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QS-ALGEBRAS DEFINED BY FUZZY POINTS

AREEJ TAWFEEQ HAMEED[1,*] AND AHLAAM ABED ABD ULKHUDHUR[2]
Department of Mathematics, College of Education for Girls, University of Kufa, Najaf, Iraq

Abstract: QS-algebras was derived from KUS-algebras. In this paper, we discussed some of the relationships and characteristics. We introduce a new idea of fuzzy QS-ideal of fuzzy point on QS-algebras and give some properties and theorems of it. We introduce the concept of the normal fuzzy QS-ideal of fuzzy point on QS-algebra and study some of the properties related thereto.

Keywords: QS-ideal; fuzzy QS-ideal; fuzzy QS-ideal of fuzzy point.

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1. Introduction

In 1999, S.S. Ahn and Kim H.S. introduced the class of QS-algebras and give some properties of QS-algebras [6] and described connections between such sub-algebra and congruences, see [4]. In 2006, A.B. Saeid considered the fuzzification of QS-sub-algebra to QS-algebras [1].

Now, we introduced a definition of the QS-ideal fuzzy QS-ideal and fuzzy QS-ideal of fuzzy point. We study some of the related properties, homomorphism fuzzy QS-ideal on fuzzy point, normal fuzzy QS-ideal on fuzzy point, homomorphism normal fuzzy QS-ideal of fuzzy point on QS-algebras.

2. Preliminaries

In this subsection, we study the definition of QS-algebra and QS-sub-algebra of QS-algebras and we give some properties of it.

Definition 2.1([4], [6]) :

*Corresponding author

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Let (X; $*$, 0) be a set with a binary operation ($*$) and a constant (0). Then (X;$*$, 0) is called a QS-algebra if it satisfies the following axioms: for all x, y, z$\in$X,

1. $x*x=0,$
2. $x*0=x,$
3. $(x*y)*z=(x*z)*y,$
4. $(x*z)*(x*y)=y*z.$

For brevity we also call X a QS-algebra, we can define a binary relation ($\leq$) by putting x$\leq$ y if and only if, $y*x=0$.

<u>Proposition 2.2 ([4],[6]):</u>

Let (X;$*$,0) be a QS-algebra, then the following hold: for any x, y, z $\in$ X,

(1) $if\ x*y=z, then\ x*z=y.$

(2) $x*y=0\ implies\ x=y.$

(3) $0*(x*y)=y*x.$

(4) $x*(0*y)=y*(0*x).$

<u>Example 2.3 ([4]):</u>

Let X ={0, a, b , c} in which ($*$) be defined by the following table:

$*$	0	a	b	c
0	0	c	b	a
a	a	0	c	b
b	b	a	0	c
c	c	b	a	0

Then (X;$*$,0) is a QS-algebra.

<u>Definition 2.4 ([1]):</u>

Let (X; $*$, 0) be a QS-algebra X and S be a nonempty subset of X. Then S is called a **QS-sub-algebra of X** if, $x*y\in$S, for any x, y $\in$ S.

<u>Definition 2.5:</u>

If ζ is the family of all fuzzy subsets on X , $x_\alpha\in\zeta$ is called a fuzzy point if and only if there exists $\alpha\in(0,1]$ such that for all y $\in$ X ,

$$x_\alpha(y)=\begin{cases}\alpha & \text{if } x=y\\ 0 & \text{otherwise}\end{cases}$$

Definition 2.6:

Let $(X;*,0)$ be a QS-algebra , the set of all fuzzy points on X denote by

$$FP(X) = \{x_\alpha | \; x \in X, \alpha \in (0, 1]\}.$$

Define a binary operation ($\odot$) on FP(X) by:

$x_\alpha \odot y_\beta = (x * y)_{\min\{\alpha,\beta\}}$, for all x_α, $y_\beta \in$ FP(X), then

($QS_{1'}$): $x_\alpha \odot x_\alpha = 0_\alpha$,

($QS_{2'}$): $x_\alpha \odot 0_\alpha = x_\alpha$,

($QS_{3'}$): $(x_\alpha \odot y_\beta) \odot z_\gamma = (x_\alpha \odot z_\gamma) \odot y_\beta$,

($QS_{4'}$): $(x_\alpha \odot z_\gamma) \odot (x_\alpha \odot y_\beta) = (y_\beta \odot z_\gamma)$.

In X we can define a binary relation ($\leq$) by : $x_\alpha \leq y_\beta$ if and only if $y_\beta \odot x_\alpha = 0_{\min\{\alpha,\beta\}}$.

Remark 2.7:

If $(X;*,0)$ is a QS-algebra and $FP_q(X)$ denote the set of all fuzzy points of X, then $(FP_q(X),\odot, 0_q)$ is a QS-algebra, which is called a fuzzy point QS-algebra, where the value q, $(0< q\leq 1)$.

Definition 2.8 :

For a fuzzy subset μ of a QS-algebra X, we define the set FP(μ) of all fuzzy points of X covered by μ to be the set $FP(\mu) = \{x_\alpha \in FP(X) \mid \mu(x) \geq \alpha, 0 < \alpha \leq 1\}$, and

$FP_q(\mu) = \{x_q \in FP_q(X) \mid \mu(x) \geq q\}$, for all $q \in (0, 1]$, $x \in X$.

Now, we give some properties and theorems of QS-algebras.

Theorem 2.9:

If $(X;*,0)$ is a QS-algebra, then the following hold: for all $x_\alpha, y_\beta, z_\gamma \in FP(X)$,

a) $x_\alpha \odot 0_\beta = x_{\min\{\alpha,\beta\}}$.

b) $(x_\alpha \odot y_\beta) \odot x_\alpha = 0_\alpha \odot y_\beta$,

c) $(x_\alpha \odot y_\beta) \odot 0_\alpha = y_\beta \odot x_\alpha$,

d) $x_\alpha \odot 0_\alpha = 0_\alpha$ implies that $x_\alpha = 0_\alpha$,

e) $x_\alpha = (x_\alpha \odot 0_\alpha) \odot 0_\alpha$,

f) $0_{\min\{\alpha,\beta\}} \odot (x_\alpha \odot y_\beta) = (0_\alpha \odot x_\alpha) \odot (0_\beta \odot y_\beta)$,

g) $z_\gamma \odot x_\alpha = z_\gamma \odot y_\beta$ implies that $0_\gamma \odot x_\alpha = 0_\gamma \odot y_\beta$.

Proof:

a) It is clear by ($QS_{2'}$).

b) $(x_\alpha \odot y_\beta) \odot x_\alpha = (x_\alpha \odot x_\alpha) \odot y_\beta = 0_\alpha \odot y_\beta$.

c) $(x_\alpha \odot y_\beta) \odot 0_\alpha = (x_\alpha \odot y_\beta) \odot (x_\alpha \odot x_\alpha) = y_\beta \odot x_\alpha$, by $(QS_{2'})$.

(d), (e) and (f) are clears by $(QS_{2'})$.

g) $0_\gamma \odot x_\alpha = (z_\gamma \odot z_\gamma) \odot x_\alpha = (z_\gamma \odot x_\alpha) \odot z_\gamma = (z_\gamma \odot y_\beta) \odot z_\gamma = (z_\gamma \odot z_\gamma) \odot y_\beta = 0_\alpha \odot y_\beta$. ⌂

Proposition 2.9:

Let $(X; *, 0)$ be a QS-algebra . Then the following holds: for any $x_\alpha, y_\beta, z_\gamma \in FP(X)$,

1. $x_\alpha \odot y_\beta \le z_\gamma$ imply $z_\gamma \odot y_\beta \le x_\alpha$,
2. $x_\alpha \le y_\beta$ implies that $z_\gamma \odot y_\beta \le z_\gamma \odot x_\alpha$,
3. $y_\beta \odot [(y_\beta \odot z_\gamma) \odot z_\gamma] = 0_{\min\{\beta,\gamma\}}$,
4. $(x_\alpha \odot z_\gamma) \odot (y_\beta \odot z_\gamma) \le (y_\beta \odot x_\alpha)$.

Proof:

1. It follows from $(QS_{4'})$.
2. By $(QS_{1'})$, we obtain $[(z_\gamma \odot x_\alpha) \odot (z_\gamma \odot y_\beta)] = (y_\beta \odot x_\alpha)$, but $x_\alpha \le y_\beta$ implies $y_\beta \odot x_\alpha = 0_{\min\{\alpha,\beta\}}$, then we get $(z_\gamma \odot x_\alpha) \odot (z_\gamma \odot y_\beta) = 0_{\min\{\alpha,\beta\}}$.
i.e., $z_\gamma \odot y_\beta \le z_\gamma \odot x_\alpha$.
3. It is clear by $(QS_{4'})$ and $(QS_{3'})$.
4. By $(QS_{3'})$, $(QS_{4'})$ and $(QS_{1'})$, we have $[(y_\beta \odot z_\gamma) \odot (x_\alpha \odot z_\gamma)] \odot (y_\beta \odot x_\alpha)$
$= [(y_\beta \odot z_\gamma) \odot (y_\beta \odot x_\alpha)] \odot (x_\alpha \odot z_\gamma) = (x_\alpha \odot z_\gamma) \odot (x_\alpha \odot z_\gamma) = 0_{\min\{\alpha,\gamma\}}$.

Thus $(x_\alpha \odot z_\gamma) \odot (y_\beta \odot z_\gamma) \le (y_\beta \odot x_\alpha)$. ⌂

3. Main results

3.1. Fuzzy point QS-sub-algebras of QS-algebras

In this section, we introduce the concept of fuzzy point QS-sub-algebra of FP(μ) and give some examples and properties of its.

Definition 3.1.1:

A subset S of $FP_q(X)$. FP(X) is called a fuzzy point QS-sub-algebra if $x_\alpha \odot y_\beta \in S$ whenever x_α , $y_\beta \in S$.

Example 3.1.2:

For the QS-algebra X = {0, a, b, c} mentioned in example (1.3), it is routine to check that $(FP_{0.3}(X), \odot, 0_{0.3})$ is a fuzzy point of QS-algebra, and that $S = \{0_{0.3}, b_{0.3}\}$ is a fuzzy point QS-sub-algebra of $FP_{0.3}(X)$.

Proposition 3.1.3:

$FP_q(X)$ is a fuzzy point QS-sub-algebra of FP(X), for every $q \in (0, 1]$.

Proof. Straightforward. □

Theorem 3.1.4:

Let μ be a fuzzy subset of a QS-algebra X. Then the following are equivalent:

(i) μ is a fuzzy QS-sub-algebra of X.

(ii) $FP_q(\mu)$ is a fuzzy point QS-sub-algebra of $FP_q(X)$, for every $q \in (0, 1]$.

(iii) $U(\mu; t)$ is a QS-sub-algebra of X when it is nonempty, for every $t \in (0, 1]$.

(iv) $FP(\mu)$ is a fuzzy point QS-sub-algebra of FP(X).

Proof.

(i) ⇒ (ii) Assume that μ is a fuzzy Q-sub-algebra of X and let x_q , $y_q \in FP_q(\mu)$ where $q \in (0, 1]$. Then $\mu(x) \geq q$ and $\mu(y) \geq q$. It follows that $\mu(x * y) \geq \min\{\mu(x), \mu(y)\} \geq q$ so that $(x_q \odot y_q) = (x * y)_q \in FP_q(\mu)$. Hence $FP_q(\mu)$ is a fuzzy point Q-sub-algebra of $FP_q(X)$.

(ii) ⇒(iii) Suppose that $FP_q(\mu)$ is a fuzzy point Q-sub-algebra of $FP_q(X)$, for every $q \in (0,1]$. Let $x, y \in U(\mu; t)$, where $t \in (0, 1]$. Then $\mu(x) \geq t$ and $\mu(y) \geq t$, and so $x_t, y_t \in FP_t(\mu)$. It follows that $(x * y)_t = (x_t \odot y_t) \in FP_t(\mu)$ so that $\mu(x * y) \geq t$, i.e. $(x * y) \in U(\mu; t)$. Therefore $U(\mu; t)$ is a Q-sub-algebra of X.

(iii) ⇒(iv) Suppose $U(\mu; t)$ ($\neq \emptyset$) is a Q-sub-algebra of X, for every $t \in (0, 1]$. Let x_p , $y_q \in FP(\mu)$ and let $t = \min\{p, q\}$. Then $\mu(x) \geq p \geq t$ and $\mu(y) \geq q \geq t$ and thus $x, y \in U(\mu; t)$. It follows that $(x * y) \in U(\mu; t)$ because $U(\mu; t)$ is a Q-sub-algebra of X. Thus $\mu(x*y) \geq t$, which implies that $(x_p \odot y_q) = (x * y)_{\min\{p,q\}} = (x_t \odot y_t) = (x * y)_t \in FP(\mu)$. Hence $FP(\mu)$ is a fuzzy point Q-sub-algebra of FP(X).

(iv)⇒(i) Assume that $FP(\mu)$ is a fuzzy point Q-sub-algebra of FP(X). For any $x, y \in X$, we have $x_t, y_t \in FP(\mu)$ which imply that $(x * y)_t = (x_t \odot y_t) \in FP(\mu)$, that is, $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$. Consequently, μ is a fuzzy Q-sub-algebra of X. □

Proposition 3.1.5:

Let μ be a fuzzy subset of a QS-algebra X. If FP(μ) is a fuzzy point QS-sub-algebra of FP(X), then $0_p \in$ FP(μ) for all p ∈ Im(μ).

Proof. Let p ∈ Im(μ). Then there exists x ∈ X such that μ(x) = p. Hence $x_p \in$ FP(μ), and so $0_p = (x * x)_p = x_p \odot x_p \in$ FP(μ). □

Corollary 3.1.6:

If μ is a fuzzy QS-sub-algebra of a QS-algebra X, then $0_p \in$ FP(μ) for all p ∈ Im(μ).

Proposition 3.1.7:

If FP_q(μ) is a fuzzy point QS-sub-algebra of FP_q(X), then $0_q \in FP_q$(μ).

Proof.

For every $x_q \in FP_q$(μ), we have $0_q = x_q \odot x_q = (x * x)_q \in$ FPq(μ). □

Corollary 3.1.8:

If μ is a fuzzy QS-sub-algebra of a QS-algebra X, then $0_q \in FP_q$(μ), for all q ∈ (0, 1].

Proposition 3.1.9:

Let μ be a fuzzy subset in a QS-algebra X and let p, q ∈ (0, 1] with p ≥ q. If $x_p \in$ FP(μ), then $x_q \in$ FP(μ).

Proof. Straightforward. □

Definition 3.1.10 ([2]):

Let X be a QS-algebra. A fuzzy subset μ of X is said to be a fuzzy QS-sub-algebra of X if it satisfies: $\mu(x*y) \geq \min\{\mu(x), \mu(y)\}$.,for all x , y ∈ X.

Example 3.1.11:

Let X = {0, a, b, c} be a set with a binary operation(∗) defined by the following table:

∗	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Then (X; ∗, 0) is a QS-algebra .Define a fuzzy subset μ : X→ [0,1] by:

X	0	a	b	c
μ	0.9	0.8	0.8	0.6

Routine calculations give that μ is a fuzzy QS-sub-algebras of QS-algebra X.

Definition 3.1.12 ([5]):

Let μ be a fuzzy subset of a set X. For t∈[0,1], the set $\mu_t = U(\mu,t) = \{x \in X | \mu(x) \geq t\}$ is called a level set (upper level cut) of μ.

Proposition 3.1.13:

Let μ be a fuzzy subset of QS-algebra X. If FP(μ) is a fuzzy point QS-sub-algebra of FP(X) if and only if, for every t ∈[0,1], μ_t is either empty or a QS-sub-algebra of QS-algebra X.

Proof:

Assume that FP (μ) is a fuzzy point QS-sub-algebra of FP (X). ⟹ $0_\lambda \in FP(\mu)$ for all $\lambda \in Im(\mu)$ and x∈X, therefore $\mu(0) \geq \mu(x) \geq t$, for $x \in \mu_t$ and so $0 \in \mu_t$.

Let x, y ∈ μ_t where t ∈ (0, 1]. Then μ(x) ≥ t and μ(y) ≥ t, and so x_t, y_t ∈ FP(μ). It follows that $(x * y)_t = x_t \odot y_t \in FP(\mu)$ so that μ(x ∗ y) ≥ t, i.e. (x ∗ y) ∈ μ_t. Therefore μ_t is a QS-sub-algebra of X.

Conversely, assume that $\mu_t \neq \emptyset$ is a QS-sub-algebra of X for every t ∈ (0, 1].

Let x_p , $y_q \in$ FP(μ) and let t = min{p, q}. Then μ (x) ≥ p ≥ t and μ (y) ≥ q ≥ t, and thus x, y ∈ μ_t. It follows that (x ∗ y) ∈ μ_t because μ_t is a QS-sub-algebra of X. Thus μ(x∗y) ≥ t, which implies that $x_p \odot y_q = (x*y)_{\min\{p, q\}} = (x * y)_t \in FP(\mu)$. Hence FP(μ) is a fuzzy point QS-sub-algebra of

3.2. Fuzzy point QS-ideal

In this section, we introduce the concept of fuzzy point QS-ideal of QS-algebra X and give some examples and properties of its as [4].

Definition 3.2.1:

Let (X;∗, 0) be a QS-algebra and I be a nonempty subset of X. I is called a QS-ideal of X if it satisfies:

i. 0∈I,

ii. $(z * y) \in I$ and $(x * y) \in I$ imply $(z * x) \in I$, for all x, y, z ∈X .

Example 3.2.2:

Let X = {0,1,2,3} be a set with a binary operation ($*$) defined by the following table:

$*$	**0**	**1**	**2**	**3**
0	**0**	**0**	**0**	**0**
1	**1**	**0**	**0**	**0**
2	**2**	**1**	**0**	**2**
3	**3**	**3**	**3**	**0**

Then (X; $*$,0) is a QS algebra. It is easy to show that I_1={0,1,2,3},I_2={0},I_3={0,1,2} are QS-ideals of X.

Definition 3.2.3:

A subset FP(μ) of FP(X) is called a fuzzy point QS-ideal of FP(X) if

QI_1) $0_\lambda \in$FP(μ), for all $\lambda \in$Im(μ) and

QI_2) $(z*y)_\beta$, $(x*y)_\alpha \in$ FP(μ) implies that $(z*x)_{\min\{\beta,\alpha\}} \in$FP(μ), for all x, y , z ∈ X and β, α∈(0, 1] .

Definition 3.2.4:

Let X be a QS-algebra. A fuzzy subset μ of X is said to be a fuzzy QS-ideal of X if it satisfies: for all x, y, z∈ X,

1. $\mu(0) \geq \mu(x)$.
2. $\mu(z*x) \geq min\{\mu(z*y), \mu(x*y)\}$.

Proposition 3.2.5:

If μ is a fuzzy QS-ideal of a QS-algebra X, then FP(μ) is a fuzzy point QS-ideal of FP(X).

Proof. Since μ(0) ≥ μ(x), for all x ∈ X, we have μ(0) ≥ λ, for all λ ∈ Im(μ). Hence $0_\lambda \in$ FP(μ). Let x, y , z ∈ X and β, α∈(0, 1] be such that $(z*y)_\beta$, ∈ FP(μ)and $(x*y)_\alpha \in$ FP(μ). Then μ(z $*$ y) ≥ β and μ(x$*$y) ≥ α . Since μ is a fuzzy QS-ideal of X, it follows that μ(z$*$x) ≥ min{μ(z$*$ y), μ(x$*$y)} ≥ min{β, α } so that $(z*x)_{\min\{\beta,\alpha\}} \in$ FP(μ). □

Proposition 3.2.6:

$FP_q(X)$ is a fuzzy point QS-ideal of FP(X), for every q ∈ (0, 1].

Proof. Straightforward. □

Theorem 3.2.7:

Let μ be a fuzzy subset of a QS-algebra X. Then the following are equivalent:

(i) μ is a fuzzy QS-ideal of X.

(ii) $FP_q(\mu)$ is a fuzzy point QS-ideal of $FP_q(X)$, for every $q \in (0, 1]$.

(iii) $U(\mu; t)$ is a QS-ideal of X when it is nonempty, for every $t \in (0, 1]$.

(iv) $FP(\mu)$ is a fuzzy point QS-ideal of $FP(X)$.

Proof.

(i) $\Rightarrow$ (ii) Assume that μ is a fuzzy QS-ideal of X and let $x, y, z \in X$ and $q \in (0, 1]$. Then $\mu(z*x) \geq q$ and $\mu(y*x) \geq q$. It follows that $\mu(z*x) \geq \min\{\mu(z*y), \mu(x*y)\} \geq q$ so that $(z*x)_q \in FP_q(\mu)$. Hence $FP_q(\mu)$ is a fuzzy point QS-ideal of $FP_q(X)$.

(ii) $\Rightarrow$(iii) Suppose that $FP_q(\mu)$ is a fuzzy point QS-ideal of $FP_q(X)$ for every $q \in (0, 1]$. Let $x, y, z \in U(\mu; t)$, where $t \in (0, 1]$. Then $\mu(z*y) \geq t$ and $\mu(x*y) \geq t$, and so $(z*y)_t$, $(x*y)_t \in FP_t(\mu)$. It follows that $(z*x)_t \in FP_t(\mu)$ so that $\mu(z*x) \geq t$, i.e. $(z*x) \in U(\mu; t)$. Therefore $U(\mu; t)$ is a QS-ideal of X.

(iii) $\Rightarrow$(iv) Suppose $U(\mu; t)$ $(\neq\emptyset)$ is a QS-ideal of X for every $t \in (0, 1]$. Let $x, y, z \in X$ and $\beta, \alpha \in (0, 1]$ and let $t = \min\{\beta, \alpha\}$. Then $\mu(z*y) \geq \beta \geq t$ and $\mu(x*y) \geq \alpha \geq t$, and thus $(z*y)$, $(x*y) \in U(\mu; t)$. It follows that $(z*x) \in U(\mu; t)$ because $U(\mu; t)$ is a QS-ideal of X. Thus $\mu(z*x) \geq t$, which implies that $(z*x)_{\min\{\beta, \alpha\}} = (z*x)_t \in FP(\mu)$. Hence $FP(\mu)$ is a fuzzy point QS-ideal of $FP(X)$.

(iv)$\Rightarrow$(i) Assume that $FP(\mu)$ is a fuzzy point QS-ideal of $FP(X)$. For any $x, y, z \in X$, we have $x, y, z \in X$ and $\beta, \alpha \in (0, 1]$ which imply that $(z*y)_\beta \in FP(\mu)$ and $(x \odot y)_\alpha \in FP(\mu)$, It follows that $(z*x)_{\min\{\beta, \alpha\}} \in FP(\mu)$ so that $\mu(z*x) \geq \min\{\beta, \alpha\}$, that is, $\mu(z*x) \geq \min\{\mu(z*y), \mu(x*y)\}$. Consequently, μ is a fuzzy QS-ideal of X. □

Proposition 3.2.8:

Every fuzzy QS-ideal of QS-algebra X is a fuzzy QS-sub-algebra of X.

Proof. Straightforward. □

Proposition 3.2.9 :

Let $\{ FP(\mu_i) \mid i \in \Lambda\}$ be a family of fuzzy point QS-ideal of QS-algebra X , then $\cap_{i\in\Lambda} FP(\mu_i)$ is a fuzzy point QS-ideal of X.

Proof:

Since { FP(μ_i) | i∈Λ} is a family of fuzzy point QS-ideal of QS-algebra X, we have

(1) $0_\lambda \in$ FP(μi), for all i ∈Λ and λ ∈ Im(μi), then $0_\lambda \in \cap_{i\in\Lambda}$FP(μi)

(2) For any x, y , z∈X, suppose $(z*y)_\beta, \in \cap_{i\in\Lambda}$FP(μi) and $(x*y)_\alpha \in \cap_{i\in\Lambda}$FP(μi) , then $(z*y)_\beta \in$FP(μ_i) and $(x*y)_\alpha \in$ FP(μ_i) , for all i ∈Λ. But FP(μ_i) is a fuzzy point QS-ideal of QS-algebra X, for all i ∈Λ, then $(z*x)_{\min\{\beta,\alpha\}} \in$FP($\mu_i$), for all i ∈Λ. Therefore, $(z*x)_{\min\{\beta,\alpha\}} \in \cap_{i\in\Lambda}$FP(μi). Hence $\cap_{i\in\Lambda}$FP(μi) is a fuzzy point QS-ideal of QS-algebra X. □

Proposition 3.2.10:

Let { FP(μ_i)| i∈Λ} be a family of fuzzy point QS-ideal of QS-algebra X, then $\cup_{i\in\Lambda}$FP(μi) is a fuzzy point QS-ideal of QS-algebra X, where FP(μ_i)⊆FP(μ_{i+1}), for all i ∈Λ .

Proof:

Since { FP(μ_i)| i∈Λ} is a family of fuzzy point QS-ideal of QS-algebra X, we have

(1) $0_\lambda \in$FP(μ_i) for some i ∈Λ and λ ∈Im(μi), then$0_\lambda \in \cup_{i\in\Lambda}$FP(μi).

(2) For any x, y ,z ∈X, suppose$(z*y)_\beta \in \cup_{i\in\Lambda}$FP(μi), and $(x*y)_\alpha \in \cup_{i\in\Lambda}$ FP(μ_i) ⟹ ∃i, j∈Λ suchthat$(z*y)_\beta \in$FP(μ_i) and$(x*y)_\alpha \in$FP(μ_j) . By assumption FP(μ_i)⊆FP(μ_k), and FP(μ_j)⊆ FP(μ_k) , k∈Λ, hence$(z*y)_\beta \in$FP(μ_k) , $(x*y)_\alpha \in$FP(μ_k) , but FP(μ_k) is a fuzzy point QS-ideal of QS-algebra X, then$(z*x)_{\min\{\beta,\alpha\}} \in$FP($\mu_k$).Therefore $(z*x)_{\min\{\beta,\alpha\}} \in \cup_{i\in\Lambda}$FP(μi). Hence $\cup_{i\in\Lambda}$FP(μi) is a fuzzy point QS-ideal of QS-algebra X.⌂

Note that: The converse of proposition (3.2.10) is not true as seen it the following example.

Example 3.2.11:

Let X = {0,1,2,3} be a set with a binary operation (∗) defined by the following table:

∗	**0**	**1**	**2**	**3**
0	**0**	**1**	**2**	**3**
1	**1**	**0**	**3**	**2**
2	**2**	**3**	**0**	**1**
3	**3**	**2**	**1**	**0**

Then (X; $*$, 0) is a fuzzy point QS-algebra. I_1={0,1} and I_2={0,2} are fuzzy QS-ideal of QS-algebra X. But $I_1 \cup I_2$={0,1,2} since $(1*0)_\alpha=(1)_\alpha \in I_1 \cup I_2$ and $(2*0)_\beta=(2)_\beta \in I_1 \cup I_2$ for all $\alpha, \beta \in (0,1]$, but $(1*2)_{\min\{\beta,\alpha\}}=(3)_{\min\{\beta,\alpha\}} \notin I_1 \cup I_2$.

Theorem 3.2.12:

Let μ be a fuzzy subset of QS-algebra X. If FP(μ) is a fuzzy point QS-ideal of FP(X) if and only if, for every t $\in$[0,1] , μ_t is either empty or a QS-ideal of QS-algebra X

Proof:

Assume that FP (μ) is a fuzzy point QS-ideal of FP (X). $\Rightarrow 0_\lambda \in FP(\mu)$, for all $\lambda \in Im(\mu)$ and $x \in X$, therefore $\mu(0) \geq \mu(x) \geq t$, for $x \in \mu_t$ and so $0 \in \mu_t$.

Let x, y, z $\in$ X be such that $(z*y)_\beta \in \mu_t$ and $(x*y)_\alpha \in \mu_t \Rightarrow \mu(z*y) \geq t$, and $\mu(x*y) \geq t$ which implies that $(z*x)_{\min\{\beta,\alpha\}} \in FP(\mu)$ and $(z*x)_{\min\{\beta,\alpha\}} = (z*x)_t \Rightarrow \mu(z*x) \geq t \Rightarrow (z*x)_{\min\{\beta,\alpha\}} \in \mu_t$. Hence μ_t is a fuzzy QS-ideal of X.

Conversely, suppose $\mu_t \neq \emptyset$ is a QS-ideal of X for every t $\in$ (0, 1]. Let x, y , z$\in$ X and β, α$\in$(0, 1] and let t = min{β, α}, then $\mu(z*y) \geq \beta \geq t$ and $\mu(x*y) \geq \alpha \geq t$, and thus $(z*y), (x*y) \in \mu_t$. It follows that $(z*x) \in \mu_t$, because μ_t is a QS-ideal of X. Thus $\mu(z*x) \geq t$, which implies that $(z*x)_t = (z*x)_{\min\{\beta,\alpha\}} \in FP(\mu)$. Hence FP(μ) is a fuzzy point QS-ideal of FP(X). □

Corollary 3.2.13:

Let μ be a fuzzy subset of QS-algebra X. If μ is a fuzzy QS-ideal, then for every $t \in Im(\mu)$, μ_t is a QS-ideal of X when $\mu_t \neq \phi$.

Proposition 3.2.14:

Every fuzzy point QS-ideal of QS-algebra X is a fuzzy point QS-sub-algebra of X.

Proof:

Since μ is fuzzy QS-ideal of a QS-algebra X, then by theorem (3.2.12), for every t $\in$[0,1], μ_t is either empty or a QS-ideal of X. By proposition (3.2.8), for every t $\in$[0,1], μ_t is either empty or a QS-sub-algebra of X. Hence μ is a fuzzy QS-sub-algebra of QS-algebra X, by theorem (3.1.13) . □

Note that: The converse of proposition (3.2.14) is not true as seen it the following example.

Example 3.2.15:

Let X = {0,1,2,3}be a set with a binary operation ($*$)define a by the following table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	1	0	1
3	3	1	1	0

Then (X;$*$,0) is a QS-algebra. Define a fuzzy subset $\mu: X \to [0,1]$ by:

x	0	1	2	3
μ	0.8	0.7	0.6	0.5

Then μ is fuzzy point QS-sub-algebra of X, but not fuzzy point QS-ideal of X. since I={0,3}$\in$ FP(μ). Iet $(3 * 0)_{\min\{0.5,0.8\}} = (3_{0,5}) \in$FP(μ) and

$((0 * 2)_{\min\{0.8,0.6\}} = (0_{0.6}) \in$ FP(μ) but $(3 * 2)_{\min\{0.5,0.6\}} = (1_{0.5}) \notin$FP(μ)

Theorem 3.2.16:

Let A be a QS-ideal of QS-algebra X. Then for any fixed number (t) in the open interval (0,1), there exists a fuzzy QS-ideal μ of X such that

Define μ: X →[0,1] by $\mu(x) = \begin{cases} t & if\ x \in \mathrm{A} \\ 0 & otherwise \end{cases}$.

Proof:

Where (t) is a fixed number in (0, 1). Clearly, $\mu(0) \geq \mu(x)$, for all x $\in$X. Let x, y, z$\in$X. If $(x * y)_\alpha \notin A$ then $(x * y)_\alpha = 0$ and so $\mu(z * x)_{\min\{\beta,\alpha\}} \geq 0 \Longrightarrow (z * x)_{\min\{\beta,\alpha\}} \in$ FP(μ) .

If $(z * x)_{\min\{\beta,\alpha\}} \epsilon A$ then clearly $t = \mu\,(z * x) \geq \min\{\beta,\alpha\} \Longrightarrow [(z * y)_\beta, (x * y)_\alpha] \in$ FP(μ)

If $(z * x)_{\min\{\beta,\alpha\}} \notin A$, $(x * y)_\alpha \in A$, $then (z * y)_\beta \notin A$, since A is a fuzzy point QS-ideal.

Thus $(z * x)_{\min\{\beta,\alpha\}} = 0 \Longrightarrow \min\{\beta,\alpha\} = 0 \Longrightarrow \left[(z * y)_\beta, (x * y)_\alpha\right] = 0$.

Hence μ is a fuzzy QS-ideal of X. It is clear that $\mu_t = \mathrm{A}$.□

3.3. Homomorphism fuzzy point QS-ideal of QS-algebras

In this section, we introduce the definition of homomorphism fuzzy point QS-ideals of QS-algebra and we study some properties of it.

Definition 3.3.1([2],[3]):

Let (X; $*$, 0) and (Y; $*'$, $0'$) be QS-algebras. A mapping $f:(X;*,0)\to(Y;*',0')$ is said to be a **homomorphism** if, $f(x*y)=f(x)*'f(y)$, for all x, y$\in$ X.

Definition 3.3.2([2],[3]) :

For any homomorphism $f:(X;*,0)\to(Y;*',0')$, the set $\{x\in X \mid f(x)=0'\}$ is called the kernel of *f*, denoted by $Ker(f)$.

Proposition 3.3.3:

Let (X; $*$,0) and (Y; $*'$, $0'$) be QS-algebras and $f:(X;*,0)\to(Y;*',0')$ be a homomorphism , then $Ker(f)$ is fuzzy point QS-ideal of QS-algebra X .

Proof:

(1) Since $f(0_\lambda)$, then $0_\lambda \in$ Ker(f) and $\lambda\in$lm(μ)

(2) For any x, y, z $\in$ X, let $(z*y)_\beta, (x*y)_\alpha \in Ker(f)$ and $f(z*y)_\beta = f(x*y)_\alpha = 0'$

$f(z*y)_\beta *' f(x*y)_\alpha = 0' *' 0' = 0'$

$f\left[(z*y)_\beta * (x*y)_\alpha\right] = f(z*x)_{\min\{\beta,\alpha\}} = 0'$

That is $(z*x)_{\min\{\beta,\alpha\}} \in Ker(f)$,then $Ker(f)$ is a fuzzy point QS-ideal of X. □

Proposition 3.3.4:

Let (X; $*$,0) and (Y; $*'$, $0'$) be QS-algebras , $f:(X;*,0)\to(Y;*',0')$ be a homomorphism , onto and A be a fuzzy point QS-ideal of X, then f(A) is fuzzy point QS-ideal of QS-algebra Y.

Proof:

(1) Since A is a fuzzy point QS-ideal of X $\Longrightarrow 0_\lambda \in$ A $\Longrightarrow f(0_\lambda)\in f(A)$ for all $\lambda\in$lm(μ) .

(2) Let x ,y ,z $\in$X and α, β $\in(0,1]$, $f(z*y)_\beta\in f(A), f(x*y)_\alpha\in f(A)$

$\Longrightarrow (z*y)_\beta\in$A, $(x*y)_\alpha\in$A, Since A is a fuzzy point QS-ideal of X $\Longrightarrow (z*x)_{\min\{\beta,\alpha\}}\in A\Longrightarrow$ $f(z*x)_{\min\{\beta,\alpha\}}\in f(A)$. Hence f(A) is fuzzy point QS-ideal of QS-algebra Y. □

Proposition 3.3.5:

Let (X; $*$,0) and (Y ; $*'$, $0'$) be QS-algebras , $f:(X;*,0)\to(Y;*',0')$ be a homomorphism and B be a fuzzy point QS-ideal of Y, then $f^{-1}(B)$ is fuzzy point QS-ideal of QS-algebra X.

Proof:

(1) Since B is a fuzzy point QS-ideal of Y $\Rightarrow(0'_{\lambda})\in B\Rightarrow f^{-1}(0'_{\lambda})\in f^{-1}(B)$ since $0_{\lambda}= f^{-1}(0'_{\lambda})$ then $0_{\lambda}\in f^{-1}(B)$ for all $\lambda\in \mathrm{lm}(\mu)$.

(2) Let x ,y ,z $\in$X and $\alpha,\beta\in(0,1]$,, $(z*y)_{\beta}\in f^{-1}(B)$,$(x*y)_{\alpha}\in f^{-1}(B)$ $\Rightarrow f(z*y)_{\beta}\in B, f(x*y)_{\alpha}\in B$, Since B is a fuzzy point QS-ideal of Y $\Rightarrow\big(f(z*y)_{\beta}*' f(x*y)_{\alpha}\big)\Rightarrow f\big((z*y)_{\beta}*(x*y)_{\alpha}\big)$ $\Rightarrow f((z*x)_{\min\{\beta,\alpha\}})\in B$ $\Rightarrow(z*x)_{\min\{\beta,\alpha\}}\in f^{-1}(B)$.

Hence $f^{-1}(B)$ is fuzzy point QS-ideal of QS-algebra X. □

Definition 3.3.6([1]):

fuzzy subset μ of X has sup property if for any subset T of X, there exist $t_0\in$T such that $\mu(t_0)=\sup_{t\in T}\mu(t)$.

Definition 3.3.7 ([1]):

Let $f:(X;*,0)\to(Y;*',0')$ be a mapping nonempty sets X and Y respectively . If μ is a fuzzy subset of X, then

$$f(\mu)(y)=\begin{cases}\sup_{x\in f^{-1}(y)}\mu(x) & f^{-1}(y)=\{x\in X/f(x)=y\}\neq\phi\\ 0 & other\ wise\end{cases}$$

is said to be the image of μ under *f*.

Definition 3.3.8 ([1]):

If β is a fuzzy subset of Y, then the fuzzy subset μ = β o *f* in X (i.e the fuzzy subset defined by: μ (x) = β (*f* (x)), for all x ∈ X) is called the pre-image of β under *f* .

Theorem 3.3.9:

A homomorphic pre-image of a fuzzy point QS-ideal is also a fuzzy point QS-ideal .

Proof:

Let $f:(X;*,0)\to(Y;*',0')$ be a homomorphism of QS-algebras, β be a fuzzy point QS-ideal of Y and μ the pre-image of β under f $\Rightarrow$ $\beta(f(x))=\mu(x)$, for all x ∈ X . Since $f(x)\in Y$ and β is a fuzzy point QS-ideal of Y, it follows that $\beta(0'_{\lambda})\geq\beta(f(x))=\mu(x)$, for every x $\in$X, where $(0'_{\lambda})$ is the zero element of Y. We get

(1) Since $\beta(0'_{\lambda}) \geq \beta(f(0_{\lambda})) = \mu(0_{\lambda})$ and $\beta(0'_{\lambda}) \geq \mu(x) \Rightarrow \mu(0_{\lambda}) = \mu(x)$, for x∈X and λ∈lm(μ)..

(2) Let x, y, z ∈ X and (β, α)∈(0,1] , then we get

$$\mu(z * x)_{\min\{\beta,\alpha\}} = \beta(f(z * x)_{\min\{\beta,\alpha\}}) = \beta\left(f\left((z)_{\min\{\beta,\alpha\}}\right) \dot{*} f\left((x)_{\min\{\beta,\alpha\}}\right)\right)$$

$$\Rightarrow \{\beta(f((z)_{\min\{\beta,\alpha\}}) \dot{*} f((y)_{\min\{\beta,\alpha\}})), \beta(f(x)_{\min\{\beta,\alpha\}} \dot{*} f(y)_{\min\{\beta,\alpha\}})\}$$

$$\Rightarrow \{\beta(f(z * y)_{\min\{\beta,\alpha\}}), \beta(f(x * y)_{\min\{\beta,\alpha\}})\}$$

$$\Rightarrow (z * y)_{\min\{\beta,\alpha\}} , (x * y)_{\min\{\beta,\alpha\}} .$$

Hence μ is a fuzzy point QS-ideal of X. □

Theorem 3.3.10:

Let $f:(X;*,0) \to (Y;*',0')$ be a homomorphism of QS-algebras. For every fuzzy point QS-ideal μ of X with sup property, $f(\mu)$ is a fuzzy point QS-ideal of Y.

Proof:

By definition $\beta(y') = f(\mu)(y') := \sup\{\mu(x) : x = f^{-1}(y')\}$, for all y'∈ Y (sup ϕ ≠0).

We have to prove that β is a fuzzy point QS-ideal of Y .

Let $f:(X;*,0) \to (Y;*',0')$ be an homomorphism of QS-algebras, μ is a fuzzy point QS-ideal of X with sup property and β the image of μ under f .

(1) Since μ is a fuzzy point QS-ideal of X , we have $\mu(0_{\lambda}) \geq \mu(x)$ for all x ∈ X , and $\lambda \in \mathrm{lm}(\mu)$. Note that $0 \in f^{-1}(0')$, where (0) and (0') are the zero elements of X and Y respectively. Thus $\gamma(0') \geq \sup_{t \in f^{-1}(0')} \mu(t) = \mu(0) \geq \mu(x')$, for all x' ∈ Y, which implies that

$\gamma(0') \geq \sup_{t \in f^{-1}(x')} \mu(t) = \gamma(x')$, for any $x' \in Y$

(2) For any x', y', z' ∈ Y , let $x_0 \in f^{-1}(x'),\ y_0 \in f^{-1}(y')$ *and* $z_0 \in f^{-1}(z')$ we have

$$\mu(z_0 * x_0)_{\min\{\beta,\alpha\}} = \beta(f(z_0 * x_0)_{\min\{\beta,\alpha\}}) = \beta((\dot{z} \dot{*} \dot{x})_{\min\{\beta,\alpha\}})$$

$$= sup_{[(z_0 * x_0)_{\min\{\beta,\alpha\}} \in f^{-1}(\dot{z} \dot{*} \dot{x})_{\min\{\beta,\alpha\}}]}\, \mu(\dot{z} \dot{*} \dot{x})_{\min\{\beta,\alpha\}}$$

And $\mu(x * y)_{\alpha} = \beta(f(x * y)_{\alpha}) = \beta(\dot{x} \dot{*} \dot{y})_{\alpha} = sup_{[(x(x*y)_{\alpha}y)_{\alpha} \in f^{-1}(\dot{y} \dot{*} \dot{x})_{\alpha}]}\, \mu(x * y)_{\alpha}$, then

$$\beta(z' \dot{*} x') = sup_{[t \in f^{-1}(z' *' x')]}\, \mu(t)$$

$$= \mu(z_0 * x_0)$$

$$\geq \min\{\mu(z_0 * y_0), \mu(x_0 * y_0)\}$$

$$= \min\left\{sup_{[t \in f^{-1}(\grave{z} \grave{*} \grave{y})]}\ \mu(t), sup_{[t \in f^{-1}(\grave{x} \grave{*} \grave{y})]}\ \mu(t)\right\}$$

$= \min\{\beta(\grave{z} \grave{*} \grave{y}), \beta(\grave{x} \grave{*} \grave{y})\}$. Hence β is a fuzzy QS-ideal of Y. □

Conflict of Interests

The authors declare that there is no conflict of interests.

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