian local ring $(R,\mathfrak{m})$ and M an R-module we have the Matlis dual module $\mathrm{D}(M) = \mathrm{Hom}_R(M,E)$ of M, where $E = \mathrm{E}(R/\mathfrak{m})$ is the injective envelope of the residue field $R/\mathfrak{m}$.

The next definition will be used in the sequence of the paper.

Definition 2.1. Let $(R,\mathfrak{m},k)$ and $(S,\mathfrak{n},l)$ be two local rings. A ring homomorphism $(R,\mathfrak{m},k) \to (S,\mathfrak{n},l)$ is a *local homomorphism* if $\mathfrak{m}S \subset \mathfrak{n}$.

In the next section, the following remark will be used.

Remark 2.2.([4, Remark 4.6]) Note that, the short exact sequence

$$0 \to \mathfrak{a}^n M/\mathfrak{a}^{n+1}M \to M/\mathfrak{a}^{n+1}M \to M/\mathfrak{a}^n M \to 0$$

induces an epimorphism $\mathrm{H}_I^i\left(M/\mathfrak{a}^{n+1}M\right) \to \mathrm{H}_I^i(M/\mathfrak{a}^n M) \to 0$, of non-zero R-modules for all $n \in \mathbb{N}$. Hence, the inverse limit $\varprojlim_{n\in\mathbb{N}} \mathrm{H}_I^i(M/\mathfrak{a}^n M)$ is not zero.

The following definition will be used in the next section.

Definition 2.3.([2, Definition 3.1]) Let R be a Noetherian ring. Let I be an ideal of R and let M be an R-module. The ith *local homology module* $\mathrm{H}_i^I(M)$ of M with respect to I is defined by, $\mathrm{H}_i^I(M) := \varprojlim_{t\in\mathbb{N}} \mathrm{Tor}_i^R(R/I^t, M)$.

3. Main Results

In this section, we have a result on formal local cohomology modules.

Theorem 3.1. *Let $(R,\mathfrak{m},k)$ be a Noetherian local ring. Let $(R,\mathfrak{m},k)\to(S,\mathfrak{n},l)$ be a local homomorphism of local rings, with S a Noetherian ring, and let $\mathfrak{a}$ be an ideal of R. Suppose that M is a finitely generated S-module. If $\mathfrak{F}^i_{\mathfrak{a},\mathfrak{m}}(M)=0$, for each $i\geq 1$, then $\mathrm{D}(M/\mathfrak{a}^nM)$ is a flat R-module, for some $n\in\mathbb{N}$.*

Proof. By the hypothesis, for all $i\geq 1$, we have that:

$$\mathfrak{F}^i_{\mathfrak{a},\mathfrak{m}}(M):=\varprojlim_{n\in\mathbb{N}}\mathrm{H}^i_{\mathfrak{m}}(M/\mathfrak{a}^nM)=0.$$

By the Remark , we have that there exists $n\in\mathbb{N}$ such that the local cohomology module $\mathrm{H}^i_{\mathfrak{m}}(M/\mathfrak{a}^nM)=0$. Therefore, it follows, as given in prerequisites, that we have:

$$\varinjlim_{t\in\mathbb{N}}\mathrm{Ext}^i_R\left(R/\mathfrak{m}^t,M/\mathfrak{a}^nM\right)=0.\quad(*)$$

Thus, applying the Matlis dual module $\mathrm{D}(\bullet)$ (see prerequisites) to $(*)$ we obtain that $\mathrm{D}\Big(\varinjlim_{t\in\mathbb{N}}\mathrm{Ext}^i_R(R/\mathfrak{m}^t,M/\mathfrak{a}$
0. Now, by [3, Theorem 2.27], it follows that $\mathrm{D}\left(\varinjlim_{t\in\mathbb{N}}\mathrm{Ext}^i_R(R/\mathfrak{m}^t,M/\mathfrak{a}^nM)\right)$, which is equal to

$$\mathrm{Hom}_R\left(\varinjlim_{t\in\mathbb{N}}\mathrm{Ext}^i_R\left(R/\mathfrak{m}^t,M/\mathfrak{a}^nM\right),\mathrm{E}(R/\mathfrak{m})\right),$$

is isomorphic to $\varprojlim_{t\in\mathbb{N}}\mathrm{Hom}_R\left(\mathrm{Ext}^i_R(R/\mathfrak{m}^t,M/\mathfrak{a}^nM),\mathrm{E}(R/\mathfrak{m})\right)$, which in turn is equal to $\varprojlim_{t\in\mathbb{N}}\mathrm{D}\big(\mathrm{Ext}^i_R(R/\mathfrak{m}^t,M$

By [5, Proposition 3.4.14 (ii)], we have that:

$$\mathrm{D}\left(\mathrm{Ext}^i_R\left(R/\mathfrak{m}^t,M/\mathfrak{a}^nM\right)\right)\cong\mathrm{Tor}^R_i\left(R/\mathfrak{m}^t,\mathrm{D}(M/\mathfrak{a}^nM)\right).$$

Therefore, by the Definition , we have that

$$\mathrm{H}^{\mathfrak{m}}_i(\mathrm{D}(M/\mathfrak{a}^nM))=\varprojlim_{t\in\mathbb{N}}\mathrm{Tor}^R_i\left(R/\mathfrak{m}^t,\mathrm{D}(M/\mathfrak{a}^nM)\right)=0.$$

By Remark , it follows that there exists $t\in\mathbb{N}$ such that

$$\mathrm{Tor}^R_i\left(R/\mathfrak{m}^t,\mathrm{D}(M/\mathfrak{a}^nM)\right)=0,\ \text{for all } i\geq 1.$$

Thus, also we have $\mathrm{Tor}^R_i(R/\mathfrak{m},\mathrm{D}(M/\mathfrak{a}^nM))=0$, for all $i\geq 1\quad(**)$.

To end the theorem, it suffices to prove that $\mathrm{Tor}_i^R(N, \mathrm{D}(M/\mathfrak{a}^n M)) = 0$ for each finitely generated R-module N, and $i \geq 1$. This we achieve by an induction on $\dim(N)$.

When $\dim(N) = 0$, let's induce on the length of N. If $\mathfrak{l}_R(N) = 1$, then $N \cong R/\mathfrak{m}$, so the desired result is the mentioned in $(**)$. When $\mathfrak{l}_R(N) \geq 2$, one can get an exact sequence of R-modules $0 \to R/\mathfrak{m} \to N \to N' \to 0$. Applying $\bullet \otimes_R \mathrm{D}(M/\mathfrak{a}^n M)$ yields an exact sequence

$$\mathrm{Tor}_i^R(R/\mathfrak{m}, \mathrm{D}(M/\mathfrak{a}^n M)) \to \mathrm{Tor}_i^R(N, \mathrm{D}(M/\mathfrak{a}^n M)) \to \mathrm{Tor}_i^R\left(N', \mathrm{D}(M/\mathfrak{a}^n M)\right).$$

Since $\mathfrak{l}_R\left(N'\right) = \mathfrak{l}_R(N) - 1$, the induction hypothesis yields the vanishing.

Let $d \geq 1$ be an integer such that for $i \geq 1$ we have that the functor $\mathrm{Tor}_i^R(\bullet, \mathrm{D}(M/\mathfrak{a}^n M))$ vanishes on finitely generated R-modules of dimension up to $d-1$. Let N be a finitely generated R-module of dimension d. Consider the exact sequence of R-modules

$$0 \to \Gamma_\mathfrak{m}(N) \to N \to N' \to 0,$$

and the induced exact sequence on $\mathrm{Tor}_i^R(\bullet, \mathrm{D}(M/\mathfrak{a}^n M))$. Since $\mathfrak{l}_R(\Gamma_\mathfrak{m}(N))$ is finite, it suffices to verify the vanishing for N'. Thus, replacing N by N', one may assume that $\mathrm{depth}(N) \geq 1$. Let x in R be an N-regular element; then $\dim(N/(x)N) = \dim(N) - 1$. In view of the induction hypothesis, the exact sequence $0 \to N \xrightarrow{x\cdot} N \to N/(x)N \to 0$ induces an exact sequence

$$\mathrm{Tor}_i^R\left(N, \mathrm{D}\left(\tilde{M}\right)\right) \xrightarrow{x\cdot} \mathrm{Tor}_i^R\left(N, \mathrm{D}\left(\tilde{M}\right)\right) \to \mathrm{Tor}_i^R\left(N/(x)N, \mathrm{D}\left(\tilde{M}\right)\right) = 0$$

for $i \geq 1$, where $\tilde{M} = M/\mathfrak{a}^n M$. As an S-module $\mathrm{Tor}_i^R(N, \mathrm{D}(M/\mathfrak{a}^n M))$ is finitely generated:compute it using a resolution of N by finitely generated free R-modules. Since, by Definition , xS is in the maximal ideal of S, the exact sequence above implies $\mathrm{Tor}_i^R(N, \mathrm{D}(M/\mathfrak{a}^n M)) = 0$ by Nakayamas lemma, for all $i \geq 1$. This completes the induction step.

Conflict of Interests

The authors declare that there is no conflict of interests.

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