Available online at http://scik.org
Algebra Letters, 2017, 2017:5
ISSN: 2051-5502

SOME RESULTS ON KÄHLER MODULES

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Abstract: Let k be an algebraically closed field of characteristic zero, R, S affine k-algebras and let$\Omega(q)(R/k)$ and $\Omega(q)(S/k)$ denote their universal finite Kähler modules of differentials over k. Our paper is concerned with the relationship between $\Omega(q)(R/k)$ and $\Omega(q)(S/k)$ when there is a ring homomorphism θ: $R \longrightarrow S$.

Keywords: Kähler module; regular ring; global dimension; Krull dimension; projective dimension.

2010 AMS Subject Classification: 13N05.

1. Introduction

The concept of a Kähler module or high order derivations of q th order was introduced by H.Osborn in 1965 [8]. Same notion has appeared in R.G. Heyneman and M.E. Sweedler [10]. J. Johnson introduced differential module structures on certain modules of Kähler differentials [3]. Y. Nakai's paper [5], he developed the fundamental theories for the calculus of high order derivations and some functorial properties of the module of high order differentials. Then many authors studied on the properties of Kähler modules [2,6,7,9].

Throughout this paper, we will let R be a commutative algebra over an algebraically closed field k with characteristic zero. When R is a k-algebra, $\Omega(q)(R/k)$ denotes the module of q-th order

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Received May 18, 2017

Kähler differentials of R over k and δ(q) denotes the canonical q-th order k derivation R$\rightarrow$ Ω(q)(R/k) of R. The pair {Ω(q)(R/k) , δ(q)} has the universal mapping property with respect to the q-th order k-derivations of R. Ω(q)(R/k) is generated by the set { δ(q)(r) : r$\in$R}. Hence if R is finitely generated k-algebra, Ω(q)(R/k) will be a finitely generated R-module.
In this paper, we will study the relationship between Ω(q)(R/k) and Ω(q)(S/k) when there is a ring homomorphism $\theta\colon R\rightarrow S$.As far as our knowledge concern, our studies are new results.

2. Preliminaries

In this section, we review some basic theorems and results. We use the following notations.

pd M : projective dimension of *M*

gld R: global dimension of *R*

dim R: Krull dimension of *R*

rk($\Omega^{(q)}(R/k)$)=rank of $\Omega^{(q)}(R/k)$

Let *R* and *S* be affine *k* algebras where $\theta\colon R\rightarrow S$ any ring homomorphism. Then any *S*-module *M* may be viewed as a *R*-module.

Lemma 2.1. [4]: Let $\theta\colon R\rightarrow S$ and M_S be given. If M_S is projective *S* module and S_R is projective then M_R is projective *R* module.

Proposition 2.2. [4]: Let $\theta\colon R\rightarrow S$ and M_S be given. Then $pdM_R\leq pd\,M_S+pd\,S_R$

Lemma 2.3. [4]: Let $R\subseteq S$ be rings with S_R faithfully flat and let *M* be a *R*-module. The map$M\rightarrow M\otimes_R S$ via $m\rightarrow m\otimes_R 1$ is an embedding.

Theorem 2.4. [4]: Let $R\subseteq S$ be rings with S_R being faithfully flat and $gld\,R<\infty$. If either

i) S_R is projective, or
ii) *R* is Noetherian and S_R is flat,

then $gld\,R\leq gld\,S$

Theorem 2.5. [4]: Let *R*,*S* be rings with $R \subseteq S$ such that *R* is an *R* bimodule direct summand of *S*. Then $\leq gld\ S + pd\ S_R$.

Proposition 2.6. [9]: Let *R* be an affine *k*-algebra. If *R* is a regular ring, then $J^{(q)}(R/k)$ is a projective *R*-module.

Theorem 2.7 Let *R* be an affine *k*-algebra. If *R* is a regular ring, then $\Omega^{(q)}(R/k)$ is a projective *R*-module.

Proof: This is immediate from the decomposition $J^{(q)}(R/k) = R \oplus \Omega^{(q)}(R/k)$ and from Proposition 2.6.

The following theorem characterizes regular rings.

Theorem 2.8[4] Let *R* be an affine *k*-algebra. *R* is regular ring if and only if $\Omega^{(1)}(R/k)$ is a projective *R*-module.

3. Main results

In this section, new results founded are given.

Theorem 3.1. Let *R*, *S* be affine *k*-algebras with $R \subseteq S$ such that *R* is an *R* bimodule direct summand of *S*. If *S* is a regular and S_R is projective then $\Omega^{(q)}(R/k)$ is projective *R*-module.

Proof: Let $S = R \oplus I$ and let $\Omega^{(q)}(R/k)$ be a q-th order Kahler *R*-module. Then

$$\Omega^{(q)}(R/k)\otimes_R S \cong \Omega^{(q)}(R/k)\otimes_{\mathrm{R}} (R \oplus I) \cong (\Omega^{(q)}(R/k)\otimes_{\mathrm{R}} R) \oplus (\Omega^{(q)}(R/k)\otimes_{\mathrm{R}} I).$$

And from the isomorphism

$$\Omega^{(q)}(R/k)\otimes_{\mathrm{R}} R \cong \Omega^{(q)}(R/k)$$

we have

$$\Omega^{(q)}(R/k)\otimes_R S \cong \Omega^{(q)}(R/k) \oplus \Omega^{(q)}(R/k)\otimes_{\mathrm{R}} I)$$

From homological properties, we obtain the following inequality.

$$pd\ \Omega^{(q)}(R/k)_R \leq \mathrm{pd}(\Omega^{(q)}(R/k)\otimes_R S)_{\mathrm{R}}$$

$\Omega^{(q)}(R/k)\otimes_R S$ is isomorphic to $\Omega^{(q)}(S/k)_{\mathrm{R}}$ as *R*-module. Hence by Proposition 2.2. we have the following inequality.

$pd\ \Omega^{(q)}(R/k)_R \leq \mathrm{pd}(\Omega^{(q)}(R/k) \otimes_R S)_{\mathrm{R}} = \mathrm{pd}(\Omega^{(q)}(S/k))_{\mathrm{R}} \leq \mathrm{pd}\ (\Omega^{(q)}(S/k))_{\mathrm{S}} + \mathrm{pd} S_R$

By Theorem 2.5 if S regular ring then $\Omega^{(q)}(S/k)$ is a projective *S*-module. That is $\mathrm{pd}\ (\Omega^{(q)}(S/k))_{\mathrm{S}} = 0$. Similarly, $\mathrm{pd} S_R = 0$since S_R is projective *R*-module.

Finally, we find that projective dimension of $\Omega^{(q)}(R/k)_R$ is zero. So$\Omega^{(q)}(R/k)_R$ is projective.

Corollary 3.2. Let *R*, *S* be affine *k*-algebras with $R \subseteq S$ such that *R* is an *R* bimodule direct summand of *S*. If *S* is a regular and S_R is projective then *R* is regular ring.

Proof: By Theorem 3.1 in the same conditions we know that $\Omega^{(q)}(R/k)_R$ is projective *R*-module for all $q > 0$. Then $\Omega^{(1)}(R/k)_R$ is projective*R*-module. If $\Omega^{(1)}(R/k)_R$ is projective *R*-module then *R* is regular ring by Theorem 2. 8.

Theorem 3.3. Let *R*, *S* be affine *k*-algebras with $R \subseteq S$ such that *R* is an *R* bimodule direct summand of *S*. If projective dimension of $(\Omega^{(q)}(S/k))_{\mathrm{S}}$is finite and S_R is projective then projective dimension of $\Omega^{(q)}(R/k)$ is finite.

Proof: Similarly, we have the following inequality

$pd\ \Omega^{(q)}(R/k)_R \leq \mathrm{pd}(\Omega^{(q)}(R/k) \otimes_R S)_{\mathrm{R}} = \mathrm{pd}(\Omega^{(q)}(S/k))_{\mathrm{R}} \leq \mathrm{pd}\ (\Omega^{(q)}(S/k))_{\mathrm{S}} + \mathrm{pd} S_R$

by using the proof of Theorem 3.1.

So $pd\ \Omega^{(q)}(R/k)_R \leq \mathrm{pd}\ (\Omega^{(q)}(S/k))_{\mathrm{S}} + \mathrm{pd} S_R$ and $\mathrm{pd} S_R = 0$since $S_R$is projective *R*-module.

We have the inequality $pd\ \Omega^{(q)}(R/k)_R \leq \mathrm{pd}\ (\Omega^{(q)}(S/k))_{\mathrm{S}}$ and therefore it is obtained$\mathrm{pd}\ (\Omega^{(q)}(R/k))_{\mathrm{R}} < \infty$ as required.

Now we have the following result.

Corollary 3.4. Let *R*, *S* be affine *k*-algebras with $R \subseteq S$ such that *R* is an *R* bimodule direct summand of *S*. If $\mathrm{pd}\ (\Omega^{(q)}(R/k))_{\mathrm{R}} = \infty$and S_R is projective then $\mathrm{pd}\ (\Omega^{(q)}(S/k))_{\mathrm{S}} = \infty$.

Proof: If we have the inequality

$pd\ \Omega^{(q)}(R/k)_R \leq \mathrm{pd}(\Omega^{(q)}(R/k)\otimes_R S)_{\mathrm{R}} = \mathrm{pd}(\Omega^{(q)}(S/k))_{\mathrm{R}} \leq \mathrm{pd}\ (\Omega^{(q)}(S/k))_{\mathrm{S}} + \mathrm{pd}S_R$

then it can be shown easily.

Theorem 3.5. Let *R*, *S* be affine *k*-algebras with $R \subseteq S$. Then there exists an exact sequence of *R*-modules

$$0 \to Ker\beta \to \Omega^{(q)}(R/k) \xrightarrow{\beta} \Omega^{(q)}(S/k) \to Coker\beta \to 0$$

If *R* is domain such that *dimR*=*dimS* then $Ker\beta$ and $Coker\beta$ are torsion modules.

Proof: Let *R*, *S* be affine *k*-algebras where $\theta: R \to S$ any *k*-algebra homomorphism with $R \subseteq S$ and $\Omega^{(q)}(R/k)$, $\Omega^{(q)}(S/k)$, denote the modules of q-th order Kähler differentials of *R* and *S* over *k* respectively and $\delta_R{}^{(q)}$, $\delta_S{}^{(q)}$ denote the canonical *q*-th order *k* derivation $R\to \Omega^{(q)}(R/k)$, $S\to \Omega^{(q)}(S/k)$ of *R* and *S* respectively.

By the universal mapping property of $\Omega^{(q)}(R/k)$ there exists a unique *R*-module homomorphism $\beta: \Omega^{(q)}(R/k) \to \Omega^{(q)}(S/k)$ such that $\beta\delta_R{}^{(q)} = \delta_S{}^{(q)}\theta$ and the following diagramcommutes.

$$\begin{array}{ccc} R & \xrightarrow{\theta} & S \\ \delta_R{}^{(q)}\downarrow & & \delta_S{}^{(q)}\downarrow \\ \Omega^{(q)}(R/k) & \xrightarrow{\beta} & \Omega^{(q)}(S/k) \end{array}$$

Therefore, we have

$$0 \to Ker\beta \to \Omega^{(q)}(R/k) \xrightarrow{\beta} \Omega^{(q)}(S/k) \to Coker\beta \to 0$$

an exact sequence of *R*-modules.

Let *dimR*=*dimS* and Q(R) be the field of fractions of *R*. Then $rk(\Omega^{(q)}(R/k)) = rk(\Omega^{(q)}(S/k))$ and therefore we have the following isomorphism

$\Omega^{(q)}(R/k) \otimes_{\mathrm{R}} \mathrm{Q(R)} \simeq \Omega^{(q)}(S/k) \otimes_{\mathrm{R}} \mathrm{Q(R)}$

This implies that

$Ker\beta \otimes_{\mathrm{R}} \mathrm{Q(R)} = 0$

and

$Coker\beta \otimes_R Q(R) = 0$

Hence $Ker\beta$ and $Coker\beta$ are torsion modules.

Corollary 3.6. Let R, S be affine k-algebras where $\theta: R \to S$ any k-algebra homomorphism with $R \subseteq S$. If θ is onto then there exists a short exact sequence of R-modules

$$0 \to Ker\beta \to \Omega^{(q)}(R/k) \xrightarrow{\beta} \Omega^{(q)}(S/k) \to 0$$

Proof: As the proof of Theorem 3.5. by the universal mapping property of $\Omega^{(q)}(R/k)$ there exists a unique R-module homomorphism $\beta: \Omega^{(q)}(R/k) \to \Omega^{(q)}(S/k)$such that $\beta\delta_R{}^{(q)} = \delta_S{}^{(q)}\theta$ and the following diagram commutes.

$$\begin{array}{ccc} R & \xrightarrow{\theta} & S \\ \delta_R{}^{(q)}\downarrow & & \delta_S{}^{(q)}\downarrow \\ \Omega^{(q)}(R/k) & \xrightarrow{\beta} & \Omega^{(q)}(S/k) \end{array}$$

The generating set of $\Omega^{(q)}(S/k)$ is $\{\delta_S{}^{(q)}(s) : \quad s \in S\}$. Since θ is onto it is clear that $\beta: \Omega^{(q)}(R/k) \to \Omega^{(q)}(S/k)$ is onto. Therefore, we have

$$0 \to Ker\beta \to \Omega^{(q)}(R/k) \xrightarrow{\beta} \Omega^{(q)}(S/k) \to 0$$

a short exact sequence of R-modules.

Corollary 3.7. If $Ker\beta$is torsion submodule of $\Omega^{(q)}(R/k)$ then we have the following R-module isomorphism.

$$Hom_R(\Omega^{(q)}(R/k), R) \cong Hom_R(\Omega^{(q)}(R/k)/Ker\beta, R)$$

Proof: Since $Ker\beta$ is submodule of $\Omega^{(q)}(R/k)$ we have the following a short exact sequence of R-modules

$$0 \to Ker\beta \to \Omega^{(q)}(R/k) \to \Omega^{(q)}(R/k)/\mathrm{Ker}\,\beta \to 0$$

From the homological algebra we have the following exact sequence of R-modules

$$0 \to Hom_R(\Omega^{(q)}(R/k)/Ker\beta, R) \to Hom_R(\Omega^{(q)}(R/k), R) \to Hom_R(Ker\beta, R) \to 0$$

$Hom_R(Ker\beta, R) = 0$ since $Ker\beta$ is torsion submodule of $\Omega^{(q)}(R/k)$.

Hence, we obtain the required result

Conflict of Interests

The authors declare that there is no conflict of interests.

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