Available online at http://scik.org
Algebra Letters, 2017, 2017:2
ISSN: 2051-5502

APPLICATIONS OF CLOSED MODELS DEFINED BY COUNTING TO GRAPH THEORY AND TOPOLOGY

TSEMO ARISTIDE

College Boreal, Yonge Street, M5E 1E5, Toronto Canada

Abstract. In this paper, we define the notion of closed models defined by counting, and we compute their homotopy categories. We apply this construction to various categories of graphs. We show that there does not exist a closed model in the category of undirected graphs which characterizes the Ihara Zeta function in the sense that, a morphism $f : X \to Y$ is a weak equivalence for this model if and only if it induces a bijection between the sets of non degenerated cycles of X and Y. Finally, we apply our construction to Galoisian complexes and dessins d'enfant.

Keywords: closed model; graph theory; category theory.

2010 AMS Subject Classification: 05C38, 05C15.

1. Introduction

The theory of closed models defined by Quillen in the context of category theory provides foundations of homotopy theory and is applied to various mathematics areas. To teach this important idea, it is necessary to have in hands examples which are easy to understand, are not trivial, and can be presented after a short introduction. A good framework to find such

*Corresponding author

E-mail address: tsemo58@yahoo.ca

Received December 30, 2016

closed models is graph theory. In our papers [1] and [2] written in collaboration with Bisson, we have defined such closed models in the category of directed graphs; one has the virtue to characterize the Zeta function and the other is adapted to symbolic dynamic. It is an interesting question to ask whether such a similar closed model characterizing the Ihara Zeta function exists in the category of undirected graphs. In this paper, we answer negatively to this question. The closed models defined in [1] and [2] are particular examples of closed models defined by counting a family of objects $(X_i)_{i\in I}$ in a topos C; that is, the class of weak equivalences of these models is the subclass W of the class of morphisms of C, such that for every $f : X \to Y \in W$, and every $i \in I$, the map $c_f : Hom_C(X_i,X) \to Hom_C(X_i,Y)$ defined by $c_f(h) = f \circ h$ is bijective. We start this paper by presenting properties of closed models defined by counting, in particular, we determine their homotopy categories. We define closed models by counting in subcategories of the category of undirected graphs, which characterize the Ihara Zeta function of objects of a large subclass of their class of objects. A particular interesting example amongst these closed models is defined in the category BC_n, whose objects are n-colored graphs. This category is equivalent to the category of G_n-sets where G_n is the group generated by $a_0,...,a_{n-1}$ such that $a_i^2 = 1, i = 0,...,n-1$. This category is studied by many authors, we can quote for example Ladegaillerie [15] who has established an equivalence between BC_{n+1} and the category of Galoisian n-complexes, we deduce the existence of closed models in these categories, and in particular in the category of Galoisian 2-complexes which is related to dessins d'enfants.

1. Plan.

1. Introduction.
2. Closed models.
3. Closed models defined by counting.
4. Closed models defined by counting in the categories of directed graphs and undirected graphs.
5. Closed models defined by counting in the category of undirected colored graphs.
6. Closed models defined by counting and topology.

2. Closed models

In this section, we are going to present the basic properties of closed models defined by counting; we start by the following definitions:

Definitions 2.1. A class W of morphisms of a category C satisfies the 2-3 property if and only if for every morphisms $f : X \to Y$ and $g : Y \to Z$ of C, if two morphisms of the triple $(f, g, g \circ f)$ is an element of W, then the third is also an element of W.

We say that the morphism $g : Y \to T$ has the right lifting property with respect to $h : X \to Z$, and that h has the left lifting property with respect to g if and only if for every commutative diagram:

$$X \xrightarrow{p} Y h \downarrow g \downarrow Z \xrightarrow{q} T$$

there exists a morphism $l : Z \to Y$ such that $l \circ h = p$ and $g \circ l = q$.

Let L and R be two classes of morphisms of C, we say that (L,R) is a weak factorization system if and only if:

- Every morphism $f \in C$ can be written $f = r \circ l$ where $l \in L$, $r \in R$;

- L is the class of morphisms which has the left lifting property in respect of every morphism of R;

- R is the class of morphisms which has the right lifting property in respect of every morphism of L.

Let C be a category complete and cocomplete; we say that C is endowed with a closed model if and only if there exist three classes of morphisms (Fib, Cof, W) such that:

- W satisfies the 2-3 property,

- Let $Fib' = W \cap Fib$, (Cof, Fib') is a weak factorization system

- Let $Cof' = W \cap Cof$, (Cof', Fib) is a weak factorization system.

We start by the following general example:

Proposition 2.2. *Let C be a category complete and cocomplete, let W be a class of morphisms of C which satisfies the* 2-3 *property. Suppose that there exists a class of morphisms Cof of C such that (Cof, W) is a weak factorization system. Then, there exists a closed model on C whose* *class of weak equivalences is W, its class of cofibrations is Cof, its class of weak cofibrations Cof' is the class $Iso(C)$ of isomorphisms of C, its class of fibrations Fib is the class $Hom(C)$ of all morphisms of C, and its class of weak fibrations Fib' is W.*

Proof. We have: $(Cof',Fib) = (Iso(C),Hom(C))$ and $(Cof,Fib') = (Cof,W)$ are weak factorization systems. We also have $Fib \cap W = Hom(C) \cap W = W = Fib'$, and $Cof \cap W = Iso(C)$ Since (Cof,W) is a factorization system. We deduce that $(Hom(C),Cof,W)$ defines a closed model on C.

3. Closed model defined by counting

We are going to apply the previous result to define closed models to count objects in categories. Let C be a category complete and cocomplete whose initial object is denoted by ϕ. For every objects X and Y of C, we denote by $X+Y$ the sum of X and Y. Let $(X_i)_{i\in I}$ be a family of objects of C and $l_i : \phi \to X_i$ the canonical morphism. There exist morphisms $j^i_1 : X_i \to X_i + X_i$ and $j^i_2 : X_i \to X_i + X_i$ such that for every morphisms $f : X_i \to Z$ and $g : X_i \to Z$, there exists a unique morphism $m(f,g) : X_i + X_i \to Z$ such that $m(f,g)\circ j^i_1 = f$ and $m(f,g)\circ j^i_2 = g$. We set $m_i = m(Id_{X_i}, Id_{X_i})$. Such a morphism is often called a folding morphism. We suppose that the class of morphisms $l_i, m_i \in I$ admits the small element argument (see [11] 12.4.13). We denote by W_I the class of morphisms which are right orthogonal to every morphisms l_i and $m_i, i \in I$.

Proposition 3.1. *A morphism $f : X \to Y$ of C is an element of W_I if and only if for every $i \in I$, the map $c^i_f : Hom_C(X_i,X) \to Hom_C(X_i,Y)$ defined by $c^i_f(h) = f \circ h$ is bijective. We deduce that W_I satisfies the* 2-3-*property and there exists a closed model on C whose class of weak equivalences is W_I.*

Proof. Let $f : X \to Y$ be a morphism of C, suppose that for every $i \in I$, f is orthogonal to l_i and m_i. Let $h,h' \in Hom_C(X_i,X)$ such that $f\circ h = f\circ h'$. The following diagram commutes:

$$X_i + X_i \xrightarrow{h+h'} X m_i \downarrow f \downarrow X_i \xrightarrow{f\circ h} Y$$

Since f is right orthogonal to m_i, we deduce the existence of a morphism $l : X_i \to X$ such that $l \circ m_i = h + h'$. We have $l \circ m_i \circ j_1^i = l \circ m_i \circ j_2^i = l$. We deduce that $h = (h+h') \circ j_1^i = l \circ m_i \circ j_1^i = l \circ m_i \circ j_2^i = (h+h') \circ j_2^i = h'$.

Let $h : X_i \to Y$ be any morphism, the following diagram commutes:

$$\phi \longrightarrow X \downarrow\downarrow f X_i \xrightarrow{h} Y$$

thus it has a filler $p : X_i \to X$ such that $f \circ p = h$. This implies that c^i_f is bijective.

We show now that W_I satisfies the 2-3 property. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of C, $c^i_{g \circ f} = c^i_g \circ c^i_f$. Since c^i_f, c^i_g and $c^i_{g \circ f}$ are morphisms of sets, we deduce that if two morphisms of the triple $(c^i_f, c^i_g, c^i_{g \circ f})$ are bijective, so is the third.

Let $cell(I)$ be the class of morphisms of C which are retracts of transfinite compositions of pushouts of $l_i, m_i, i \in I$, the propositions 12.4.14 and 12.4.20 of [11] imply that $(cell(I), W_I)$ is a factorization system. We deduce from the proposition 2.2 the existence of a closed model on C, whose class of weak equivalences is W_I.

Remarks. Recall that a closed model (Fib, Cof, W) on a category is cofibrantly generated (see [11] 13.2.2.) if there exist sets of morphisms $(f_i)_{i \in I}$ and $(g_j)_{j \in J}$ both which allow the small element argument, such that the class of fibrations is the class of morphisms which are right orthogonal to every morphism of the family $(g_j)_{j \in J}$, and the class of weak fibrations is the class of morphisms which are right orthogonal to every morphism of the family $(f_i)_{i \in I}$. Thus, the closed model defined by counting in the previous proposition is cofibrantly generated. In the sequel, we will only consider such closed models defined by counting.

Let K be a subset of I, we denote by X_K the sum of objects of the family $(X_k)_{k \in K}$. The morphism $\phi \to X_K$ is a cofibration, since it is transfinite composition of pushouts of elements of $(l_k)_{k \in K}$.

The homotopy category of a closed model defined by counting.

One of the main purpose of the theory of closed models is to find a proper framework to localize classes of morphisms. In this perspective, we are going to compute the homotopy category of a closed model defined by counting. We start by remarking the fact that since every morphism is a fibration in a closed model defined by counting, every object is fibrant. Let us determine now the cofibrant replacement of an object:

Proposition 3.2. *Let U be a Grothendieck universe, and C be an U-category endowed with a closed model defined by counting a set of objects $(X_i)_{i\in I}$, where I is an U-set. Let Z be an object of C, then Z has a cofibrant replacement QZ, isomorphic to a transfinite composition of a subset of $l_i, m_i, i \in I$.*

Proof. Let $I_Z = \{f \in Hom_C(X_i, Z), i \in I\}$. It is an U-set. We denote by X_{I_Z} the pushout of elements of I_Z, and by $i_f : X_i \to X_{I_Z}$ the morphisms satisfying the universal property of the pushout. There exists a morphism $d_Z : X_{I_Z} \to Z$ such that for every $f \in I_Z$, $d_Z \circ i_f = f$. For every $i \in I$, $c_{d_Z} : Hom_C(X_i, X_{I_Z}) \to Hom_C(X_i, Z)$ defined by $c_{d_Z}(g) = d_Z \circ g$ is surjective since if $h \in Hom_C(X_i, Z)$, we have $d_Z \circ i_h = h$. Consider $L(Z)$ the U-set whose elements are morphisms $p_V : X_{I_Z} \to V$ such that p_V is a transfinite composition of pushouts of a subset of $m_i, i \in I$, and there exists a morphism $f_V : V \to Z$ such that $f_V \circ p_V = d_Z$. There exist a relation of order define on $L(Z)$ such that $p_V \geq p_W$ if and only if there exists a morphism $h_{V,W} : W \to V$ such that $p_V = h_{V,W} \circ p_W$. Let $(p_{V_j})_{j\in J}$ be an ordered family of $L(Z)$; $lim_{j\in J} p_{V_j}$ is a lower bound of $(p_{V_j})_{j\in J}$. The Zorn's lemma implies that $L(Z)$ has a maximal $c_Z : QZ \to Z$ which is a weak equivalence.

Remark. In the rest of this section, we are going to suppose that the category C is U-small, where U is a Grothendieck universe. Let V and W be objects of C, every morphism $f : V \to W$ induces a morphism $d(f) : X_{I_V} \to X_{I_W}$; $d(f)$ also induces a morphism $Qf : QV \to QW$ such that f is a weak equivalence if and only if Qf is an isomorphism. (See also [6] Lemma 5.1).

Definitions 3.3. A path object of Z is an object Z^I such that there exists a weak equivalence $i_Z : Z \to Z^I$, a morphism $p_Z : Z^I \to Z \times Z$ such that $(id_Z, id_Z) = p_Z \circ i_Z$.

Two morphisms $f, g : Y \to Z$ are right homotopic if and only if there exists a path object Z^I, and a morphism $H : Y \to Z^I$ such that $(f, g) = p_Z \circ H$.

Proposition 3.4. *Let C be a category endowed with a closed model defined by counting, and Y a cofibrant object of C. Two morphisms $f, g : Y \to Z$ of C are right homotopic if and only if they are equal.*

Proof. If Y is cofibrant and f,g are right homotopic, we can suppose that there exists a path object Z^I such that $i_Z : Z \to Z^I$ is an acyclic cofibration and $(f,g) = p_Z \circ H$ (See [6] Lemma 4.15). Since i_Z is an acyclic cofibration of a closed model defined by counting, it is an isomorphism. We can thus suppose that $Z^I = Z$ and $p_Z = (id_Z, id_Z)$. This implies that $f = p_1 \circ p_Z \circ H = p_2 \circ p_Z \circ H = g$ where $p_1, p_2 : Z \times Z \to Z$ are the projections on the first and second factors.

Remark. Let Y and Z be two objects of C, we denote by $C(Y)$ (resp. $C(Z)$) the cofibrant replacement of Y (resp. Z). The objects $C(Y)$ and $C(Z)$ are also fibrant. The homotopy category of C is the category which have the same class of objects than C, and the set $Hom_{Hot}(Y,Z)$ of morphisms of the homotopy category between the objects Y and Z is the set $\pi(X,Y)$ whose elements are right homotopy classes of morphisms between $C(Y)$ and $C(Z)$. (See [6] 4.22 and definition 5.6). We deduce:

Proposition 3.5. *Let C be a category endowed with a closed model defined by counting, for every objects X and Y of C, we have* $Hom_{Hot}(Y,Z) = \pi(C(Y),C(Z)) = Hom_C(C(Y),C(Z))$.

4. Closed models defined by counting in the categories of directed graphs and undirected graphs

We will define now various closed models in different categories of graphs. We start by the category of directed graphs.

Let C_D be the category which has two objects that we denote by 0 and 1. We suppose that $Hom_{C_D}(0,1)$ contains two elements s,t, $Hom_{C_D}(0,0)$, $Hom_{C_D}(1,1)$ contain one element and $Hom_{C_D}(1,0)$ empty.

Definition 4.1. A directed graph is a presheaf defined on C_D. Let X be such a presheaf; X is defined by two sets $X(0)$ and $X(1)$, and two maps $X(s), X(t) : X(1) \to X(0)$.

The set $X(0)$ is called the space of nodes of X and the set $X(1)$ the space of directed arcs of X.

Definition 4.2. A morphism $f : X \to Y$ between the graphs X and Y is a natural transformation between the presheaves X and Y; thus f is defined by a morphism $f_0 : X(0) \to Y(0)$, $f_1 : X(1) \to Y(1)$ such that $f_0 \circ X(s) = Y(s) \circ f_1$, $f_0 \circ X(t) = Y(t) \circ f_1$.

We denote by Gph the category of directed graphs. The category of directed graphs is complete and cocomplete since it is a Grothendieck topos.

Examples of directed graphs are: the directed dot graph D. It is the graph such that $D(0)$ is a singleton and $D(1)$ is empty.

The directed arc graph A is the graph defined by: $A(0) = \{u, v\}$, $A(1) = \{a\}$ and $A(s)(a) = u$, $A(t)(a) = v$.

Let p be a strictly positive integer. We denote by c_p the graph whose set of nodes is Z/pZ, let $[n]$ be the class of the integer n in Z/p, there exists a unique arc a_n such that $c_p(s)(a_n) = [n]$ and $c_p(t)(a_n) = [n+1]$.

We can define on Gph the closed model obtained by counting the elements of the set $Cycl = \{c_p, p \in N - \{0\}\}$.

Remark. The closed model obtained here have the same class of weak equivalences than the closed model defined in [1], but the classes of cofibrations, weak cofibrations, fibrations, weak fibrations of these closed models are different.

Definition 4.3. Let X be a directed graph, for every non zero integer p, we denote by $n_p(X)$ the cardinality of $Hom_{Gph}(c_p, X)$. Suppose that for every strictly positive integer p, $n_p(X)$ is finite. The Zeta serie $Z_X(t)$ of X is:

$$exp\left(\sum_{p=1}^{p=\infty} n_p(X)\frac{t^p}{p}\right).$$

Proposition 4.4. *Let X and Y be two finite directed graphs, $Z_X(t) = Z_Y(t)$ if and only if there exists an isomorphism f in $Hom_{Hot}(X,Y)$.*

Proof. Let X and Y be two finite graphs; the proposition 3.5 implies that there exists an isomorphism in $Hom_{Hot}(X,Y)$ if and only if there exists an isomorphism of graphs $f : C(X) \to C(Y)$. The proposition 3.2 implies that the cofibrant replacement $C(X)$ is a sum of cycles such that $Z_X(t) = Z_{C(X)}(t)$. We deduce that $Z_X(t) = Z_{C(X)}(t) = Z_{C(Y)}(t) = Z_Y(t)$.

Undirected graphs.

It is natural to try to generalize this closed model to others categories of graphs, unfortunately straightforward generalizations do not have the same natural properties, for example we do not obtain the same characterization of the weak equivalences with the corresponding Zeta series. We will consider the category $UGph$ of undirected graphs.

Let C_U be the category which has two objects that we denote by 0 and 1. We suppose that $Hom_{C_U}(0,1)$ contains two elements s,t, $Hom_{C_U}(0,0)$ contains one element, $Hom_{C_U}(1,1)$ contains the identity and an involution i such that $i \circ s = t$, and $Hom_{C_U}(1,0)$ is empty.

Definition 4.5. An undirected graph is a presheaf defined on C_U. Let X be such a presheaf, X is defined by two sets $X(0)$ and $X(1)$, two maps $X(s), X(t) : X(1) \to X(0)$ and an involution $X(i)$ of $X(1)$ such that $X(s) \circ X(i) = X(t)$.

The set $X(0)$ is called the space of nodes, and the space $X(1)$ the space of half-arcs. For an half-arc $a \in X(1)$, $X(s)(a)$ is the source of a and $X(t)(a)$ is the target of a.

Remark that $X(i)$ is an involution of $X(1)$, and the source of the half-arc a is the target of $X(i)(a)$ since $i \circ s = t$.

We have not assume that $X(i)$ acts freely, this implies the existence of undirected graphs X with degenerated loops; these are half-arcs fixed by $X(i)$.

An arc of the graph X is defined by a couple $(u, X(i)(u))$ where $u \in X(1)$. We denote by $Arc(X)$ the space of arcs of the undirected graph X. The source or the target of u will often be called an end of the arc $(u, X(i)(u))$.

Geometrically, if the set of half arcs of an undirected graph X does not contain a degenerated loop, it can be represented by a set of points corresponding to its nodes, and an arc $(u, X(i)(u))$ is an unoriented segment connecting $X(s)(u)$ and $X(t)(u)$.

Definition 4.6. A morphism $f : X \to Y$ between the undirected graphs X and Y is a natural transformation between the presheaves X and Y; thus f is defined by morphisms $f_0 : X(0) \to Y(0)$, and $f_1 : X(1) \to Y(1)$ such that $f_0 \circ X(s) = Y(s) \circ f_1$, $f_0 \circ X(t) = Y(t) \circ f_1$ and $f_1 \circ X(i) = Y(i) \circ f_1$.

The morphism of graphs $f : X \to Y$ induces a morphism $a(f) : Arc(X) \to Arc(Y)$. If there is no confusion, we will often denote $a(f)$ by f_1.

Examples of undirected graphs are: the undirected dot graph D_U. It is the graph such that $D_U(0)$ is a singleton and $D_U(1)$ is empty.

The undirected arc graph A_U is the graph defined by $A_U(0) = \{u_1, u_2\}, A_U(1) = \{a_1, a_2\}$ such that $A_U(i)(a_1) = a_2, A_U(s)(a_1) = u_1$ and $A_U(t)(a_1) = u_2$.

The graph V_U is the graph defined by $V_U(0) = \{v_1, v_2, v_3\}$, $V_U(1) = \{b_1, b_2, c_1, c_2\}$ such that $V_U(s)(b_1) = V_U(s)(c_1) = v_1, V_U(t)(b_1) = v_2, V_U(t)(c_1) = v_3, V_U(i)(b_1) = b_2$ and $V_U(i)(c_1) = c_2$.

The path graph P_n is the graph whose set of nodes is $\{0, ..., n\}, P_n(1) = \{p^+, p^-, p = 0, ..., n-1\}$ such that $P_n(s)(p^+) = p, P_n(t)(p^+) = p+1, P_n(i)(p^+) = p^-$.

There is a morphism $f : V_U \to A_U$ such that $f_0(v_1) = u_1, f_0(v_2) = f_0(v_3) = u_2$, $f_1(b_1) = f_1(c_1) = a_1$. This morphism is called the elementary folding.

Let p be a strictly positive integer. We denote by c_U^p the undirect graph whose set of nodes is Z/pZ. Let $[n]$ be the class of the integer n in Z/pZ, we have $c_U^p(1) = \{[n]^+, [n]^-, [n] \in Z/p\}$, $c_U^p(s)([n]^+) = [n], c_U^p(s)([n]^-) = [n+1]$, and $c_U^p(i)([n]^+) = [n]^-$. The graph c_U^p is called the undirected p-cycle.

Let X and Y two undirected graphs isomorphic to the 1-cycle c_U^1. Remark that there exists a unique morphism $f : D_U \to X$ (resp. $g : D_U \to Y$). The pushout of f and g is the eight graph. Geometrically, it corresponds to two circles attached in one point.

Definition. 4.7. Let X be an undirected graph, a p-cycle of X is a morphism $f : c_U^p \to X$. We say that the p-cycle f has a backtracking if and only if there exists an integer n such that $f_1([n+1]^+) = f_1([n]^-)$. We denote by $Cycl_p(X)$ the set of p-cycles of the undirected X without a backtracking.

We denote by W_U the class of morphisms of $UGph$ such that for every $f : X \to Y$ in W_U, for every integer $p > 0$, the morphism $c_p(f) : Hom(c_U^p, X) \to Hom(c_U^p, Y)$ which sends the morphism h to $f \circ h$ induces a bijection on cycles without a backtracking. The class of morphisms W_U satisfies the 2-3-property.

Let X be a finite undirected graph, we denote by $c_p(X)$ the cardinality of the set of morphisms $c_U^p \to X$ without a backtracking. The Ihara zeta function of X is defined by:

$$exp(\sum_{p \geq 1} \frac{c_p(X)}{p} t^p)$$

Remark that if $f : X \to Y$ is a morphism between finite undirected graphs in W_U, the graphs X and Y have the same Ihara zeta function. We want to find a closed model for which W_U is the class of weak equivalences. We will see that such a model does not exist.

Remark. We can naively adapt the previous closed model defined in the category of directed graphs to the category of undirected graphs: so, we define the closed model obtained by counting elements of the family $(c^p_U)_{p\in N-\{0\}}$. A morphism $f : X \to Y$ of $UGph$ is a weak equivalence for this closed model if and only if for every strictly positive integer p, the map $Hom(c^p_U,X) \to Hom(c^p_U,Y)$ induced by f is bijective. Thus, f induces a bijection between the p-cycles of X, and the p-cycles of Y for every strictly positive integer, but the image of a cycle without backtracking by f is not necessarily a cycle without backtracking. This closed model is essentially trivial as shows the following result:

Proposition 4.8. *A weak equivalence $f : X \to Y$ between two undirected finite and connected graphs for the closed model defined by counting elements of the family $(c^p_U)_{p\in N-\{0\}}$ is an isomorphism.*

Proof. Firstly, we show that $f_1 : X(1) \to Y(1)$ is injective. Let a,b be two distinct arcs such that $f_1(a) = f_1(b)$. There exist morphisms $h_i : c^2_U \to X$, $i = 1,2$ such that the image of h_1 is a and the image of h_2 is b, the square diagram:

$$c^2_U + c^2_U \overset{h_1+h_2}{\longrightarrow} X \downarrow j_2 \downarrow f c^2_U \overset{f\circ h_1}{\longrightarrow} Y$$

is commutative and does not have a filler this is a contradiction.

We show now that f_1 is surjective. Let a be an arc of Y, there exists a morphism $h : c^2_U \to Y$ whose image is a. Since f is right orthogonal to $i_2 : \phi \to c^2_U$, we deduce the existence of a morphism $h' : c^2_U \to X$ such that $h = f \circ h'$. This implies that f_1 is surjective on arcs.

We show now that $f_0 : X(0) \to Y(0)$ is injective. Let x and y be two distinct nodes of X such that $f_0(x) = f_0(y)$. Since X is connected, there exists a path h between x and y, $f(h)$ is a p-cycle where $p > 0$. Since f induces a bijection on p-cycles, we deduce the existence of a p-cycle c of X such that $f(c) = f(h)$. This is in contradiction with the fact that f is bijective on arcs.

We show now that f_0 is surjective.

Let y be a node of Y since Y is connected, there exists an arc b of Y which has y as an end. Since f_1 is bijective, we deduce the existence of an arc a of X such that $b = f_1(a)$. This implies that a has an end x such that $f_0(x) = y$.

Theorem 4.9. *There does not exist a closed model on $UGph$ whose class of weak equivalences is the class W_U.*

Proof. Suppose that such a closed model exists, then the elementary folding $f : V_U \to A_U$ would be a weak equivalence, and we can write $f = g \circ h$ where $h : V_U \to X$ is a weak cofibration and $g : X \to A_U$ is a fibration. The 2-3 property implies that g is a weak fibration.

Suppose that $h_1(b_1) = h_1(c_1)$, and consider the morphism $l : V_U \to c_U^2$ defined by $l_0(v_1) = [0], l_0(v_2) = l_0(v_3) = [1]$, $l_1(b_1) = [0]^+$ and $l_1(c_1) = [1]^-$. We can define the pushout diagram:

$$V_U \xrightarrow{h} X l \downarrow p \downarrow c_U^2 \xrightarrow{q} Z$$

Remark that X does not have any cycle without backtracking since h is a weak equivalence and V_U does not have any cycle without backtrackings. We deduce that Z does not have any cycle without backtrackings since Z is isomorphic to X. This implies that q is not a weak equivalence. This is a contradiction with the fact that in a closed model, the pushout of a weak cofibration is a weak cofibration. Thus $h_1(b_1)$ is distinct of $h_1(c_1)$. Remark that the image of V_U by h cannot be isomorphic to a 2-cycle since h is a weak equivalence; we deduce that this image is isomorphic to V_U.

Now, consider the morphism $m : c_U^2 \to A_U$ defined by $m_0([0]) = u_1, m_0([1]) = u_2$ and $m_1([0]^+) = m_1([1]^-) = a_1$. Consider the pullback diagram:

$$U \xrightarrow{p'} X q' \downarrow g \downarrow c_U^2 \xrightarrow{m} A_U$$

The morphism $q' : U \to c_U^2$ must be a weak equivalence since in a closed model, the pullback of a weak fibration must be a weak fibration. But, there exists a subgraph of U isomorphic to the pullback of the elementary folding by $m : c_U^2 \to A_U$. Such a subgraph is isomorphic to the graph obtained by identifying two nodes of two distinct unoriented 2-cycles. So q' cannot be a weak equivalence. This is a contradiction.

5. Closed models defined by counting in the category of undirected colored graphs

We are going to define closed models in subcategories of $UGph$, and in particular in the category of undirected colored graphs.

Definitions 5.1. Let X be an undirected graph, for every node x of X, we denote by $X(x,*)$ the set of arcs $(u,X(i)(u))$ such that $X(s)(u)=x$ or $X(t)(u)=x$.

A morphism $f:X\rightarrow Y$ between undirected graphs is a covering if and only if for every $x\in X_0$, the morphism $f_x:X(x,*)\rightarrow Y(f_0(x),*)$ induced by f is bijective. Remark that for every undirected graph, the morphism $\phi\rightarrow X$ is a covering.

Let X be an undirected graph, we denote by C_X the category whose objects are coverings $f:Y\rightarrow X$. A morphism between the objects $f:Y\rightarrow X$ and $g:Z\rightarrow X$ is a covering morphism $h:Y\rightarrow Z$ such that $g\circ h=f$.

Proposition 5.2. *Limits and colimits exist in C_X.*

Proof. The class Cov of covering morphisms of $UGph$ is the class of morphisms which are right orthogonal to the elementary folding $f_U:V_U\rightarrow A_U$ and to $i_U:D_U\rightarrow A_U$. This implies that the pullback of a covering morphism is a covering morphism. Since the products in C_X are pullbacks of covering morphisms, we deduce that products and pullbacks exist in C_X, and henceforth that limits exist in C_X. (See SGA 4.1 proposition 2.3).

Let $(f_i:Y_i\rightarrow X)_{i\in I}$ be a family of elements of C_X. The morphism $f:\sum_{i\in I}Y_i\rightarrow X$ whose restriction to Y_i is f_i is a covering; this implies that sums exist in C_X.

We show now that pushouts exist in C_X. Let $h:Z\rightarrow X$ and $h':Z'\rightarrow X$ be two objects of C_X; consider an object $p:Y\rightarrow X$, $f:Y\rightarrow Z$ and $g:Y\rightarrow Z'$ two morphisms of C_X. Without restricting the generality, we can suppose that Y,Z and Z' are connected. The morphisms f and g are surjective on nodes and arcs since they are coverings and the pushout of f and g is defined by the graph L whose set of nodes, L_0 is the quotient of $Z_0\bigcup Z'_0$ by the equivalence relation generated by: let $x\in Z_0$ and $x'\in Z'_0$, $x\simeq x'$ if and only if there exists $x"\in Y_0$ such that $f_0(x")=x$ and $g_0(x")=x'$; L_1 is the quotient of $Z_1\bigcup Z'_1$ by the equivalence relation generated by: let $a\in Z_1$ and $b\in Z'_1$, $a\simeq b$ if and only if there exists $c\in Y_1$ such that $f_1(c)=a$ and $g_1(c)=b$. We denote by $p_l:Y\rightarrow L$ the quotient morphism. There exists a morphism $l:L\rightarrow X$

such that $p = l \circ p_l$ since $p = h \circ f = h' \circ g$. The morphism l is a covering since f and g are elements of C_X; it is the pushout of f and g.

We consider R^p_X the set of graphs in C_X such that an element of R^p_X is obtained by attaching a forest to a p-cycle. We define on C_X the closed model obtained by counting elements of $R_X = \{R^p_X, p \in N - \{0\}\}$. Thus a morphism $f : X \to Y$ of C_X is a weak equivalence if and only if for every for every $p > 0$, for every $U_p \in R^p_X$, $Hom_{C_X}(U_p, X) \to Hom_{C_X}(U_p, Y)$ is bijective.

Proposition 5.3. *Let $f : Y \to X$ be an object of C_X such that Y does not have loops, then for every p-cycle $h : c^p_U \to Y, p > 1$, without backtrackings there exists an element U of R^p_X and a morphism $g : U \to Y$ of C_X whose restriction to the p-cycle is h.*

Proof. Let $h : c^p_U \to Y$ be a p-cycle. Since Y does not have loops, for every integer n, $h_1([n]^+)$ is distinct of $h_1([n+1]^+)$. This implies that we can attach a tree to every node of c^U_p to obtain a graph U for which there exists a covering $g : U \to Y$ whose restriction to c^U_p is h.

Remark. The previous proposition is not true if there are 1-cycles in Y. Consider the following example: $X = Y$ is the 1-cycle, Consider the morphism defined by $h : c^2_U \to X$ such that $h_1([0]^+) = h_1([1]^+)$. This cycle does not have backtracking, but it is impossible extend h to an element of R^2_X such that it becomes a covering, since the restriction of h_1 to $c^2_U([1], *)$ is not injective.

Proposition 5.4. *Let $f : Y \to X$ and $g : Z \to X$ be objects of C_X such that Y and Z are finite and do not have loops. If there exists a weak equivalence between f and g, then Y and Z have the same Ihara Zeta function.*

Proof. A weak equivalence between $f : Y \to X$ and $g : Z \to X$ is defined by a covering $h : Y \to Z$ such that $g \circ h = f$. We are going to show that h induces a bijection on p-cycles without backtracking. Let $u, u' : c^p_U \to Y$ be two p-cycles of Y without backtrackings such that $h \circ u = h \circ u'$. Since Z does not have loops, there exists an element $v : V \to X \in R^p_X$ and a morphism between v and g whose restriction to the p-cycle of V coincide with $h \circ u$. Since h is a covering, we can lift v to morphisms $v_1 : V \to Y$ (resp $v_2 : V \to Y$) whose restriction the p-cycle is u (resp. u') and such that v_1, v_2 are morphisms of C_X respectively between v and f. Since h is a weak equivalence, we deduce that $v_1 = v_2$, and henceforth that $u = u'$. We deduce that h is injective on p-cycles. The fact that f is surjective on p-cycles results from the fact that for every p-cycle without backtracking $l : c^p_U \to Z$ there exists an element $v : V \to X \in R^p_X$ and a morphism d of C_X between v and g whose restriction to the p-cycle of V is l. We can lift $d : V \to Z$ to a morphism $d' : V \to Y$ since h is a covering, the restriction of d' to the p-cycle of V is a preimage of l.

Remarks. Let $f : Y \to X$ be an object of C_X without loops. A p-cycle $u : c^p_U \to Y$ without backtracking, is primitive if and only if for every q-cycle $u' : c^q_U \to Y$ without backtracking, if there exists a morphism $f : c^p_U \to c^q_U$ such that $u = u' \circ f$, then $p = q$. Two primitive p-cycles $u, u' : c^p_U \to Y$ are equivalent if there exists an isomorphism f of c^p_U such that $u' = u \circ f$. We denote by $E_p(Y)$ the set whose elements are equivalence classes of primitive p-cycles without backtracking, and by $E(Y) = \bigcup_{p \in N - \{0\}} E_p(Y)$. For a primitive p-cycle without backtracking $u : c^p_U \to Y$, we will denote by $[u]$ its equivalence class.

For every element $[u] \in E_p(Y)$, we choose an element $u : c^p_U \to Y$ in this class, and consider the element V_u of R^p_X such that there exists a morphism $v_u : V_u \to Y$ of C_X whose restriction to the cycle of V_u coincide with u. Let V be the direct summand of the graphs V_u, there exists a covering $c : V \to Y$ whose restriction to V_u is v. The morphism c is a weak equivalence and the morphism $\phi \to c(Y)$ is a cofibration, thus $c(Y)$ is a cofibrant replacement of Y.

Let B_n be the undirected graph which has one node $*$, and n undirected loops. Let X be an undirected graph, there exists a covering $f : X \to B_n$ if and only if X is a n-regular graph and the edges of X can be colored by n-distinct colors. The proposition 5.4 implies that there exists a closed model on the category of n-regular graphs whose edges can be colored by n distinct colors such that, if there exists a weak equivalence between two finite graphs without loops in this category, then they have the same Zeta function.

Proposition 5.5. *Suppose that $X = B_n$, let $f : Y \to X$ and $g : Z \to X$ be two objects of C_X such that Y and Z are finite and do not have loops. Moreover, suppose that Y and Z have the same Zeta function, and there exist an isomorphism $H_p : E_p(Y) \to E_p(Z)$ such that each element $[c] \in E_p(Y)$ there exists a morphism $c : c^p_U \to Y$ representing $[c]$, such that $f \circ c = g \circ H_p(c)$ where $H_p(c)$ represents $H_p([c])$. Then there exists a graph L, coverings $p : L \to Y$ and $p' : L \to Z$ which are weak equivalences.*

Proof. If $Y \to X$ and $Z \to X$ are two objects of C_X such that the Zeta series of Y and Z are equal, then any isomorphism between their respective sets of primitive cycles which respects their colors as described above induces an isomorphism between their cofibrant replacements.

Remarks. Let X be an undirected graph; X is n-regular if and only if for every node x of X, the cardinal of $X(x,*)$ is n. Remark that an n-colored graph is an n-regular but not every n-regular graph is n-colored as shows the snark graph. The Quillen model that we have just defined in the category of n-regular colored graphs cannot be naively extended to the category whose objects are n-regular graphs and the morphisms are the coverings morphisms, since pushouts do not exist in this category.

We are going to relate n-colored graphs to Cayley graphs. Let G be a group and S a set of generators of G, for every G-set X, the Cayley graph $C(X,S,G)$ is the directed graph whose set of nodes is X, and for every elements x and y of X, the set of arcs between x and y is in bijection with $\{s \in S, s(x) = y\}$.

Let G_n be the group generated by $S_n = \{a_0,...,a_n\}$ such that $a_i^2 = 1$ for every $i = 0,...,n$. For every G_n-set X, the Cayley graph $C(X,S_n,G_n)$ is endowed with the structure of an undirected regular colored graph defined as follows: let x be a node of X, there exists an half edge between x and $a_i(x)$ colored by a_i. The symmetric of this half edge is the half edge defined by $a_i(a_i(x)) = x$. We denote by UC_{G_n} the category of G_n-sets.

Proposition 5.6. *The correspondence $C(S_n,G_n)$ which associates to X, $C(X,S_n,G_n)$ induces an isomorphism between UC_{G_n} and the category of $n+1$-regular colored graphs C_{B_n}.*

Proof. We have only to construct the inverse of $C(S_n,G_n)$. Let X be an $n+1$-regular colored graph. We assume that the colors are labeled by $a_0,..,a_n$. We associate to X its set of nodes X_0 endowed with the action of G_n defined as follows: if $x \in X_0$ and there exists an arc between x and y colored by a_i, we set $a_i(x) = y$.

Remarks. The closed model defined on C_{B_n} induces a closed model on UC_{G_n}.

Let LUC_{G_n} be the full subcategory of UC_{G_n} such that for every object X of LUC_{G_n}, every $x \in X$, and for every $i = 0,...,n$, $a_i(x) \neq x$. The functor $C(S_n,G_n)$ establishes an isomorphism between LUC_{G_n} and the category of $n+1$-regular colored graphs without a loop.

Closed models can also be defined in others interesting comma categories associated to $UGph$, here is an example:

Definition 5.7. The category of bipartite $BUGph$ graphs is the comma category $UGph/A_U$. Thus a bipartite graph is a morphism $f : X \to A_U$. A morphism between the objects $f : X \to A_U$ and $g : Y \to A_U$ of $UGph_n/A_U$ is a morphism $h : X \to Y$ such that $f = g \circ h$.

Consider the undirected graph D_n which has two nodes 0 and 1, and such that there exist $n+1$-arcs $a_0,...,a_n$ between 0 and 1. The category $BC_n = C_{D_n}$ of coverings of D_n is the category of graphs which are bipartite and $n+1$-colored. We deduce the existence of a closed model on BC_n obtained by counting objects of BC_n obtained by attaching a forest to a cycle.

6. Closed models by counting and topology

We are going to use the closed model defined on BC_n to study the Galoisian complexes introduced by Ladegaillerie [15]:

Definitions 6.1. Let S_n^+ be the oriented standard affine n-simplex whose vertices are labeled $A_0,...,A_n$. We denote by S_n^- the corresponding simplex with the opposite orientation. Let I be a set (not necessarily numerable), and $(S_i^+)_{i\in I}$ a set of examples of S_n^+ and $(S_i^-)_{i\in I}$ the corresponding set of examples of S_n^-. The elements of $(S_i^+)_{i\in I}$ are called the direct simplexes, and the elements of $(S_i^-)_{i\in I}$ are called the undirect simplexes. A Galoisian n-complex C is obtained by gluing elements of $(S_i^+)_{i\in I}$ with elements of $(S_i^-)_{i\in I}$ such that the gluing respect the labeling, affine structures and inverse orientations. Moreover,we suppose that each face of C belongs to exactly two simplexes; one direct and the other undirect.

A morphism $f : X \to Y$ between two Galoisian n-complexes is a continuous map which sends a direct simplex to a direct simplex, an undirect simplex to an undirect simplex, respects the labelings, and the affine structures. This defines the category CG_n, whose objects are Galoisian n-complexes and the morphisms are morphisms between Galoisian n-complexes.

Let X be a Galoisian n-complex. We denote by $\Omega_n^+(X)$ the union of elements of $(S_i^+)_{i\in I}$, by $\Omega_n^-(X)$ the union of elements of $(S_i^-)_{i\in I}$, and by $\Omega_n(X)$ the union of $\Omega_n^+(X)$ and $\Omega_n^-(X)$. We can define $s_j, j = 0,...,n$ the involution of $\Omega(X)$ such that for an element S_i^+ of $\Omega_n^+(X)$,

$a_j(S_i^+)$ is the unique undirected simplex of $\Omega_n^-(X)$ whose j-face is identified with the j-face of S_i^+. If G_n is the group generated by $\{a_0,...,a_n\}$ with the relations $a_j^2 = 1, j = 0,...,n$, the proof of Ladegaillerie [15] 1.2 shows that the correspondence Ω_n between CG_n and the category of G_n-sets, which sends a Galoisian n-complex X to the G_n-set $\Omega_n(X)$ endowed with the action that we have just defined is an isomorphism between CG_n and the category of G_n-sets. Remark that the composition of $C(S_n,G_n)\circ\Omega_n$ defines an isomorphism between CG_n and the category of undirected $n+1$-colored bipartite graphs BC_n. This shows the existence of a closed model on CG_n. We will denote by $L_n(X)$ the Cayley graph of the G_n-set $\Omega_n(X)$, defined by the generators of G_n, $a_0,...,a_n$.

Remarks. Let X and Y be finite Galoisian complexes, since $L_n(X)$ is a bipartite graph, it does not have loops, we deduce that if there exists a weak equivalence $L_n(X)\rightarrow L_n(Y)$, then $L_n(X)$ and $L_n(Y)$ have the same Ihara Zeta function.

Question. Is it possible to provide a geometric interpretation of the coefficient of the Ihara Zeta function of $L_n(X)$?

The Galoisian complex X_0^n defined by two elements $S^+_{n\,0}$ and $S^-_{n\,0}$ is homeomorphic to the n-sphere S^n. For every complex X defined by $\Omega_n^+(X)\bigcup\Omega_n^-(X)$, there exists a morphism of Galoisian complexes $p : X\rightarrow X_0^n$ which identifies the elements of $\Omega_n^+(X)$ to $S^+_{n\,0}$, and the elements of $\Omega_n^-(X)$ to $S^-_{n\,0}$. This map is ramified at a $(n-2)$-subcomplex of C (see [15] 1.3).

The subgroup of G_n generated by $\{a_ia_n, i = 0,..,n-1\}$ is isomorphic to the free subgroup generated by n elements, F_n. Let X be a Galoisian n-complex defined by $\Omega(X) = \Omega_n^+(X)\bigcup\Omega_n^-(X)$. The previous action of G_n on $\Omega(X)$ induces an action of F_n on $\Omega_n^+(X)$. The proof of Ladegaillerie [15] (p.1725-1726) shows this action induces an isomorphism between CG_n and the category of F_n-sets.

For every F_n-set X, we can define the Cayley graph $L_n^+(X)$ defined by the set of generators $a_0a_1,...,a_0a_n$. The isomorphism between CG_n and the category of F_n-sets E_{F_n} induces a closed closed model on E_{F_n}. Others closed models can be defined on E_{F_n}. In the next section we are going to present a general construction to transfer the closed model of Gph, to the category of G-sets for any group G.

Closed model and G-sets.

Let G be a group, consider the category C_G which has only one object that we denote by $*$, we suppose that $Hom_{C_G}(*,*) = G$. An object of the category $\hat{C_G}$, of presheaves over C_G is a set E, endowed with an action of G. Thus, $\hat{C_G}$ is the category of G-sets. We are going to transfer the closed model defined at the section 4 in the category of directed graphs, to the category of G-sets. On this purpose, we firstly recall the definition of some canonical functors.

The category C_D (see section 4) can be embedded in $\hat{C_D}$ by using the Yoneda embedding as follows: to 0, we associate the presheaf $\hat{0}$ defined by $\hat{0}(0) = Hom_{C_D}(0,0), \hat{0}(1) = Hom_{C_D}(1,0)$.We see that $\hat{0}$ is D_D the dot graph. To 1, we associate the presheaf $\hat{1}$ defined by $\hat{1}(0) = Hom_{C_D}(0,1), \hat{1}(1) = Hom_{C_D}(1,1)$. We remark that $\hat{1}$ is the arc graph A_D. Let X be a directed graph, we denote by C_D/X the category whose objects are morphisms of presheaves between the objects of C_D and X. The objects of C_D/X are morphisms $D_D \to X$ and $A_D \to X$. Thus, the class of objects of C_D/X can be identified with the union of the set of nodes of X, and its set of arcs. Let $f : U \to X$ and $g : V \to X$ be two objects of C_D/X, a morphism between f and g is a morphism $h : U \to V$ such that $f = g \circ h$. Let a be an arc of X, there exists a morphism $f_a : A_D \to X$ whose image is a. We also have morphisms $s_a : D_D \to s(a) \to X$ and $t_a : D_D \to t(a) \to X$, where $s(a)$ and $t(a)$ are respectively the source and the target of a. The source and the target morphisms $D_D \to A_D$ induces morphisms of C_D/X between s_a and f_a and between t_a and f_a. Remark that if a is a loop these two morphisms are distinct.

Let A be a set of generators of G. For every G-set S, recall that we have denoted by $C(A,G)(S)$ the Cayley graph of S associated to G and A.

Let U be the terminal object of the category of G-sets. U is the G-set which has a unique element n. We denote by B_A the Cayley graph of U defined by A. The objects of the category C_D/B_A are the unique morphism $i_a : D_D \to B_A$ and the morphisms $c_a : A_D \to B_A$ which sends A_D to the loop of B_A corresponding to a. The morphisms of C_D/B_A are the isomorphisms and the morphisms $\hat{s}_a : i_a \to c_a$ induced by $\hat{s} : \hat{0} \to \hat{1}$ and $\hat{t}_a : i_a \to c_a$ induced by $\hat{t}$. We have a functor $F_A : C_D/B_A \to C_G$ such that $F_A(i_a) = F_A(c_a) = *$, $F_A(\hat{s}_a) = Id$, $F_A(\hat{t}_a) = a$. In [8] p. 33, Grothendieck defines an equivalence of categories $e_{B_A} : \widehat{C_D/B_A} \to Gph/B_A$.

Proposition 6.2. *The composition: $e_{B_A}\circ\hat{F}_A:\hat{C}_G\rightarrow Gph/B_A$ is the functor which associates to a G-set S, the canonical morphism $C(A,G)(S)\rightarrow B_A$ and henceforth $D(A,G)=e_{B_A}\circ\hat{F}_A$, has a left adjoint.*

Proof. Let E^G be a G-set, we denote by E the image of E^G by the forgetful functor $G\text{-}Sets\rightarrow Set$. We have: $\hat{F}_A(E^G)(i_A)=\hat{F}_A(E^G)(c_a)=E$. The construction in [8] 5.10.1 shows that:

$$e_{B_A}\circ\hat{F}_A(E^G)(0)=E,$$

and

$$e_{B_A}\circ\hat{F}_A(E^G)(1)=\bigcup_{a\in A}\hat{F}_A(E^G(c_a))=\bigcup_{a\in A}E_a,$$

where $E_a=E$. This shows that the set of nodes (resp. the set of arcs) of the Cayley graph of E^G coincide with the set of nodes (resp. the set of arcs) of $e_{B_A}\circ\hat{F}_A(E)$.

The restriction of $e_{B_A}\circ\hat{F}_A(E^G)(s)$ to E_a is the identity, and the restriction of $e_{B_A}\circ\hat{F}_A(E^G)(t)$ to E_a is the multiplication by a. This shows that the Cayley graph of E^G is $e_{B_A}\circ\hat{F}_A(E^G)$.

We deduce that the functor $D(A,G)$ has a left and right adjoint since e_{B_A} is an equivalence of categories and $\hat{F}_A$ has left and right adjoint see [8] proposition 5.1.

Remark. The existence of a left adjoint of $D(A,G)$ can be shown directly, by showing that $D(A,G)$ commutes with limits. The functor $\hat{C}_G\rightarrow Gph$ which sends a G-set to its Cayley graph does not have always a left adjoint since it does not commutes always with limits.

We consider the closed model defined on Gph/B_A obtained by counting the object $c_n\rightarrow B_A$, where c_n is an n-cycle.

We present now the transfer theorem that we are going to use see [5] theorem 3.3. Let C and D be categories in which limits and colimits exist. Suppose that C is endowed with a closed model. We denote by W_C the class of weak equivalences, Cof_C the class of cofibrations and Fib_C the class of fibrations of this closed model. Suppose that there exists a functor $F:C\rightarrow D$ which has a right adjoint functor G.

We denote W_D the class of morphisms of D such that for every morphism $f\in W_D$, $G(f)$ is a weak equivalence,

We denote Fib_D the class of morphisms of D such that for every $f\in Fib_D$, $G(f)$ is a fibration,

We denote by Fib'_D the intersection of W_D and Fib_D.

An arrow of D is a cofibration if and only if it has the left lifting property with respect to every element of Fib'_D. We denote by Cof_D the class of cofibrations.

Theorem 6.3. *With the notations above, suppose that D allows the small object argument, and suppose that for every morphism d of D which is a transfinite composition of pushouts of coproducts of morphisms $F(c)$ where c is a weak cofibration, $G(d)$ is a weak equivalence. Then, there exists a closed model on D whose class of weak equivalences is W_D, the class of fibrations is Fib_D and the class of cofibrations is Cof_D.*

We can deduce the following result:

Corollary 6.4. *Let G be a group, A a set of generators of G and $\hat{C}_G$ the category of G-sets. There exists a closed model on $\hat{C}_G$ such that the morphism of G-sets $f : X \to Y$ is a weak equivalence if and only if $D(A,G)(f) : D(X,A,G) \to D(Y,A,G)$ is a weak equivalence.*

Proof. To show this result, we apply the theorem 6.4 to transfer the model defined on the category of directed graphs at paragraph 4 with the functor $D(A,G)$. The category of G-sets allows the small object argument. Let F be the left adjoint of $D(A,G)$ and d be a morphism of $\hat{C}_G$ which is a transfinite composition of pushouts of coproducts of morphisms $F(c)$, where c is a weak cofibration. Since weak cofibrations in Gph are isomorphisms, we deduce that d and $D(A,G)(d)$ are isomorphisms.

We are going to apply this construction to the free group F_n. Let $A_n = \{a_1,...,a_n\}$ be a set of generators of F_n; we denote by R_n the set of F_n-sets such that for every F_n-set X in R_n, $C(A_n,F_n)(X)$ is obtained by attaching a forest to a sum of cycles.

Proposition 6.5. *A F_n-set X is cofibrant for the closed model obtained by transferring the closed model of Gph/B_A to the category of F_n-sets with $D(A_n,F_n)$ if and only if it is an element of R_n.*

Proof. Let X be a cofibrant F_n-set. The source of the cofibrant replacement X' of $D(A_n,F_n)(X)$ is the sum $\sum_{i\in I} X'_i$ of cycles. Let h_i be the restriction of the cofibrant morphism $h : X' \to D(A_n,F_n)(X)$ to X'_i. There exists an element of R_n, $Y_i = D(A_n,F_n)(X_i)$, a morphism $g_i : X_i \to X$ such that h_i is the restriction of $D(A_n,F_n)(g_i)$ to the cycle of Y_i. To see this, consider $x^i_1,...,x^i_{i_n}$ be the nodes of X'_i. We suppose that there exists an arc between $x^i_{i_j}$ and $x^i_{i_{j+1}}$ if $i_j < i_n$ and an arc between $x^i_{i_n}$ and x^i_1. There exists a generator a_{i_j} such that $a_{i_j}(h(x^i_{i_j})) = h(x^i_{i_{j+1}})$, $i_j <$

i_n and $a_{i_n}(h(x^i_{i_n}) = h(x^i_1)$. The set F_n-set X_i is the unique set which contains the elements $x^i_1,..,x^i_{i_n}, a_{i_j}(x^i_{i_j}) = x^i_{i_{j+1}},\ j < n$ and $a_{i_n}(x^i_{i_n}) = x^i_1$ and the Cayley graph of X_i is obtained by attaching a minimal forest to the cycle $(x^i_1,..,x^i_{i_n})$. The sum $\sum_{i\in I} g_i$ is a weak equivalence.

Consider the commutative diagram:

$$\phi \longrightarrow \sum_{i\in I} X_i \downarrow\downarrow \sum_{i\in I} g_i X \stackrel{Id_X}{\longrightarrow} X$$

Since $\sum_{i\in I} h_i$ is a weak equivalence or equivalently a weak fibration, we deduce that the existence of a morphism $f : X \to \sum_{i\in I} X_i$ which fills the previous commutative diagram. This implies that X is an element of R_n.

Conversely, let X be an element of R_n, let $f : Y \to Z$ be a weak equivalence or equivalently a weak fibration such that there exists a commutative diagram:

$$\phi \longrightarrow Y \downarrow\downarrow f X \stackrel{g}{\longrightarrow} Z$$

Since the source of $D(A_n,F_n)(X)$ is obtained by attaching a forest to a union of cycles, and f is a weak equivalence the morphism $D(A_n,F_n)(g)$ can be lifted to a morphism $D(A_n,F_n)(h)$: $D(A_n,F_n)(X) \to D(A_n,F_n)(Y)$ which is the image by the functor $D(A_n,F_n)$ of a morphism h : $X \to Y$ which makes the previous diagram commutes.

Remarks. Let X and Y be F_n-sets, if $c(X)$ is a cofibrant replacement of X, it is also a fibrant replacement of X since every morphism of F_n-sets is a fibration.

There exists a functor $c : F_n\text{-}sets \to F_n\text{-}sets$ such that $c(X)$ is a cofibrant replacement of X. To construct $c(X)$, consider a cofibrant replacement $c(X)$ of X and suppose that every connected component of $C(A_n,F_n)(c(X))$ is not isomorphic to a tree.

Proposition. 6.6. *Let X and Y be F_n-sets, the set of morphisms $Hom_{Hot}(X,Y)$ between X and Y in the homotopy category is $Hom_{F_n-sets}(c(Y),c(Y))$ where $c(Y)$ and $c(Y)$ are respectively cofibrant replacements of X and Y.*

Proof. We are going to show that the category Hot_n whose objects are F_n-sets and such that for every objects X and Y of Hot_n, $Hom_{Hot_n}(X,Y) = Hom_{F_n-sets}(c(X),c(Y))$ is a localization of the category of F_n-sets by the class of weak equivalences. The morphism $f : X \to Y$ is a weak equivalence between F_n-sets if and only if $c(f) : c(X) \to c(Y)$ is a weak equivalence. This is equivalent to saying that $D(F_n, A_n)(c(f))$ is a weak equivalence. Since the sources of $D(F_n, A_n)(c(X))$ and $D(F_n, A_n)(c(Y))$ are sum of cycles, this is equivalent to say that $D(F_n, A_n)(c(f))$ and $c(f)$ are isomorphisms. We deduce that Hot_n is a localization of the closed model that we have defined on the category of F_n-sets. We can conclude by using the remark at the first line of [6] p.29.

Closed model and Dessins d'enfants.

In this section, we are going to recall the definition of a dessin d'enfant and see how it is a particular case of the construction above.

Let FCG_2 be the category of finite Galoisian 2-complexes, and X an object of FCG_2. The morphism $p_X : X \to S^2$ is a covering of the 2-sphere ramified at three elements that we denote A_0, A_1 and A_2. We can identify $S^2 - \{A_0, A_1, A_2\}$ with $C - \{0, 1\}$, the complex line without two points. Let $[0, 1]$ be the segment drawn between 0 and 1 in the complex line, $X_C = p_X^{-1}([0, 1])$ is an undirected graph. Remark that p induces a morphism $X_C \to [0, 1]$; thus X_C is a bipartite graph.

Let $U_X^0 = p_X^{-1}(0)$ and $U_X^1 = p_X^{-1}(1)$. The fundamental group F_2 of $\pi_1(C - \{0, 1\})$ is the free group generated by two elements. We denote by s_0 and s_1 its generators. Without restricting the generality, we suppose that for each $x \in p_X^{-1}(0)$, the monodromy of s_0 induces an action on $X_C(x, *)$ and for every $x \in p_X^{-1}(1)$ the monodromy of s_1 induces an action on $X_C(x, *)$. This action is nothing but the restriction of the action of F_2 on $\Omega^+(X)$ (see [17]). A bipartite graph endowed with such an action of F_2 is called a dessin d'enfant. Conversely, any finite F_2-set define a dessin d'enfant. Let FS_2 be the category of finite F_2-sets; the functor $F : FCG_2 \to FS_2$ which associates to a finite Galoisian 2-complex the F_2-set defined by its dessin d'enfant induces an isomorphism between FCG_2 and FS_2 (see [17]).

There exists a one to one correspondence between finite dessin d'enfants and algebraic curves defined over the algebraic closure $\bar{Q}$ of the field of rational numbers. The action of the Galois group $Gal(\bar{Q}/Q)$ on algebraic curves defined over $\bar{Q}$ induces an action of $Gal(\bar{Q}/Q)$ on dessins d'enfants.

We denote by $Hot(FS_2)$ the homotopy category of the closed model defined on FS_2-sets.

Remark. Let D_0 be the Dessin d'enfant whose underlying graph is A_U and such that F_2 acts trivially on the nodes. The Cayley graph $Cal(D_0)$ associated to this action is B_2: the graph which has one node n an two loops a, b.

We denote by D_1 the dessin d'enfant whose underlying graph is P_2. Let s_1 and s_2 be the generators of F_2. We suppose that s_1 acts trivially on the arcs of P_2 and s_2 defines a non trivial involution on them. The Cayley graph $Cal(D_1)$ associated to this action is a directed graph which has two nodes x and y, there exists one directed arc u between x and y, one loop u_x at x, one directed arc v between y and x and one loop v_y at y. The morphism $f : Cal(D_1) \to Cal(D_0)$ defined by $f_0(x) = f_0(y) = n$, $f_1(u) = f_1(u_x) = a, f_1(v) = f_1(v_y) = b$ is a weak equivalence for the closed model defined on Gph by counting the cycles but not an isomorphism. This implies that D_0 and D_1 are weak equivalent.

Questions. Is the action of $Gal(\bar{Q}/Q)$ on the category of dessins d'enfant induces an action of $Gal(\bar{Q}/Q)$ on the image of FS_2 in the homotopy category of the closed model defined on the category of F_2-sets ?

We have constructed (see p. 15) a closed model in CG_2 induced by the closed model defined on the category of colored 3-regular graphs. Is the action of $Gal(\bar{Q}/Q)$ on dessins d'enfant induces an action on the image of the category of finite Galoisian 2-complexes in the homotopy category of this closed model ?

Conflict of Interests

The authors declare that there is no conflict of interests.

References

[1] T. Bisson and A. Tsemo, A homotopical algebras of graphs related to zeta series, Homology, Homotopy and its Applications, 10 (2008), 1-13.

[2] Bisson and A. Tsemo, Symbolic dynamic and the category of graphs, Theory and Applications of Categories, 25 (22) (2011), 614-640.

[3] T. Bisson and A. Tsemo, A Homotopy equivalence of isospectral graphs New York J. Math. 17 (2011) 295320.

[4] P. Boldi and S. Vigna, Fibrations of graphs, Discrete Math., 243 (2002), 21-66.

[5] S.E. Crans, Quillen closed model structures for sheaves. J. Pure Appl. Algebra 101 (1995), no. 1, 35-57.

[6] W.G. Dwyer and J. Spalinski, Homotopy theories and model categories, in Handbook of Algebraic Topology, Editor I. M. James, 73-126, Elsevier, 1995.

[7] M. Ferri, Crystallisations of 2-fold branched coverings of S3, Proc. Amer. Math. Soc. 73 (2) (1979), 271-276.

[8] A. Grothendieck, and al. Theorie des topos et cohomologie etale des schemas, Seminaire de Geometrie Algebrique du Bois-Marie 1963/64 (SGA 4), Tomes 1, 2, rev. ed., Lecture Notes in Mathematics, vols. 269, 270, Springer-Verlag, Berlin, 1972. .

[9] Grothendieck, A., Esquisse d'un programme. London Math. Soc. Lecture Note Ser., 242, Geometric Galois actions, 1, 5-48, Cambridge Univ. Press, Cambridge, 1997.

[10] Hatcher, Allen, Algebraic topology, Cambridge University Press (2002)

[11] S. Hirschhorn, Model Categories and Their Localizations, Mathematical Surveys and Monographs, 99, American Mathematical Society, Providence, RI, 2003.

[12] G. A. Jones and D. Singerman, Theory of maps on orientable surfaces, Proc. London Math. Soc. 37 (1978), 273307.

[13] A. Joyal and M. Tierney, Quasi-categories vs Segal spaces, arXiv:math/06/07/820v1

[14] M. Kotani and T. Sunada, Zeta functions of finite graphs, J. Math. Sci. Univ. Tokyo, 7 (2000), 7-25.

[15] Ladegaillerie Y. Complexes Galoisiens. Transactions of the American Mathematical Society. Volume 352, Number 4, Pages 17231741

[16] D.G. Quillen, Homotopical Algebra, Lecture Notes in Mathematics no. 43, Springer-Verlag, Berlin, 1967.

[17] L. Schneps: Dessins d'enfants on the Riemann Sphere, in The Grothendieck Theory of Dessins d'Enfants, LMS Lecture Notes 200, Cambridge U. Press, 1994.

[18] Serre, J-P. Arbres, amalgames, SL(2), Astrisque, vol. 46, SMF, Paris, 1977

[19] Terras, Audrey (2010). Zeta Functions of Graphs: A Stroll through the Garden. Cambridge Studies in Advanced Mathematics 128. Cambridge University Press.

[20] S. Vigna, A guided tour in the topos of graphs. Technical Report 199-97. Universit'a di Milano. Dipartimento di Scienze dell Informazione, 1997.