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QS-ALGEBRAS DEFINED BY FUZZY POINTS

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Abstract: QS-algebras was derived from KUS-algebras. In this paper, we discussed some of the relationships and characteristics. We introduce a new idea of fuzzy QS-ideal of fuzzy point on QS-algebras and give some properties and theorems of it. We introduce the concept of the normal fuzzy QS-ideal of fuzzy point on QS-algebra and study some of the properties related thereto.

Keywords: QS-ideal; fuzzy QS-ideal; fuzzy QS-ideal of fuzzy point.

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1. Introduction

In 1999, S.S. Ahn and Kim H.S. introduced the class of QS-algebras and give some properties of QS-algebras [6] and described connections between such sub-algebra and congruences, see [4].

In 2006, A.B. Saeid considered the fuzzification of QS-sub-algebra to QS-algebras [1].

Now, we introduced a definition of the QS-ideal fuzzy QS-ideal and fuzzy QS-ideal of fuzzy point. We study some of the related properties, homomorphism fuzzy QS-ideal on fuzzy point, normal fuzzy QS-ideal on fuzzy point, homomorphism normal fuzzy QS-ideal of fuzzy point on QS-algebras.

2. Preliminaries

In this subsection, we study the definition of QS-algebra and QS-sub-algebra of QS-algebras and we give some properties of it.

Definition 2.1([4], [6]) :

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Let $(X; *, 0)$ be a set with a binary operation $(*)$ and a constant (0) . Then $(X; *, 0)$ is called a QS-algebra if it satisfies the following axioms: for all $x, y, z \in X$,

1. $x * x = 0$,
2. $x * 0 = x$,
3. $(x * y) * z = (x * z) * y$,
4. $(x * z) * (x * y) = y * z$.

For brevity we also call X a QS-algebra, we can define a binary relation (\leq) by putting $x \leq y$ if and only if, $y * x = 0$.

Proposition 2.2 ([4],[6]):

Let $(X; *, 0)$ be a QS-algebra, then the following hold: for any $x, y, z \in X$,

- (1) if $x * y = z$, then $x * z = y$.
- (2) $x * y = 0$ implies $x = y$.
- (3) $0 * (x * y) = y * x$.
- (4) $x * (0 * y) = y * (0 * x)$.

Example 2.3 ([4]):

Let $X = \{0, a, b, c\}$ in which $(*)$ be defined by the following table:

*	0	a	b	c
0	0	c	b	a
a	a	0	c	b
b	b	a	0	c
c	c	b	a	0

Then $(X; *, 0)$ is a QS-algebra.

Definition 2.4 ([1]):

Let $(X; *, 0)$ be a QS-algebra X and S be a nonempty subset of X . Then S is called a **QS-sub-algebra of X** if, $x * y \in S$, for any $x, y \in S$.

Definition 2.5:

If ζ is the family of all fuzzy subsets on X , $x_\alpha \in \zeta$ is called a fuzzy point if and only if there exists $\alpha \in (0, 1]$ such that for all $y \in X$,

$$x_\alpha(y) = \begin{cases} \alpha & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

Definition 2.6:

Let $(X; *, 0)$ be a QS-algebra, the set of all fuzzy points on X denote by

$$FP(X) = \{x_\alpha \mid x \in X, \alpha \in (0, 1]\}.$$

Define a binary operation (\odot) on $FP(X)$ by:

$x_\alpha \odot y_\beta = (x * y)_{\min\{\alpha, \beta\}}$, for all $x_\alpha, y_\beta \in FP(X)$, then

$$(QS_1): x_\alpha \odot x_\alpha = 0_\alpha,$$

$$(QS_2): x_\alpha \odot 0_\alpha = x_\alpha,$$

$$(QS_3): (x_\alpha \odot y_\beta) \odot z_\gamma = (x_\alpha \odot z_\gamma) \odot y_\beta,$$

$$(QS_4): (x_\alpha \odot z_\gamma) \odot (x_\alpha \odot y_\beta) = (y_\beta \odot z_\gamma).$$

In X we can define a binary relation (\leq) by : $x_\alpha \leq y_\beta$ if and only if $y_\beta \odot x_\alpha = 0_{\min\{\alpha, \beta\}}$.

Remark 2.7:

If $(X; *, 0)$ is a QS-algebra and $FP_q(X)$ denote the set of all fuzzy points of X , then

$(FP_q(X), \odot, 0_q)$ is a QS-algebra, which is called a fuzzy point QS-algebra, where the value q , $(0 < q \leq 1)$.

Definition 2.8 :

For a fuzzy subset μ of a QS-algebra X , we define the set $FP(\mu)$ of all fuzzy points of X covered by μ to be the set $FP(\mu) = \{x_\alpha \in FP(X) \mid \mu(x) \geq \alpha, 0 < \alpha \leq 1\}$, and

$FP_q(\mu) = \{x_q \in FP_q(X) \mid \mu(x) \geq q\}$, for all $q \in (0, 1]$, $x \in X$.

Now, we give some properties and theorems of QS-algebras.

Theorem 2.9:

If $(X; *, 0)$ is a QS-algebra, then the following hold: for all $x_\alpha, y_\beta, z_\gamma \in FP(X)$,

- a) $x_\alpha \odot 0_\beta = x_{\min\{\alpha, \beta\}}$.
- b) $(x_\alpha \odot y_\beta) \odot x_\alpha = 0_\alpha \odot y_\beta$,
- c) $(x_\alpha \odot y_\beta) \odot 0_\alpha = y_\beta \odot x_\alpha$,
- d) $x_\alpha \odot 0_\alpha = 0_\alpha$ implies that $x_\alpha = 0_\alpha$,
- e) $x_\alpha = (x_\alpha \odot 0_\alpha) \odot 0_\alpha$,
- f) $0_{\min\{\alpha, \beta\}} \odot (x_\alpha \odot y_\beta) = (0_\alpha \odot x_\alpha) \odot (0_\beta \odot y_\beta)$,
- g) $z_\gamma \odot x_\alpha = z_\gamma \odot y_\beta$ implies that $0_\gamma \odot x_\alpha = 0_\gamma \odot y_\beta$.

Proof:

- a) It is clear by (QS_2) .

- b) $(x_\alpha \odot y_\beta) \odot x_\alpha = (x_\alpha \odot x_\alpha) \odot y_\beta = 0_\alpha \odot y_\beta$.
- c) $(x_\alpha \odot y_\beta) \odot 0_\alpha = (x_\alpha \odot y_\beta) \odot (x_\alpha \odot x_\alpha) = y_\beta \odot x_\alpha$, by (QS₂).
- (d), (e) and (f) are clears by (QS₂).
- g) $0_\gamma \odot x_\alpha = (z_\gamma \odot z_\gamma) \odot x_\alpha = (z_\gamma \odot x_\alpha) \odot z_\gamma = (z_\gamma \odot y_\beta) \odot z_\gamma = (z_\gamma \odot z_\gamma) \odot y_\beta = 0_\alpha \odot y_\beta$. \triangle

Proposition 2.9:

Let $(X; *, 0)$ be a QS-algebra . Then the following holds: for any $x_\alpha, y_\beta, z_\gamma \in FP(X)$,

1. $x_\alpha \odot y_\beta \leq z_\gamma$ imply $z_\gamma \odot y_\beta \leq x_\alpha$,
2. $x_\alpha \leq y_\beta$ implies that $z_\gamma \odot y_\beta \leq z_\gamma \odot x_\alpha$,
3. $y_\beta \odot [(y_\beta \odot z_\gamma) \odot z_\gamma] = 0_{\min\{\beta, \gamma\}}$,
4. $(x_\alpha \odot z_\gamma) \odot (y_\beta \odot z_\gamma) \leq (y_\beta \odot x_\alpha)$.

Proof:

1. It follows from (QS₄).
2. By (QS_{1'}), we obtain $[(z_\gamma \odot x_\alpha) \odot (z_\gamma \odot y_\beta)] = (y_\beta \odot x_\alpha)$, but $x_\alpha \leq y_\beta$ implies $y_\beta \odot x_\alpha = 0_{\min\{\alpha, \beta\}}$, then we get $(z_\gamma \odot x_\alpha) \odot (z_\gamma \odot y_\beta) = 0_{\min\{\alpha, \beta\}}$.
i.e., $z_\gamma \odot y_\beta \leq z_\gamma \odot x_\alpha$.
3. It is clear by (QS_{4'}) and (QS_{3'}) .
4. By (QS_{3'}) , (QS_{4'}) and (QS_{1'}) , we have $[(y_\beta \odot z_\gamma) \odot (x_\alpha \odot z_\gamma)] \odot (y_\beta \odot x_\alpha) = [(y_\beta \odot z_\gamma) \odot (y_\beta \odot x_\alpha)] \odot (x_\alpha \odot z_\gamma) = (x_\alpha \odot z_\gamma) \odot (x_\alpha \odot z_\gamma) = 0_{\min\{\alpha, \gamma\}}$.

Thus $(x_\alpha \odot z_\gamma) \odot (y_\beta \odot z_\gamma) \leq (y_\beta \odot x_\alpha)$. \triangle

3. Main results

3.1. Fuzzy point QS-sub-algebras of QS-algebras

In this section, we introduce the concept of fuzzy point QS-sub-algebra of $FP(\mu)$ and give some examples and properties of its.

Definition 3.1.1:

A subset S of $FP_q(X)$. $FP(X)$ is called a fuzzy point QS-sub-algebra if $x_\alpha \odot y_\beta \in S$ whenever $x_\alpha, y_\beta \in S$.

Example 3.1.2:

For the QS-algebra $X = \{0, a, b, c\}$ mentioned in example (1.3), it is routine to check that $(FP_{0.3}(X), \odot, 0_{0.3})$ is a fuzzy point of QS-algebra, and that $S = \{0_{0.3}, b_{0.3}\}$ is a fuzzy point QS-sub-algebra of $FP_{0.3}(X)$.

Proposition 3.1.3:

$FP_q(X)$ is a fuzzy point QS-sub-algebra of $FP(X)$, for every $q \in (0, 1]$.

Proof. Straightforward. \square

Theorem 3.1.4:

Let μ be a fuzzy subset of a QS-algebra X . Then the following are equivalent:

- (i) μ is a fuzzy QS-sub-algebra of X .
- (ii) $FP_q(\mu)$ is a fuzzy point QS-sub-algebra of $FP_q(X)$, for every $q \in (0, 1]$.
- (iii) $U(\mu; t)$ is a QS-sub-algebra of X when it is nonempty, for every $t \in (0, 1]$.
- (iv) $FP(\mu)$ is a fuzzy point QS-sub-algebra of $FP(X)$.

Proof.

(i) \Rightarrow (ii) Assume that μ is a fuzzy Q-sub-algebra of X and let $x_q, y_q \in FP_q(\mu)$ where $q \in (0, 1]$. Then $\mu(x) \geq q$ and $\mu(y) \geq q$. It follows that $\mu(x * y) \geq \min\{\mu(x), \mu(y)\} \geq q$ so that $(x_q \odot y_q) = (x * y)_{q \in FP_q(\mu)}$. Hence $FP_q(\mu)$ is a fuzzy point Q-sub-algebra of $FP_q(X)$.

(ii) \Rightarrow (iii) Suppose that $FP_q(\mu)$ is a fuzzy point Q-sub-algebra of $FP_q(X)$, for every $q \in (0, 1]$. Let $x, y \in U(\mu; t)$, where $t \in (0, 1]$. Then $\mu(x) \geq t$ and $\mu(y) \geq t$, and so $x_t, y_t \in FP_t(\mu)$. It follows that $(x * y)_t = (x_t \odot y_t) \in FP_t(\mu)$ so that $\mu(x * y) \geq t$, i.e. $(x * y) \in U(\mu; t)$. Therefore $U(\mu; t)$ is a Q-sub-algebra of X .

(iii) \Rightarrow (iv) Suppose $U(\mu; t) (\neq \emptyset)$ is a Q-sub-algebra of X , for every $t \in (0, 1]$. Let $x_p, y_q \in FP(\mu)$ and let $t = \min\{p, q\}$. Then $\mu(x) \geq p \geq t$ and $\mu(y) \geq q \geq t$ and thus $x, y \in U(\mu; t)$. It follows that $(x * y) \in U(\mu; t)$ because $U(\mu; t)$ is a Q-sub-algebra of X . Thus $\mu(x * y) \geq t$, which implies that $(x_p \odot y_q) = (x * y)_{\min\{p, q\}} = (x_t \odot y_t) = (x * y)_t \in FP(\mu)$. Hence $FP(\mu)$ is a fuzzy point Q-sub-algebra of $FP(X)$.

(iv) \Rightarrow (i) Assume that $FP(\mu)$ is a fuzzy point Q-sub-algebra of $FP(X)$. For any $x, y \in X$, we have $x_t, y_t \in FP(\mu)$ which imply that $(x * y)_t = (x_t \odot y_t) \in FP(\mu)$, that is,

$\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$. Consequently, μ is a fuzzy Q-sub-algebra of X . \square

Proposition 3.1.5:

Let μ be a fuzzy subset of a QS-algebra X . If $FP(\mu)$ is a fuzzy point QS-sub-algebra of $FP(X)$, then $0_p \in FP(\mu)$ for all $p \in Im(\mu)$.

Proof. Let $p \in Im(\mu)$. Then there exists $x \in X$ such that $\mu(x) = p$. Hence $x_p \in FP(\mu)$, and so $0_p = (x * x)_p = x_p \odot x_p \in FP(\mu)$. \square

Corollary 3.1.6:

If μ is a fuzzy QS-sub-algebra of a QS-algebra X , then $0_p \in FP(\mu)$ for all $p \in Im(\mu)$.

Proposition 3.1.7:

If $FP_q(\mu)$ is a fuzzy point QS-sub-algebra of $FP_q(X)$, then $0_q \in FP_q(\mu)$.

Proof.

For every $x_q \in FP_q(\mu)$, we have $0_q = x_q \odot x_q = (x * x)_q \in FP_q(\mu)$. \square

Corollary 3.1.8:

If μ is a fuzzy QS-sub-algebra of a QS-algebra X , then $0_q \in FP_q(\mu)$, for all $q \in (0, 1]$.

Proposition 3.1.9:

Let μ be a fuzzy subset in a QS-algebra X and let $p, q \in (0, 1]$ with $p \geq q$. If $x_p \in FP(\mu)$, then $x_q \in FP(\mu)$.

Proof. Straightforward. \square

Definition 3.1.10 ([2]):

Let X be a QS-algebra. A fuzzy subset μ of X is said to be a fuzzy QS-sub-algebra of X if it satisfies: $\mu(x*y) \geq \min\{\mu(x), \mu(y)\}$, for all $x, y \in X$.

Example 3.1.11:

Let $X = \{0, a, b, c\}$ be a set with a binary operation $(*)$ defined by the following table:

*	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Then $(X; *, 0)$ is a QS-algebra. Define a fuzzy subset $\mu : X \rightarrow [0, 1]$ by:

X	0	a	b	c
μ	0.9	0.8	0.8	0.6

Routine calculations give that μ is a fuzzy QS-sub-algebras of QS-algebra X.

Definition 3.1.12 ([5]):

Let μ be a fuzzy subset of a set X. For $t \in [0,1]$, the set $\mu_t = U(\mu, t) = \{x \in X | \mu(x) \geq t\}$ is called a level set (upper level cut) of μ .

Proposition 3.1.13:

Let μ be a fuzzy subset of QS-algebra X. If $FP(\mu)$ is a fuzzy point QS-sub-algebra of $FP(X)$ if and only if, for every $t \in [0,1]$, μ_t is either empty or a QS-sub-algebra of QS-algebra X.

Proof:

Assume that $FP(\mu)$ is a fuzzy point QS-sub-algebra of $FP(X)$. $\implies 0_\lambda \in FP(\mu)$ for all $\lambda \in Im(\mu)$ and $x \in X$, therefore $\mu(0) \geq \mu(x) \geq t$, for $x \in \mu_t$ and so $0 \in \mu_t$.

Let $x, y \in \mu_t$ where $t \in (0, 1]$. Then $\mu(x) \geq t$ and $\mu(y) \geq t$, and so $x_t, y_t \in FP(\mu)$. It follows that $(x * y)_t = x_t \odot y_t \in FP(\mu)$ so that $\mu(x * y) \geq t$, i.e. $(x * y) \in \mu_t$. Therefore μ_t is a QS-sub-algebra of X.

Conversely, assume that $\mu_t \neq \emptyset$ is a QS-sub-algebra of X for every $t \in (0, 1]$.

Let $x_p, y_q \in FP(\mu)$ and let $t = \min\{p, q\}$. Then $\mu(x) \geq p \geq t$ and $\mu(y) \geq q \geq t$, and thus $x, y \in \mu_t$.

It follows that $(x * y) \in \mu_t$ because μ_t is a QS-sub-algebra of X. Thus $\mu(x * y) \geq t$, which implies that $x_p \odot y_q = (x * y)_{\min\{p, q\}} = (x * y)_t \in FP(\mu)$. Hence $FP(\mu)$ is a fuzzy point QS-sub-algebra of

3.2. Fuzzy point QS-ideal

In this section, we introduce the concept of fuzzy point QS-ideal of QS-algebra X and give some examples and properties of its as [4].

Definition 3.2.1:

Let $(X; *, 0)$ be a QS-algebra and I be a nonempty subset of X. I is called a QS-ideal of X if it satisfies:

- i. $0 \in I$,
- ii. $(z * y) \in I$ and $(x * y) \in I$ imply $(z * x) \in I$, for all $x, y, z \in X$.

Example 3.2.2:

Let $X = \{0,1,2,3\}$ be a set with a binary operation $(*)$ defined by the following table:

*	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	1	0	2
3	3	3	3	0

Then $(X; *, 0)$ is a QS algebra. It is easy to show that $I_1 = \{0,1,2,3\}, I_2 = \{0\}, I_3 = \{0,1,2\}$ are QS-ideals of X .

Definition 3.2.3:

A subset $FP(\mu)$ of $FP(X)$ is called a fuzzy point QS-ideal of $FP(X)$ if

QI₁) $0_\lambda \in FP(\mu)$, for all $\lambda \in Im(\mu)$ and

QI₂) $(z * y)_\beta, (x * y)_\alpha \in FP(\mu)$ implies that $(z * x)_{\min\{\beta, \alpha\}} \in FP(\mu)$, for all $x, y, z \in X$ and $\beta, \alpha \in (0, 1]$.

Definition 3.2.4:

Let X be a QS-algebra. A fuzzy subset μ of X is said to be a fuzzy QS-ideal of X if it satisfies:

for all $x, y, z \in X$,

1. $\mu(0) \geq \mu(x)$.
2. $\mu(z * x) \geq \min\{\mu(z * y), \mu(x * y)\}$.

Proposition 3.2.5:

If μ is a fuzzy QS-ideal of a QS-algebra X , then $FP(\mu)$ is a fuzzy point QS-ideal of $FP(X)$.

Proof. Since $\mu(0) \geq \mu(x)$, for all $x \in X$, we have $\mu(0) \geq \lambda$, for all $\lambda \in Im(\mu)$. Hence $0_\lambda \in FP(\mu)$.

Let $x, y, z \in X$ and $\beta, \alpha \in (0, 1]$ be such that $(z * y)_\beta \in FP(\mu)$ and $(x * y)_\alpha \in FP(\mu)$. Then $\mu(z * y) \geq \beta$ and $\mu(x * y) \geq \alpha$. Since μ is a fuzzy QS-ideal of X , it follows that

$\mu(z * x) \geq \min\{\mu(z * y), \mu(x * y)\} \geq \min\{\beta, \alpha\}$ so that $(z * x)_{\min\{\beta, \alpha\}} \in FP(\mu)$. \square

Proposition 3.2.6:

$FP_q(X)$ is a fuzzy point QS-ideal of $FP(X)$, for every $q \in (0, 1]$.

Proof. Straightforward. \square

Theorem 3.2.7:

Let μ be a fuzzy subset of a QS-algebra X . Then the following are equivalent:

- (i) μ is a fuzzy QS-ideal of X .
- (ii) $FP_q(\mu)$ is a fuzzy point QS-ideal of $FP_q(X)$, for every $q \in (0, 1]$.
- (iii) $U(\mu; t)$ is a QS-ideal of X when it is nonempty, for every $t \in (0, 1]$.
- (iv) $FP(\mu)$ is a fuzzy point QS-ideal of $FP(X)$.

Proof.

(i) \Rightarrow (ii) Assume that μ is a fuzzy QS-ideal of X and let $x, y, z \in X$ and $q \in (0, 1]$. Then $\mu(z * x) \geq q$ and $\mu(y * x) \geq q$. It follows that $\mu(z * x) \geq \min\{\mu(z * y), \mu(x * y)\} \geq q$ so that $(z * x)_q \in FP_q(\mu)$. Hence $FP_q(\mu)$ is a fuzzy point QS-ideal of $FP_q(X)$.

(ii) \Rightarrow (iii) Suppose that $FP_q(\mu)$ is a fuzzy point QS-ideal of $FP_q(X)$ for every $q \in (0, 1]$. Let $x, y, z \in U(\mu; t)$, where $t \in (0, 1]$. Then $\mu(z * y) \geq t$ and $\mu(x * y) \geq t$, and so $(z * y)_t, (x * y)_t \in FP_t(\mu)$. It follows that $(z * x)_t \in FP_t(\mu)$ so that $\mu(z * x) \geq t$, i.e. $(z * x) \in U(\mu; t)$. Therefore $U(\mu; t)$ is a QS-ideal of X .

(iii) \Rightarrow (iv) Suppose $U(\mu; t) (\neq \emptyset)$ is a QS-ideal of X for every $t \in (0, 1]$. Let $x, y, z \in X$ and $\beta, \alpha \in (0, 1]$ and let $t = \min\{\beta, \alpha\}$. Then $\mu(z * y) \geq \beta \geq t$ and $\mu(x * y) \geq \alpha \geq t$, and thus $(z * y), (x * y) \in U(\mu; t)$. It follows that $(z * x) \in U(\mu; t)$ because $U(\mu; t)$ is a QS-ideal of X . Thus $\mu(z * x) \geq t$, which implies that $(z * x)_{\min\{\beta, \alpha\}} = (z * x)_t \in FP(\mu)$. Hence $FP(\mu)$ is a fuzzy point QS-ideal of $FP(X)$.

(iv) \Rightarrow (i) Assume that $FP(\mu)$ is a fuzzy point QS-ideal of $FP(X)$. For any $x, y, z \in X$, we have $x, y, z \in X$ and $\beta, \alpha \in (0, 1]$ which imply that $(z * y)_\beta \in FP(\mu)$ and $(x * y)_\alpha \in FP(\mu)$. It follows that $(z * x)_{\min\{\beta, \alpha\}} \in FP(\mu)$ so that $\mu(z * x) \geq \min\{\beta, \alpha\}$, that is, $\mu(z * x) \geq \min\{\mu(z * y), \mu(x * y)\}$. Consequently, μ is a fuzzy QS-ideal of X . \square

Proposition 3.2.8:

Every fuzzy QS-ideal of QS-algebra X is a fuzzy QS-sub-algebra of X .

Proof. Straightforward. \square

Proposition 3.2.9 :

Let $\{FP(\mu_i) \mid i \in \Lambda\}$ be a family of fuzzy point QS-ideal of QS-algebra X , then $\bigcap_{i \in \Lambda} FP(\mu_i)$ is a fuzzy point QS-ideal of X .

Proof:

Since $\{ FP(\mu_i) \mid i \in \Lambda \}$ is a family of fuzzy point QS-ideal of QS-algebra X , we have

- (1) $0_\lambda \in FP(\mu_i)$, for all $i \in \Lambda$ and $\lambda \in \text{Im}(\mu_i)$, then $0_\lambda \in \bigcap_{i \in \Lambda} FP(\mu_i)$
- (2) For any $x, y, z \in X$, suppose $(z * y)_\beta \in \bigcap_{i \in \Lambda} FP(\mu_i)$ and $(x * y)_\alpha \in \bigcap_{i \in \Lambda} FP(\mu_i)$, then $(z * y)_\beta \in FP(\mu_i)$ and $(x * y)_\alpha \in FP(\mu_i)$, for all $i \in \Lambda$. But $FP(\mu_i)$ is a fuzzy point QS-ideal of QS-algebra X , for all $i \in \Lambda$, then $(z * x)_{\min\{\beta, \alpha\}} \in FP(\mu_i)$, for all $i \in \Lambda$. Therefore, $(z * x)_{\min\{\beta, \alpha\}} \in \bigcap_{i \in \Lambda} FP(\mu_i)$. Hence $\bigcap_{i \in \Lambda} FP(\mu_i)$ is a fuzzy point QS-ideal of QS-algebra X . \square

Proposition 3.2.10:

Let $\{ FP(\mu_i) \mid i \in \Lambda \}$ be a family of fuzzy point QS-ideal of QS-algebra X , then $\bigcup_{i \in \Lambda} FP(\mu_i)$ is a fuzzy point QS-ideal of QS-algebra X , where $FP(\mu_i) \subseteq FP(\mu_{i+1})$, for all $i \in \Lambda$.

Proof:

Since $\{ FP(\mu_i) \mid i \in \Lambda \}$ is a family of fuzzy point QS-ideal of QS-algebra X , we have

- (1) $0_\lambda \in FP(\mu_i)$ for some $i \in \Lambda$ and $\lambda \in \text{Im}(\mu_i)$, then $0_\lambda \in \bigcup_{i \in \Lambda} FP(\mu_i)$.
- (2) For any $x, y, z \in X$, suppose $(z * y)_\beta \in \bigcup_{i \in \Lambda} FP(\mu_i)$, and $(x * y)_\alpha \in \bigcup_{i \in \Lambda} FP(\mu_i) \implies \exists i, j \in \Lambda$ such that $(z * y)_\beta \in FP(\mu_i)$ and $(x * y)_\alpha \in FP(\mu_j)$. By assumption $FP(\mu_i) \subseteq FP(\mu_k)$, and $FP(\mu_j) \subseteq FP(\mu_k)$, $k \in \Lambda$, hence $(z * y)_\beta \in FP(\mu_k)$, $(x * y)_\alpha \in FP(\mu_k)$, but $FP(\mu_k)$ is a fuzzy point QS-ideal of QS-algebra X , then $(z * x)_{\min\{\beta, \alpha\}} \in FP(\mu_k)$. Therefore $(z * x)_{\min\{\beta, \alpha\}} \in \bigcup_{i \in \Lambda} FP(\mu_i)$. Hence $\bigcup_{i \in \Lambda} FP(\mu_i)$ is a fuzzy point QS-ideal of QS-algebra X . \triangle

Note that: The converse of proposition (3.2.10) is not true as seen in the following example.

Example 3.2.11:

Let $X = \{0, 1, 2, 3\}$ be a set with a binary operation $(*)$ defined by the following table:

*	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then $(X; *, 0)$ is a fuzzy point QS-algebra. $I_1 = \{0, 1\}$ and $I_2 = \{0, 2\}$ are fuzzy QS-ideal of QS-algebra X . But $I_1 \cup I_2 = \{0, 1, 2\}$ since $(1 * 0)_\alpha = (1)_\alpha \in I_1 \cup I_2$ and $(2 * 0)_\beta = (2)_\beta \in I_1 \cup I_2$ for all $\alpha, \beta \in (0, 1]$, but $(1 * 2)_{\min\{\beta, \alpha\}} = (3)_{\min\{\beta, \alpha\}} \notin I_1 \cup I_2$.

Theorem 3.2.12:

Let μ be a fuzzy subset of QS-algebra X . If $FP(\mu)$ is a fuzzy point QS-ideal of $FP(X)$ if and only if, for every $t \in [0, 1]$, μ_t is either empty or a QS-ideal of QS-algebra X

Proof:

Assume that $FP(\mu)$ is a fuzzy point QS-ideal of $FP(X)$. $\Rightarrow 0_\lambda \in FP(\mu)$, for all $\lambda \in Im(\mu)$ and $x \in X$, therefore $\mu(0) \geq \mu(x) \geq t$, for $x \in \mu_t$ and so $0 \in \mu_t$.

Let $x, y, z \in X$ be such that $(z * y)_\beta \in \mu_t$ and $(x * y)_\alpha \in \mu_t \Rightarrow \mu(z * y) \geq t$, and $\mu(x * y) \geq t$ which implies that $(z * x)_{\min\{\beta, \alpha\}} \in FP(\mu)$ and $(z * x)_{\min\{\beta, \alpha\}} = (z * x)_t \Rightarrow \mu(z * x) \geq t \Rightarrow (z * x)_{\min\{\beta, \alpha\}} \in \mu_t$. Hence μ_t is a fuzzy QS-ideal of X .

Conversely, suppose $\mu_t \neq \emptyset$ is a QS-ideal of X for every $t \in (0, 1]$. Let $x, y, z \in X$ and $\beta, \alpha \in (0, 1]$ and let $t = \min\{\beta, \alpha\}$, then $\mu(z * y) \geq \beta \geq t$ and $\mu(x * y) \geq \alpha \geq t$, and thus $(z * y), (x * y) \in \mu_t$. It follows that $(z * x) \in \mu_t$, because μ_t is a QS-ideal of X . Thus $\mu(z * x) \geq t$, which implies that $(z * x)_t = (z * x)_{\min\{\beta, \alpha\}} \in FP(\mu)$. Hence $FP(\mu)$ is a fuzzy point QS-ideal of $FP(X)$. \square

Corollary 3.2.13:

Let μ be a fuzzy subset of QS-algebra X . If μ is a fuzzy QS-ideal, then for every $t \in Im(\mu)$, μ_t is a QS-ideal of X when $\mu_t \neq \emptyset$.

Proposition 3.2.14:

Every fuzzy point QS-ideal of QS-algebra X is a fuzzy point QS-sub-algebra of X .

Proof:

Since μ is fuzzy QS-ideal of a QS-algebra X , then by theorem (3.2.12), for every $t \in [0, 1]$, μ_t is either empty or a QS-ideal of X . By proposition (3.2.8), for every $t \in [0, 1]$, μ_t is either empty or a QS-sub-algebra of X . Hence μ is a fuzzy QS-sub-algebra of QS-algebra X , by theorem (3.1.13). \square

Note that: The converse of proposition (3.2.14) is not true as seen it the following example.

Example 3.2.15:

Let $X = \{0,1,2,3\}$ be a set with a binary operation $(*)$ define a by the following table:

*	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	1	0	1
3	3	1	1	0

Then $(X;*, 0)$ is a QS-algebra. Define a fuzzy subset $\mu: X \rightarrow [0,1]$ by:

x	0	1	2	3
μ	0.8	0.7	0.6	0.5

Then μ is fuzzy point QS-sub-algebra of X , but not fuzzy point QS-ideal of X . since

$I=\{0,3\} \in \text{FP}(\mu)$. Let $(3 * 0)_{\min\{0.5,0.8\}} = (3_{0.5}) \in \text{FP}(\mu)$ and

$((0 * 2)_{\min\{0.8,0.6\}} = (0_{0.6}) \in \text{FP}(\mu)$ but $(3 * 2)_{\min\{0.5,0.6\}} = (1_{0.5}) \notin \text{FP}(\mu)$

Theorem 3.2.16:

Let A be a QS-ideal of QS-algebra X . Then for any fixed number (t) in the open interval $(0,1)$, there exists a fuzzy QS-ideal μ of X such that

Define $\mu: X \rightarrow [0,1]$ by $\mu(x) = \begin{cases} t & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$.

Proof:

Where (t) is a fixed number in $(0, 1)$. Clearly, $\mu(0) \geq \mu(x)$, for all $x \in X$. Let $x, y, z \in X$. If

$(x * y)_{\alpha} \notin A$ then $(x * y)_{\alpha} = 0$ and so $\mu(z * x)_{\min\{\beta,\alpha\}} \geq 0 \Rightarrow (z * x)_{\min\{\beta,\alpha\}} \in \text{FP}(\mu)$.

If $(z * x)_{\min\{\beta,\alpha\}} \in A$ then clearly $t = \mu(z * x) \geq \min\{\beta, \alpha\} \Rightarrow [(z * y)_{\beta}, (x * y)_{\alpha}] \in \text{FP}(\mu)$

If $(z * x)_{\min\{\beta,\alpha\}} \notin A, (x * y)_{\alpha} \in A$, then $(z * y)_{\beta} \notin A$, since A is a fuzzy point QS-ideal.

Thus $(z * x)_{\min\{\beta,\alpha\}} = 0 \Rightarrow \min\{\beta, \alpha\} = 0 \Rightarrow [(z * y)_{\beta}, (x * y)_{\alpha}] = 0$.

Hence μ is a fuzzy QS-ideal of X . It is clear that $\mu_t = A$. \square

3.3. Homomorphism fuzzy point QS-ideal of QS-algebras

In this section, we introduce the definition of homomorphism fuzzy point QS-ideals of QS-algebra and we study some properties of it.

Definition 3.3.1([2],[3]):

Let $(X; *, 0)$ and $(Y; *', 0')$ be QS-algebras. A mapping $f : (X; *, 0) \rightarrow (Y; *', 0')$ is said to be a **homomorphism** if $f(x * y) = f(x) *' f(y)$, for all $x, y \in X$.

Definition 3.3.2([2],[3]) :

For any homomorphism $f : (X; *, 0) \rightarrow (Y; *', 0')$, the set $\{x \in X \mid f(x) = 0'\}$ is called the kernel of f , denoted by $Ker(f)$.

Proposition 3.3.3:

Let $(X; *, 0)$ and $(Y; *', 0')$ be QS-algebras and $f : (X; *, 0) \rightarrow (Y; *', 0')$ be a homomorphism, then $Ker(f)$ is fuzzy point QS-ideal of QS-algebra X .

Proof:

(1) Since $f(0_\lambda)$, then $0_\lambda \in Ker(f)$ and $\lambda \in \text{Im}(\mu)$

(2) For any $x, y, z \in X$, let $(z * y)_\beta, (x * y)_\alpha \in Ker(f)$ and $f(z * y)_\beta = f(x * y)_\alpha = 0'$

$$f(z * y)_\beta *' f(x * y)_\alpha = 0' *' 0' = 0'$$

$$f[(z * y)_\beta * (x * y)_\alpha] = f(z * x)_{\min\{\beta, \alpha\}} = 0'$$

That is $(z * x)_{\min\{\beta, \alpha\}} \in Ker(f)$, then $Ker(f)$ is a fuzzy point QS-ideal of X . \square

Proposition 3.3.4:

Let $(X; *, 0)$ and $(Y; *', 0')$ be QS-algebras, $f : (X; *, 0) \rightarrow (Y; *', 0')$ be a homomorphism, onto and A be a fuzzy point QS-ideal of X , then $f(A)$ is fuzzy point QS-ideal of QS-algebra Y .

Proof:

(1) Since A is a fuzzy point QS-ideal of $X \Rightarrow 0_\lambda \in A \Rightarrow f(0_\lambda) \in f(A)$ for all $\lambda \in \text{Im}(\mu)$.

(2) Let $x, y, z \in X$ and $\alpha, \beta \in (0, 1]$, $f(z * y)_\beta \in f(A), f(x * y)_\alpha \in f(A)$

$\Rightarrow (z * y)_\beta \in A, (x * y)_\alpha \in A$, Since A is a fuzzy point QS-ideal of $X \Rightarrow (z * x)_{\min\{\beta, \alpha\}} \in A \Rightarrow$

$f(z * x)_{\min\{\beta, \alpha\}} \in f(A)$. Hence $f(A)$ is fuzzy point QS-ideal of QS-algebra Y . \square

Proposition 3.3.5:

Let $(X; *, 0)$ and $(Y; *', 0')$ be QS-algebras, $f : (X; *, 0) \rightarrow (Y; *', 0')$ be a homomorphism and B be a fuzzy point QS-ideal of Y , then $f^{-1}(B)$ is fuzzy point QS-ideal of QS-algebra X .

Proof:

(1) Since B is a fuzzy point QS-ideal of $Y \Rightarrow (0'_\lambda) \in B \Rightarrow f^{-1}(0'_\lambda) \in f^{-1}(B)$ since $0_\lambda = f^{-1}(0'_\lambda)$ then $0_\lambda \in f^{-1}(B)$ for all $\lambda \in \text{Im}(\mu)$.

(2) Let $x, y, z \in X$ and $\alpha, \beta \in (0, 1]$, $(z * y)_\beta \in f^{-1}(B)$, $(x * y)_\alpha \in f^{-1}(B) \Rightarrow f(z * y)_\beta \in B, f(x * y)_\alpha \in B$, Since B is a fuzzy point QS-ideal of $Y \Rightarrow (f(z * y)_\beta *' f(x * y)_\alpha) \in B \Rightarrow f((z * y)_\beta * (x * y)_\alpha) \in B \Rightarrow (z * x)_{\min\{\beta, \alpha\}} \in f^{-1}(B)$.

Hence $f^{-1}(B)$ is fuzzy point QS-ideal of QS-algebra X . \square

Definition 3.3.6([1]):

fuzzy subset μ of X has sup property if for any subset T of X , there exist $t_0 \in T$ such that

$$\mu(t_0) = \sup_{t \in T} \mu(t).$$

Definition 3.3.7 ([1]):

Let $f : (X; *, 0) \rightarrow (Y; *', 0')$ be a mapping nonempty sets X and Y respectively. If μ is a fuzzy subset of X , then

$$f(\mu)(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \mu(x) & f^{-1}(y) = \{x \in X / f(x) = y\} \neq \emptyset \\ 0 & \text{other wise} \end{cases}$$

is said to be the image of μ under f .

Definition 3.3.8 ([1]):

If β is a fuzzy subset of Y , then the fuzzy subset $\mu = \beta \circ f$ in X (i.e the fuzzy subset defined by: $\mu(x) = \beta(f(x))$, for all $x \in X$) is called the pre-image of β under f .

Theorem 3.3.9:

A homomorphic pre-image of a fuzzy point QS-ideal is also a fuzzy point QS-ideal.

Proof:

Let $f : (X; *, 0) \rightarrow (Y; *', 0')$ be a homomorphism of QS-algebras, β be a fuzzy point QS-ideal of Y and μ the pre-image of β under $f \Rightarrow \beta(f(x)) = \mu(x)$, for all $x \in X$. Since $f(x) \in Y$ and β is a fuzzy point QS-ideal of Y , it follows that $\beta(0'_\lambda) \geq \beta(f(x)) = \mu(x)$, for every $x \in X$, where $(0'_\lambda)$ is the zero element of Y . We get

(1) Since $\beta(0'_\lambda) \geq \beta(f(0_\lambda)) = \mu(0_\lambda)$ and $\beta(0'_\lambda) \geq \mu(x) \Rightarrow \mu(0_\lambda) = \mu(x)$, for $x \in X$ and $\lambda \in \text{Im}(\mu)$.

(2) Let $x, y, z \in X$ and $(\beta, \alpha) \in (0, 1]$, then we get

$$\begin{aligned} \mu(z * x)_{\min\{\beta, \alpha\}} &= \beta(f(z * x)_{\min\{\beta, \alpha\}}) = \beta\left(f\left((z)_{\min\{\beta, \alpha\}} \dot{*} f\left((x)_{\min\{\beta, \alpha\}}\right)\right)\right) \\ &\Rightarrow \left\{\beta\left(f\left((z)_{\min\{\beta, \alpha\}}\right) \dot{*} f\left((y)_{\min\{\beta, \alpha\}}\right)\right), \beta\left(f\left(x\right)_{\min\{\beta, \alpha\}} \dot{*} f\left(y\right)_{\min\{\beta, \alpha\}}\right)\right\} \\ &\Rightarrow \left\{\beta\left(f\left(z * y\right)_{\min\{\beta, \alpha\}}\right), \beta\left(f\left(x * y\right)_{\min\{\beta, \alpha\}}\right)\right\} \\ &\Rightarrow (z * y)_{\min\{\beta, \alpha\}}, (x * y)_{\min\{\beta, \alpha\}}. \end{aligned}$$

Hence μ is a fuzzy point QS-ideal of X . \square

Theorem 3.3.10:

Let $f : (X; *, 0) \rightarrow (Y; *, 0')$ be a homomorphism of QS-algebras. For every fuzzy point QS-ideal μ of X with sup property, $f(\mu)$ is a fuzzy point QS-ideal of Y .

Proof:

By definition $\beta(y') = f(\mu)(y') := \sup\{\mu(x) : x = f^{-1}(y')\}$, for all $y' \in Y$ ($\sup \phi \neq 0$).

We have to prove that β is a fuzzy point QS-ideal of Y .

Let $f : (X; *, 0) \rightarrow (Y; *, 0')$ be an homomorphism of QS-algebras, μ is a fuzzy point QS-ideal of X with sup property and β the image of μ under f .

(1) Since μ is a fuzzy point QS-ideal of X , we have $\mu(0_\lambda) \geq \mu(x)$ for all $x \in X$, and $\lambda \in \text{Im}(\mu)$. Note that $0 \in f^{-1}(0')$, where (0) and $(0')$ are the zero elements of X and Y respectively. Thus $\beta(0') \geq \sup_{t \in f^{-1}(0')} \mu(t) = \mu(0) \geq \mu(x')$, for all $x' \in Y$, which implies that

$$\beta(0') \geq \sup_{t \in f^{-1}(x')} \mu(t) = \beta(x'), \text{ for any } x' \in Y$$

(2) For any $x', y', z' \in Y$, let $x_0 \in f^{-1}(x')$, $y_0 \in f^{-1}(y')$ and $z_0 \in f^{-1}(z')$ we have

$$\begin{aligned} \mu(z_0 * x_0)_{\min\{\beta, \alpha\}} &= \beta\left(f\left(z_0 * x_0\right)_{\min\{\beta, \alpha\}}\right) = \beta\left(\left(z \dot{*} x\right)_{\min\{\beta, \alpha\}}\right) \\ &= \sup_{[(z_0 * x_0)_{\min\{\beta, \alpha\}} \in f^{-1}(\dot{z} \dot{*} \dot{x})_{\min\{\beta, \alpha\}}]} \mu\left(\dot{z} \dot{*} \dot{x}\right)_{\min\{\beta, \alpha\}} \end{aligned}$$

And $\mu(x * y)_\alpha = \beta(f(x * y)_\alpha) = \beta(\dot{x} \dot{*} \dot{y})_\alpha = \sup_{[(x(x*y))_\alpha \in f^{-1}(\dot{y} \dot{*} \dot{x})_\alpha]} \mu(x * y)_\alpha$, then

$$\begin{aligned} \beta(z' \dot{*} x') &= \sup_{[t \in f^{-1}(z' \dot{*} x')]} \mu(t) \\ &= \mu(z_0 * x_0) \\ &\geq \min\{\mu(z_0 * y_0), \mu(x_0 * y_0)\} \end{aligned}$$

$$\begin{aligned}
&= \min \left\{ \sup_{t \in f^{-1}(\hat{z} \hat{*} \hat{y})} \mu(t), \sup_{t \in f^{-1}(\hat{x} \hat{*} \hat{y})} \mu(t) \right\} \\
&= \min \{ \beta(\hat{z} \hat{*} \hat{y}), \beta(\hat{x} \hat{*} \hat{y}) \}. \text{ Hence } \beta \text{ is a fuzzy QS-ideal of } Y. \quad \square
\end{aligned}$$

Conflict of Interests

The authors declare that there is no conflict of interests.

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