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NEW RESULTS ON EXTERIOR DERIVATION OF UNIVERSAL MODULES

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Abstract. This study is interested in the exterior and symmetric derivation on universal modules. In this paper, we investigated some interesting homological results about universal modules of second order derivations and we gave some examples related to universal modules of second order derivations.

Keywords: exterior derivation; universal module; commutative rings.

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1. Introduction

In order to prove conclusions about algebraic sets and their coordinate rings, one of the methods is to study the universal module of differential operators. This ideas of studying the universal module may decrease questions about algebras to questions of module theory. The idea of using the universal module goes as far back as [1] which was proved some propeties of $\Omega_1(R)$. The universal modules of higher differential operators of an algebra were introduced by [2]. After on the like thought has appeared in [3] and [4]. During the recent years, subject of universal modules of high order differential operators has studied by [5]. The knowledge of a Kahler

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module of q-th order was present by Osborn in [10]. Same notion has emerged in Heyneman and Sweedler in [3]. They mentioned differential operators on a commutative algebra which spread out the notion of derivations. Hart introduced the higher derivations and universal differentials operators of Kahler modules in [6]. Nakai advanced the fundemental theories for the calculus of high order derivations and some functorial properties of the module of high order differentials in his work in [4]. Johnson considered Kahler differentials and differential algebra in [11]. Olgun take care of the universal modules of finitely generated algebras in [9]. It was researched the homological dimension of universal modules of differential operators. Erdogan and Olgun essentially interested in some problems related to the universal modules of high order derivations introduced and developed by Osborn, Heyneman and Sweedler and Nakai in [2], [3] and [4], respectively. Komatsu introduced right differential operators on a non-commutative ring extensions in [12]. In [8], the authors characterized the homological dimension of n-th order Kahler differentials of R over k and examined functorial properties of the module of n-th order Kahler differentials of $R \otimes S$ over k in [8].

In this study, we have inspired from Hart's [6] and Osborn's [10]. They gave split exact sequences related to first order derivations of Kahler modules. Using these, first of all we have described a new isomorphism and given an example about it. Secondly, we have identified new exact sequences and a new split short exact sequence related to second order derivations of Kahler modules and proved them. Then, we have assigned that $\ker \alpha$ is isomorphic to what? where α is a map from $J_2(\Omega_2(R))$ to $\Omega_2(R)$ and R is a finitely generated k-algebra over algebraically closed field k with characteristic zero. Finally, we have given some examples.

2. Preliminaries

Throughout this paper, unless the contrary is stated explicitly, we will let R be a commutative algebra over an algebraically closed field k with charasteristic zero and by a k-algebra, we mean commutative ring which contains a field k of characteristic zero. All modules will be unitary. We use the following notation: The universal module of n-th order derivations of a ring R will be denoted by $\Omega_n(R)$. When M is an R-module, $J_n(M)$ will denote the universal module of differential operators of order less than or equal to n on M. When in a diagram, we fail to specify a homomorphism, this will mean that there is a natural homomorphism.

Definition 2.1. [10] Let R be any k-algebra commutative with unit, $R \to \Omega_1(R)$ be first order Kahler derivation of R and let $S(\Omega_1(R))$ be the symmetric algebra $\oplus_{p\geq 0}(S^p(\Omega_1(R)))$ generated over R by $\Omega_1(R)$. A symmetric derivation is any linear map D of $S(\Omega_1(R))$ into itself such that

$i-)$ $D(S^p(\Omega_1(R))) \subset S^{p+1}(\Omega_1(R))$

$ii-)$ D is a first order derivation over k

$iii-)$ the restriction of D to R is the Kahler derivation $d_1 : R \to \Omega_1(R)$.

Theorem 2.1. [10] Let R be an affine k-algebra. Then, there exist a short exact sequence of R-modules

$$0 \to \ker\theta \to \Omega_2(R) \xrightarrow{\theta} \Omega_1(R) \to 0$$

such that $\theta(d_2(f)) = d_1(f)$ for all $f \in R$ and $\ker\theta$ is generated by the set $\{d_2(ab) - ad_2(b) - bd_2(a) : a, b \in R\}$.

Definition 2.2. [13] Let R be any k-algebra commutative with unit, $R \to \Omega_q(R)$ be q-th order Kahler derivation of R and let $S\left(\Omega_q(R)\right)$ be the symmetric algebra $\oplus_{p\geq 0}\left(S^p\left(\Omega_q(R)\right)\right)$ generated over R by $\Omega_q(R)$. A symmetric derivation is any k-linear map D of $S^p\left(\Omega_q(R)\right)$ into itself such that

$i-)$ $D\left(S^p\left(\Omega_q(R)\right)\right) \subset S^{p+1}\left(\Omega_q(R)\right)$

$ii-)$ D is a first order derivation over k

$iii-)$ the restriction of D to R is the Kahler derivation $d_q : R \to \Omega_q(R)$.

Theorem 2.2. [13] Let R be an affine k-algebra. Then, there exist a long exact sequence of R-modules

$$0 \to \ker\theta \to \Omega_{2q}(R) \xrightarrow{\theta} J_q\left(\Omega_q(R)\right) \to co\ker\theta \to 0$$

for all $q \geq 0$.

Theorem 2.3. [13] Let R be an affine k-algebra. Then, there exist a long exact sequence of R-modules

$$0 \to \ker\beta \to J_q\left(\Omega_q(R)\right) \xrightarrow{\beta} S^2\left(\Omega_q(R)\right) \to co\ker\beta \to 0$$

for all $q \geq 0$.

Lemma 2.1. [10] Let R be a commutative k-algebra. Assume that, $\Omega_1(R)$ is the universal module of derivations of R with universal derivation $d: R \to \Omega_1(R)$. Then, the map

$$D: \Omega_1(R) \to \Lambda^2(\Omega_1(R))$$

$$D\left(\sum_i a_i d(b_i)\right) = \sum_i d(a_i) \Lambda d(b_i)$$

is a differential operator of order 1 on $\Omega_1(R)$ where $a_i, b_i \in R$.

Corollary 2.1. [6] Let R be an affine k-algebra. Then, there exist a split exact sequence of R-modules

$$0 \to \Omega_2(R) \xrightarrow{\phi} J_1(\Omega_1(R)) \to \Lambda^2(\Omega_1(R)) \to 0.$$

Lemma 2.2. [14] Let R be a commutative k-algebra. Suppose that $\Omega_2(R)$ is the universal module of derivations of R with universal derivations $d_2: R \to \Omega_2(R)$.Then, the map

$$D_2: \Omega_2(R) \to \Lambda^2(\Omega_2(R)), \sum_i a_i d_2(b_i) \to D_2\left(\sum_i a_i d_2(b_i)\right) = \sum_i d_2(a_i) \Lambda d_2(b_i)$$

is a differential operator of order 2 on $\Omega_2(R)$ where $a_i, b_i \in R$.

3. Main Conclusions

Theorem 3.1. Let $R = k[x_1, x_2, ..., x_s]$ be a commutative k-algebra. Then, there exist a split short exact sequence of R-modules

$$0 \to \ker\alpha \to J_2(\Omega_2(R)) \xrightarrow{\alpha} \Lambda^2(\Omega_2(R)) \to 0$$

and

$$\ker\alpha \approxeq \Omega_2(R) \oplus S^2(\Omega_2(R)).$$

Before the prove theorem, we will give some useful Lemma's about our main aim.

Lemma 3.1. Let $R = k[x_1, x_2, ..., x_s]$ be an affine k-algebra. Then,

$$\Omega_2(R) \xrightarrow{\Delta_2} J_2(\Omega_2(R)) \xrightarrow{\alpha_1} S^2(\Omega_2(R)) \to 0$$

is an exact sequence of R-modules.

Proof. Let R be a commutative affine k-algebra with unit and $\Omega_2(R)$ be the universal modules of second order derivations of R. Let $J_2(\Omega_2(R))$ be the universal module of differential operators of order less than or equal to 2 on $\Omega_2(R)$ with universal differential operator $\Delta_2 : \Omega_2(R) \to J_2(\Omega_2(R))$. Then, the map

$$\breve{A} : \Omega_2(R) \to S^2(\Omega_2(R))$$

$$\breve{A}\left(\sum_{i,j,k,l} x_i^m x_j^n d_2\left(x_k^t x_l^r\right)\right) = \sum_{i,j,k,l} d_2\left(x_i^m x_j^n\right) \vee d_2\left(x_k^t x_l^r\right)$$

is a differential operator of order 2 on $\Omega_2(R)$ where $x_i, x_j x_k, x_l \in R$, $m,n,t,r = 0,1,2$ and $1 \leq i,j,k,l \leq s$. By the universal property of $J_2(\Omega_2(R))$, there is a unique R-module homomorphism

$$\alpha_1 : J_2(\Omega_2(R)) \to S^2(\Omega_2(R))$$

such that the following diagram

$$\begin{array}{ccc} \Omega_2(R) & \overset{\breve{A}}{\to} & S^2(\Omega_2(R)) \\ \Delta_2 \searrow & & \nearrow \alpha_1 \\ & J_2(\Omega_2(R)) & \end{array}$$

commutes and $\alpha_1 \Delta_2 = \breve{A}$. Since,

$$\alpha_1 \Delta_2\left(x_i^m x_j^n d_2\left(x_k^t x_l^r\right)\right) = \breve{A}\left(x_i^m x_j^n d_2\left(x_k^t x_l^r\right)\right) = d_2\left(x_i^m x_j^n\right) \vee d_2\left(x_k^t x_l^r\right),$$

α_1 is surjective. Therefore, we have

$$\Omega_2(R) \overset{\Delta_2}{\to} J_2(\Omega_2(R)) \overset{\alpha_1}{\to} S^2(\Omega_2(R)) \to 0$$

an exact sequence of R-modules. It suffice to show that the sequence is exact at $J_2(\Omega_2(R))$. Note that, $Im\Delta_2$ is generated by the set

$$\left\{\Delta_2(d_2(x_i)), \Delta_2\left(d_2\left(x_i x_j\right)\right)\right\}$$

for $i,j = 1,2,...,s$ and that

$$d_2(x_i) \vee d_2\left(x_j\right) - d_2\left(x_j\right) \vee d_2(x_i) = 0.$$

Therefore, we have

$$\alpha_1\Delta_2\left(d_2\left(x_i\right)\right)=\breve{A}\left(d_2\left(x_i\right)\right)=d_2\left(1\right)\vee d_2\left(x_i\right)=0$$

$$\alpha_1\Delta_2\left(d_2\left(x_ix_j\right)\right)=\breve{A}\left(d_2\left(x_ix_j\right)\right)=d_2\left(1\right)\vee d_2\left(x_ix_j\right)=0$$

this implies that $Im\Delta_2$ is contained in $\ker\alpha_1$. Hence, we have an induced map

$$\gamma_1:J_2\left(\Omega_2\left(R\right)\right)/Im\Delta_2\rightarrow S^2\left(\Omega_2\left(R\right)\right)$$

defined by

$$\gamma_1\left(\overline{\Delta_2\left(x_i^mx_j^nd_2\left(x_k^tx_l^r\right)\right)}\right)=d_2\left(x_i^mx_j^n\right)\vee d_2\left(x_k^tx_l^r\right).$$

Now, $S^2\left(\Omega_2\left(R\right)\right)$ is generated by the set

$$\left\{d_2\left(x_i^mx_j^n\right)\vee d_2\left(x_k^tx_l^r\right):m,n,t,r=0,1,2\text{ and }i,j,k,l=1,2,...,s\right\}.$$

We can define a map

$$\gamma_2:S^2\left(\Omega_2\left(R\right)\right)\rightarrow J_2\left(\Omega_2\left(R\right)\right)/Im\Delta_2$$

defined by

$$\gamma_2\left(d_2\left(x_i^mx_j^n\right)\vee d_2\left(x_k^tx_l^r\right)\right)=\overline{\Delta_2\left(x_i^mx_j^nd_2\left(x_k^tx_l^r\right)\right)}.$$

It is clear that, $\gamma_1\gamma_2$ and $\gamma_2\gamma_1$ are identities and so, $\ker\gamma_1=\ker\alpha_1/Im\Delta_2=0$ and then, $\ker\alpha_1=Im\Delta_2$. Therefore, the sequence is exact.

Lemma 3.2. Let $R=k\left[x_1,x_2,...,x_s\right]$ be an affine k-algebra. Then, there is a short exact squence of R-modules

$$0\rightarrow\ker\theta_1\rightarrow J_2\left(\Omega_2\left(R\right)\right)\xrightarrow{\theta_1}\Omega_2\left(R\right)\rightarrow 0.$$

Proof. Let $J_2\left(\Omega_2\left(R\right)\right)$ be the universal module of differential operators of order less than or equal to 2 on $\Omega_2\left(R\right)$ with the universal differential operator Δ_2. Then, the following diagram

$$\begin{array}{ccc}
\Omega_2\left(R\right) & \xrightarrow{1_{\Omega_2(R)}} & \Omega_2\left(R\right)\\
\Delta_2\searrow & & \nearrow\theta_1\\
 & J_2\left(\Omega_2\left(R\right)\right) &
\end{array}$$

commutes and that $\theta_1\Delta_2 = 1_{\Omega_2(R)}$. Since

$$\theta_1\left(\Delta_2\left(\sum_{i,j,k,l} x_i^m x_j^n d_2\left(x_k^t x_l^r\right)\right)\right) = \sum_{i,j,k,l} x_i^m x_j^n d_2\left(x_k^t x_l^r\right)$$

in $\Omega_2(R)$ for $m,n,t,r = 0,1,2$ and $1 \leq i,j,k,l \leq s$, α is surjective. Hence, we have

$$0 \to \ker\theta_1 \to J_2\left(\Omega_2(R)\right) \xrightarrow{\theta_1} \Omega_2(R) \to 0$$

a short exact sequence of R-modules.

Lemma 3.3. Let $R = k[x_1, x_2, ..., x_s]$ be an affine k-algebra. Then,

$$\Omega_2(R) \xrightarrow{\Delta_2} J_2\left(\Omega_2(R)\right) \xrightarrow{\alpha} \Lambda^2\left(\Omega_2(R)\right) \to 0$$

is an exact sequence of R-modules.

Proof. Let R be a commutative affine k-algebra with unit and $\Omega_2(R)$ be the universal modules of second order derivations of R. Let $J_2(\Omega_2(R))$ be the universal module of differential operators of order less than or equal to 2 on $\Omega_2(R)$ with universal differential operator $\Delta_2 : \Omega_2(R) \to J_2(\Omega_2(R))$. Then, the map

$$\widehat{A} : \Omega_2(R) \to \Lambda^2\left(\Omega_2(R)\right)$$
$$\widehat{A}\left(\sum_{i,j,k,l} x_i^m x_j^n d_2\left(x_k^t x_l^r\right)\right) = \sum_{i,j,k,l} d_2\left(x_i^m x_j^n\right) \Lambda d_2\left(x_k^t x_l^r\right)$$

is a differential operator of order 2 on $\Omega_2(R)$ where $x_i, x_j x_k, x_l \in R$, $m,n,t,r = 0,1,2$ and $1 \leq i,j,k,l \leq s$. By the universal property of $J_2(\Omega_2(R))$, there is a unique R-module homomorphism

$$\widehat{A} : \Omega_2(R) \to \Lambda^2\left(\Omega_2(R)\right)$$

such that the following diagram

$$\begin{array}{ccc} \Omega_2(R) & \overset{\widehat{A}}{\to} & \Lambda^2\left(\Omega_2(R)\right) \\ \Delta_2 \searrow & & \nearrow \alpha \\ & J_2\left(\Omega_2(R)\right) & \end{array}$$

commutes and $\alpha\Delta_2 = \widehat{A}$.

Since,

$$\alpha\Delta_2\left(x_i^m x_j^n d_2\left(x_k^t x_l^r\right)\right) = \widehat{A}\left(x_i^m x_j^n d_2\left(x_k^t x_l^r\right)\right) = d_2\left(x_i^m x_j^n\right) \Lambda d_2\left(x_k^t x_l^r\right),$$

α is surjective. Therefore, we have

$$\Omega_2(R) \xrightarrow{\Delta_2} J_2(\Omega_2(R)) \xrightarrow{\alpha} \Lambda^2(\Omega_2(R)) \to 0$$

an exact sequence of R-modules. It suffice to show that the sequence is exact at $J_2(\Omega_2(R))$. Note that, $Im\Delta_2$ is generated by the set

$$\left\{\Delta_2(d_2(x_i)), \Delta_2\left(d_2\left(x_i x_j\right)\right)\right\}$$

for $i, j = 1, 2, ..., s$ and that

$$d_2(x_i)\Lambda d_2\left(x_j\right) + d_2\left(x_j\right)\Lambda d_2(x_i) = 0.$$

Therefore, we have

$$\alpha\Delta_2(d_2(x_i)) = \widehat{A}(d_2(x_i)) = d_2(1)\Lambda d_2(x_i) = 0$$

$$\alpha\Delta_2\left(d_2\left(x_i x_j\right)\right) = \widehat{A}\left(d_2\left(x_i x_j\right)\right) = d_2(1)\Lambda d_2\left(x_i x_j\right) = 0$$

this implies that $Im\Delta_2$ is contained in $\ker\alpha$. Hence, we have an induced map

$$\gamma_3 : J_2(\Omega_2(R))/Im\Delta_2 \to \Lambda^2(\Omega_2(R))$$

defined by

$$\gamma_3\left(\overline{\Delta_2\left(x_i^m x_j^n d_2\left(x_k^t x_l^r\right)\right)}\right) = d_2\left(x_i^m x_j^n\right)\Lambda d_2\left(x_k^t x_l^r\right).$$

Now, $\Lambda^2(\Omega_2(R))$ is generated by the set

$$\left\{d_2\left(x_i^m x_j^n\right)\Lambda d_2\left(x_k^t x_l^r\right) : m, n, t, r = 0, 1, 2 \text{ and } i, j, k, l = 1, 2, ..., s\right\}.$$

We can define a map

$$\gamma_4 : \Lambda^2(\Omega_2(R)) \to J_2(\Omega_2(R))/Im\Delta_2$$

defined by

$$\gamma_4\left(d_2\left(x_i^m x_j^n\right)\Lambda d_2\left(x_k^t x_l^r\right)\right) = \overline{\Delta_2\left(x_i^m x_j^n d_2\left(x_k^t x_l^r\right)\right)}.$$

It is clear that, $\gamma_3\gamma_4$ and $\gamma_4\gamma_3$ are identities and so, $\ker\gamma_3 = \ker\alpha/Im\Delta_2 = 0$ and then, $\ker\alpha = Im\Delta_2$. Therefore, the sequence is exact.

Now, we are ready to proof of the main theorem:

Proof. We can define the map $\Delta_2 : \Omega_2(R) \to J_2(\Omega_2(R))$ defined by

$$\Delta_2\left(\sum_{i,j,k,l} x_i^m x_j^n d_2\left(x_k^t x_l^r\right)\right) = 1/2\left[\begin{array}{c} \Delta_2\left(x_i^m x_j^n d_2\left(x_k^t x_l^r\right)\right) + \Delta_2\left(x_k^t x_l^r d_2\left(x_i^m x_j^n\right)\right) + x_i^m x_j^n \Delta_2\left(d_2\left(x_k^t x_l^r\right)\right) \\ -x_k^t x_l^r \Delta_2\left(d_2\left(x_i^m x_j^n\right)\right) \end{array}\right]$$

and similarly the map

$$i_1 : S^2(\Omega_2(R)) \to J_2(\Omega_2(R))$$

defined by

$$i_1\left(x_i^m x_j^n d_2\left(x_k^t x_l^r\right)\right) = 1/2\left[\begin{array}{c} \Delta_2\left(x_i^m x_j^n d_2\left(x_k^t x_l^r\right)\right) - \Delta_2\left(x_k^t x_l^r d_2\left(x_i^m x_j^n\right)\right) - x_i^m x_j^n \Delta_2\left(d_2\left(x_k^t x_l^r\right)\right) \\ +x_k^t x_l^r \Delta_2\left(d_2\left(x_i^m x_j^n\right)\right) \end{array}\right].$$

Then, we define the following map

$$h : \Omega_2(R) \oplus S^2(\Omega_2(R)) \to J_2(\Omega_2(R))$$

$$h(a,b) = \alpha\Delta_2(a) + \alpha_1 i_1(b)$$

$$= 0 + 0 = 0$$

where $a \in \Omega_2(R)$ and $b \in S^2(\Omega_2(R))$. Therefore,

$$\ker\alpha \approxeq \Omega_2(R) \oplus S^2(\Omega_2(R)).$$

Now, let us prove the splitting at the short exact sequence. Let,

$$\frown{A} : \Omega_2(R) \to \Lambda^2(\Omega_2(R))$$

defined by

$$\frown{A}\left(\sum_{i,j,k,l} x_i^m x_j^n d_2\left(x_k^t x_l^r\right)\right) = \sum_{i,j,k,l} d_2\left(x_i^m x_j^n\right) \Lambda d_2\left(x_k^t x_l^r\right)$$

and

$$\smile{A} : \Omega_2(R) \to S^2(\Omega_2(R))$$

defined by

$$\smile{A}\left(\sum_{i,j,k,l} x_i^m x_j^n d_2\left(x_k^t x_l^r\right)\right) = \sum_{i,j,k,l} d_2\left(x_i^m x_j^n\right) \vee d_2\left(x_k^t x_l^r\right).$$

Therefore, $\breve{A}$ and $\widehat{A}$ are second order derivations. So, by the universal property of $J_2(\Omega_2(R))$, there exist R-module homomorphism

$$\alpha : J_2(\Omega_2(R)) \to \Lambda^2(\Omega_2(R))$$

$$\alpha_1 : J_2(\Omega_2(R)) \to S^2(\Omega_2(R))$$

such that the following diagrams

$$\begin{array}{ccc} \Omega_2(R) & \overset{\widehat{A}}{\to} & \Lambda^2(\Omega_2(R)) \\ \Delta_2 \downarrow & & \downarrow 1_{\Lambda^2(\Omega_2(R))} \\ J_2(\Omega_2(R)) & \underset{\alpha}{\to} & \Lambda^2(\Omega_2(R)) \end{array}$$

and

$$\begin{array}{ccc} \Omega_2(R) & \overset{\breve{A}}{\to} & S^2(\Omega_2(R)) \\ \Delta_2 \downarrow & & \downarrow 1_{S^2(\Omega_2(R))} \\ J_2(\Omega_2(R)) & \underset{\alpha_1}{\to} & S^2(\Omega_2(R)) \end{array}$$

are commutes.So, we can write

$$\alpha\Delta_2\left(\sum_{i,j,k,l} x_i^m x_j^n d_2\left(x_k^t x_l^r\right)\right) = \widehat{A}\left(\sum_{i,j,k,l} x_i^m x_j^n d_2\left(x_k^t x_l^r\right)\right) = \sum_{i,j,k,l} d_2\left(x_i^m x_j^n\right)\Lambda d_2\left(x_k^t x_l^r\right)$$

and

$$\alpha_1\Delta_2\left(\sum_{i,j,k,l} x_i^m x_j^n d_2\left(x_k^t x_l^r\right)\right) = \breve{A}\left(\sum_{i,j,k,l} x_i^m x_j^n d_2\left(x_k^t x_l^r\right)\right) = \sum_{i,j,k,l} d_2\left(x_i^m x_j^n\right)\vee d_2\left(x_k^t x_l^r\right).$$

From here, we obtain

$$\alpha\Delta_2 = 1_{\Lambda^2(\Omega_2(R))}$$

$$\alpha_1 i_1 = 1_{S^2(\Omega_2(R))}$$

$$\alpha i_1 = 0$$

$$\alpha_1\Delta_2 = 0.$$

So, the sequence is splits.

Example 3.1. Let $R = k[a,b]$ be a polynomial algebra of dimension 2. Then, $\Omega_2(R)$ is a free R-module of rank $\begin{pmatrix} 4 \\ 2 \end{pmatrix} - 1 = 5$ with basis

$$\left\{d_2(a), d_2(b), d_2(ab), d_2\left(a^2\right), d_2\left(b^2\right)\right\}$$

and $S^2(\Omega_2(R))$ is a free R-module of rank $\begin{pmatrix} 6 \\ 4 \end{pmatrix} = 15$ with basis

$$\left\{\begin{array}{c} d_2(a)\vee d_2(a), d_2(a)\vee d_2(b), d_2(a)\vee d_2(ab), d_2(a)\vee d_2\left(a^2\right), d_2(a)\vee d_2\left(b^2\right), d_2(b)\vee d_2(b) \\ , d_2(b)\vee d_2(ab), d_2(b)\vee d_2\left(a^2\right), d_2(b)\vee d_2\left(b^2\right), d_2(ab)\vee d_2(ab), d_2(ab)\vee d_2\left(a^2\right) \\ , d_2(ab)\vee d_2\left(b^2\right), d_2\left(a^2\right)\vee d_2\left(a^2\right), d_2\left(a^2\right)\vee d_2\left(b^2\right), d_2\left(b^2\right)\vee d_2\left(b^2\right) \end{array}\right\}$$

and $\Lambda^2(\Omega_2(R))$ is a free R-module of rank $\begin{pmatrix} 5 \\ 2 \end{pmatrix} = 10$ with basis

$$\left\{\begin{array}{c} d_2(a)\Lambda d_2(b), d_2(a)\Lambda d_2(ab), d_2(a)\Lambda d_2\left(a^2\right), d_2(a)\Lambda d_2\left(b^2\right), d_2(b)\Lambda d_2(ab), d_2(b)\Lambda d_2\left(a^2\right) \\ , d_2(b)\Lambda d_2\left(b^2\right), d_2(ab)\Lambda d_2\left(a^2\right), d_2(ab)\Lambda d_2\left(b^2\right), d_2\left(a^2\right)\Lambda d_2\left(b^2\right) \end{array}\right\}$$

and $J_2(\Omega_2(R))$ is generated by the set

$$\left\{\begin{array}{c} \Delta_2(d_2(a)), \Delta_2(d_2(b)), \Delta_2(d_2(ab)), \Delta_2\left(d_2\left(a^2\right)\right), \Delta_2\left(d_2\left(b^2\right)\right), \Delta_2(ad_2(a)) \\ , \Delta_2(ad_2(b)), \Delta_2(ad_2(ab)), \Delta_2\left(ad_2\left(a^2\right)\right), \Delta_2\left(ad_2\left(b^2\right)\right), \Delta_2(bd_2(a)), \Delta_2(bd_2(b)) \\ , \Delta_2(bd_2(ab)), \Delta_2\left(bd_2\left(a^2\right)\right), \Delta_2\left(bd_2\left(b^2\right)\right), \Delta_2(abd_2(a)), \Delta_2(abd_2(b)) \\ , \Delta_2(abd_2(ab)), \Delta_2\left(abd_2\left(a^2\right)\right), \Delta_2\left(abd_2\left(b^2\right)\right), \Delta_2\left(a^2d_2(a)\right), \Delta_2\left(a^2d_2(b)\right) \\ , \Delta_2\left(a^2d_2(ab)\right), \Delta_2\left(a^2d_2\left(a^2\right)\right), \Delta_2\left(a^2d_2\left(b^2\right)\right), \Delta_2\left(b^2d_2(a)\right) \\ , \Delta_2\left(b^2d_2(b)\right), \Delta_2\left(b^2d_2(ab)\right), \Delta_2\left(b^2d_2\left(a^2\right)\right), \Delta_2\left(b^2d_2\left(b^2\right)\right) \end{array}\right\}$$

so, we see that the isomorphism from ranks of $\Omega_2(R), S^2(\Omega_2(R)), \Lambda^2(\Omega_2(R))$. Therefore, we obtained the isomorphism

$$J_2(\Omega_2(R)) \cong \Omega_2(R) \oplus S^2(\Omega_2(R)) \oplus \Lambda^2(\Omega_2(R)).$$

Example 3.2. Let $R=k[a,b,c]$ be a polynomial algebra of dimension 3. Then, $\Omega_2(R)$ is a free R-module of rank $\begin{pmatrix} 5 \\ 3 \end{pmatrix}-1=9$ with basis

$$\left\{d_2(a),d_2(b),d_2(c),d_2(ab),d_2(ac),d_2(bc),d_2\left(a^2\right),d_2\left(b^2\right),d_2\left(c^2\right)\right\}$$

and $S^2(\Omega_2(R))$ is a free R-module of rank $\begin{pmatrix} 10 \\ 8 \end{pmatrix}=45$ with basis

$$\left\{\begin{array}{c}
d_2(a)\vee d_2(a),d_2(a)\vee d_2(b),d_2(a)\vee d_2(c),d_2(a)\vee d_2(ab),d_2(a)\vee d_2(ac),d_2(a)\vee d_2(bc)\\
,d_2(a)\vee d_2\left(a^2\right),d_2(a)\vee d_2\left(b^2\right),d_2(a)\vee d_2\left(c^2\right),d_2(b)\vee d_2(b),d_2(b)\vee d_2(c),d_2(b)\vee d_2(ab)\\
,d_2(b)\vee d_2(ac),d_2(b)\vee d_2(bc),d_2(b)\vee d_2\left(a^2\right),d_2(b)\vee d_2\left(b^2\right),d_2(b)\vee d_2\left(c^2\right),d_2(c)\vee d_2(c)\\
,d_2(c)\vee d_2(ab),d_2(c)\vee d_2(ac),d_2(c)\vee d_2(bc),d_2(c)\vee d_2\left(a^2\right),d_2(c)\vee d_2\left(b^2\right),d_2(c)\vee d_2\left(c^2\right)\\
,d_2(ab)\vee d_2(ab),d_2(ab)\vee d_2(ac),d_2(ab)\vee d_2(bc),d_2(ab)\vee d_2\left(a^2\right),d_2(ab)\vee d_2\left(b^2\right),d_2(ab)\vee d_2\left(c^2\right)\\
,d_2(ac)\vee d_2(ac),d_2(ac)\vee d_2(bc),d_2(ac)\vee d_2\left(a^2\right),d_2(ac)\vee d_2\left(b^2\right),d_2(ac)\vee d_2\left(c^2\right),d_2(bc)\vee d_2(bc)\\
,d_2(bc)\vee d_2\left(a^2\right),d_2(bc)\vee d_2\left(b^2\right),d_2(bc)\vee d_2\left(c^2\right),d_2\left(a^2\right)\vee d_2\left(a^2\right),d_2\left(a^2\right)\vee d_2\left(b^2\right),d_2\left(a^2\right)\vee d_2\left(c^2\right)\\
,d_2\left(b^2\right)\vee d_2\left(b^2\right),d_2\left(b^2\right)\vee d_2\left(c^2\right),d_2\left(c^2\right)\vee d_2\left(c^2\right)
\end{array}\right\}$$

and $\Lambda^2(\Omega_2(R))$ is a free R-module of rank $\begin{pmatrix} 9 \\ 2 \end{pmatrix}=36$ with basis

$$\left\{\begin{array}{c}
d_2(a)\Lambda d_2(b),d_2(a)\Lambda d_2(c),d_2(a)\Lambda d_2(ab),d_2(a)\Lambda d_2(ac),d_2(a)\Lambda d_2(bc),d_2(a)\Lambda d_2\left(a^2\right)\\
,d_2(a)\Lambda d_2\left(b^2\right),d_2(a)\Lambda d_2\left(c^2\right),d_2(b)\Lambda d_2(c),d_2(b)\Lambda d_2(ab),d_2(b)\Lambda d_2(ac),d_2(b)\Lambda d_2(bc)\\
,d_2(b)\Lambda d_2\left(a^2\right),d_2(b)\Lambda d_2\left(b^2\right),d_2(b)\Lambda d_2\left(c^2\right),d_2(c)\Lambda d_2(ab),d_2(c)\Lambda d_2(ac),d_2(c)\Lambda d_2(bc)\\
,d_2(c)\Lambda d_2\left(a^2\right),d_2(c)\Lambda d_2\left(b^2\right),d_2(c)\Lambda d_2\left(c^2\right),d_2(ab)\Lambda d_2(ac),d_2(ab)\Lambda d_2(bc),d_2(ab)\Lambda d_2\left(a^2\right)\\
,d_2(ab)\Lambda d_2\left(b^2\right),d_2(ab)\Lambda d_2\left(c^2\right),d_2(ac)\Lambda d_2(bc),d_2(ac)\Lambda d_2\left(a^2\right),d_2(ac)\Lambda d_2\left(b^2\right),d_2(ac)\Lambda d_2\left(c^2\right)\\
,d_2(bc)\Lambda d_2\left(a^2\right),d_2(bc)\Lambda d_2\left(b^2\right),d_2(bc)\Lambda d_2\left(c^2\right),d_2\left(a^2\right)\Lambda d_2\left(b^2\right),d_2\left(a^2\right)\Lambda d_2\left(c^2\right),d_2\left(b^2\right)\Lambda d_2\left(c^2\right)
\end{array}\right\}$$

and $J_2(\Omega_2(R))$ is generated by the set

$$\left\{\begin{array}{c}
\Delta_2(d_2(a)),\Delta_2(d_2(b)),\Delta_2(d_2(c)),\Delta_2(d_2(ab)),\Delta_2(d_2(ac)),\Delta_2(d_2(bc))\\
,\Delta_2(d_2(a^2)),\Delta_2(d_2(b^2)),\Delta_2(d_2(c^2)),\Delta_2(ad_2(a)),\Delta_2(ad_2(b)),\Delta_2(ad_2(c))\\
,\Delta_2(ad_2(ab)),\Delta_2(ad_2(ac)),\Delta_2(ad_2(bc)),\Delta_2(ad_2(a^2)),\Delta_2(ad_2(b^2)),\Delta_2(ad_2(c^2))\\
,\Delta_2(bd_2(a)),\Delta_2(bd_2(b)),\Delta_2(bd_2(c)),\Delta_2(bd_2(ab)),\Delta_2(bd_2(ac)),\Delta_2(bd_2(bc))\\
,\Delta_2(bd_2(a^2)),\Delta_2(bd_2(b^2)),\Delta_2(bd_2(c^2)),\Delta_2(cd_2(a)),\Delta_2(cd_2(b)),\Delta_2(cd_2(c))\\
,\Delta_2(cd_2(ab)),\Delta_2(cd_2(ac)),\Delta_2(cd_2(bc)),\Delta_2(cd_2(a^2)),\Delta_2(cd_2(b^2)),\Delta_2(cd_2(c^2))\\
,\Delta_2(abd_2(a)),\Delta_2(abd_2(b)),\Delta_2(abd_2(c)),\Delta_2(abd_2(ab)),\Delta_2(abd_2(ac)),\Delta_2(abd_2(bc))\\
,\Delta_2(abd_2(a^2)),\Delta_2(abd_2(b^2)),\Delta_2(abd_2(c^2)),\Delta_2(acd_2(a)),\Delta_2(acd_2(b)),\Delta_2(acd_2(c))\\
,\Delta_2(acd_2(ab)),\Delta_2(acd_2(ac)),\Delta_2(acd_2(bc)),\Delta_2(acd_2(a^2)),\Delta_2(acd_2(b^2)),\Delta_2(acd_2(c^2))\\
,\Delta_2(bcd_2(a)),\Delta_2(bcd_2(b)),\Delta_2(bcd_2(c)),\Delta_2(bcd_2(ab)),\Delta_2(bcd_2(ac)),\Delta_2(bcd_2(bc))\\
,\Delta_2(bcd_2(a^2)),\Delta_2(bcd_2(b^2)),\Delta_2(bcd_2(c^2)),\Delta_2(a^2d_2(a)),\Delta_2(a^2d_2(b)),\Delta_2(a^2d_2(c))\\
,\Delta_2(a^2d_2(ab)),\Delta_2(a^2d_2(ac)),\Delta_2(a^2d_2(bc)),\Delta_2(a^2d_2(a^2)),\Delta_2(a^2d_2(b^2)),\Delta_2(a^2d_2(c^2))\\
,\Delta_2(b^2d_2(a)),\Delta_2(b^2d_2(b)),\Delta_2(b^2d_2(c)),\Delta_2(b^2d_2(ab)),\Delta_2(b^2d_2(ac)),\Delta_2(b^2d_2(bc))\\
,\Delta_2(b^2d_2(a^2)),\Delta_2(b^2d_2(b^2)),\Delta_2(b^2d_2(c^2)),\Delta_2(c^2d_2(a)),\Delta_2(c^2d_2(b)),\Delta_2(c^2d_2(c))\\
,\Delta_2(c^2d_2(ab)),\Delta_2(c^2d_2(ac)),\Delta_2(c^2d_2(bc)),\Delta_2(c^2d_2(a^2)),\Delta_2(c^2d_2(b^2)),\Delta_2(c^2d_2(c^2))
\end{array}\right\}$$

so, we see that the isomorphism from ranks of $\Omega_2(R), S^2(\Omega_2(R)), \Lambda^2(\Omega_2(R))$. Therefore, we obtained the isomorphism

$$J_2(\Omega_2(R)) \cong \Omega_2(R) \oplus S^2(\Omega_2(R)) \oplus \Lambda^2(\Omega_2(R)).$$

Conflict of Interests

The authors declare that there is no conflict of interests.

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