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A NOTE ON ISBELL'S ZIGZAG THEOREM FOR COMMUTATIVE SEMIGROUPS

NOOR ALAM* AND NOOR MOHAMMAD KHAN

Department of Mathematics, Aligarh Muslim University, Aligarh-202002, India

Abstract. We have given a new and short proof of the Isbell's Zigzag Theorem for the category of all commutative semigroups.

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1. Introduction

In [5], Howie and Isbell have extended Isbell's Zigzag Theorem, by using free products of commutative semigroups, for the category of all commutative semigroups. Stenstrom [8], by using tensor product of monoids, provided a new proof of the celebrated Isbell's Zigzag Theorem in the category of all semigroups. In this paper, we provide, based on Stenstrom's approach, a new algebraic proof of the Howie and Isbell's result [6, Theorem 1.1] for the category of all commutative semigroups.

2. Preliminaries

Let U and S be any semigroups with U a subsemigroup of S in a category C of semigroups. We say that U dominates an element d of S in C if for every semigroup $T \in$ C and for all homomorphisms $\alpha, \beta : S \to T$, $u\alpha = u\beta$ for all $u \in U$ implies $d\alpha = d\beta$. The set of all elements of S dominated by U is called the dominion of U in S, and we denote it by $Dom_{\mathrm{C}}(U, S)$. It can be easily seen that $Dom_{\mathrm{C}}(U, S)$ is a subsemigroup of S containing U. A morphism $\alpha : S \to T$ in C is said to be an epimorphism (epi for short) if for all morphims β, γ, $\alpha\beta = \alpha\gamma$ in C implies $\beta = \gamma$ (where β, γ are semigroup morphisms). It

*Corresponding author

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can be easily checked that $\alpha : S \to T$ is epi in C if and only if $i : S\alpha \to T$ is epi in C and the inclusion map $i : U \to S$ is epi in C if and only if $Dom_{\mathrm{C}}(U, S) = S$. Note that when C is the category of all semigroups, the above definitions which have been first given by Hall and Jones [2] are precisely those of Howie and Isbell [5] and Isbell [6].

Let S be a semigroup with identity 1 and A be any non-empty set. Then A is said to be a right S-system if there exists a mapping $(x, s) \mapsto xs$ from $A \times S$ into A such that $(xs)t = x(st)$ for all $x \in A, s, t \in S$ and $x1 = x$ for all $x \in A$. Dually, we may define a left S-system B.

Let A be a right S-system and B be a left S-system and let τ be the equivalence relation on $A \times B$ generated by the relation $T = \{((as, b), (a, sb)) : a \in A, b \in B, s \in S\}$. Then $A \times B/\tau$ is called the tensor product of A and B over S and is denoted by $A \otimes_S B$. We also denote an element $(a, b)\tau$ of $A \otimes_S B$ by $a \otimes b$.

For any unexplained notations and conventions, one may refer to Clifford and Preston [1] and Howie [4]. We shall also use the notation $Dom(U, S)$, when it is clear from the context, for the dominion of U in S both in the category of all semigroups as well as in the category of all commutative semigroups.

A most useful characterization of semigroup dominions is provided by Isbell's Zigzag Theorem.

Result 2.1([4, Theorem 8.3.4]). Let U be a submonoid of a monoid S and let $d \in S$. Then $d \in Dom(U, S)$ if and only if $d \in U$ or there exists a series of factorizations of d as follows:
$d = a_1 s_1 = a_1 t_1 b_1 = a_2 s_2 b_1 = a_2 t_2 b_2 = \ldots = a_{n-1} t_{n-1} b_{n-1} = s_n b_{n-1}$,
where $n \geq 1, s_i, t_i \in U$, $a_i, b_i \in S$ and

$$\begin{aligned} d &= a_1 s_1, & s_1 &= t_1 b_1 \\ a_i t_i &= a_{i+1} s_{i+1}, & s_{i+1} b_i &= t_{i+1} b_{i+1} \quad (i = 2, \ldots, n-2) \\ a_{n-1} t_{n-1} &= s_n, & s_n b_{n-1} &= d. \end{aligned}$$

Such a series of factorization is called a zigzag in S over U with value d, length n and spine $s_1, \ldots, s_n, t_1, \ldots, t_{n-1}$. We refer to the equations in Result 1.1 as *the zigzag equations*.

Result 2.2([4, Theorem 8.1.8]). Two elements $a \otimes b$ and $c \otimes d$ in $A \otimes_S B$ are equal if and only if $(a, b) = (c, d)$ or there exist $a_1, a_2, \ldots, a_{n-1}$ in A, $b_1, b_2, \ldots, b_{n-1}$ in B, $s_1, s_2, \ldots, s_n, t_1, t_2, \ldots, t_{n-1}$ in S such that

$$\begin{aligned} a &= a_1 s_1, & s_1 b &= t_1 b_1 \\ a_1 t_1 &= a_2 s_2, & s_2 b_1 &= t_2 b_2 \\ a_i t_i &= a_{i+1} s_{i+1}, & s_{i+1} b_i &= t_{i+1} b_{i+1} \quad (i = 2, \ldots, n-2) \\ a_{n-1} t_{n-1} &= c s_n, & s_n b_{n-1} &= d. \end{aligned}$$

3. Main Result

Theorem 3.1. *Let U be a submonoid of a commutative monoid S, Then d is in $Dom(U,S)$ if and only if either $d \in U$ or there exists a zigzag in S over U with value d.*

Proof. To prove the theorem, we, by Result 1.2, essentially show that if $d \in S$, then $d \in Dom(U,S)$ if and only if $d \otimes 1 = 1 \otimes d$ in $A = S \otimes_U S$, where 1 is the identity of S. So let us suppose first that $d \in S$ and $d \otimes 1 = 1 \otimes d$ in $A = S \otimes_U S$. Then, by Result 1.2, we have

$$\begin{aligned} d &= a_1 s_1, & s_1 &= t_1 b_1 \\ a_1 t_1 &= a_2 s_2, & s_2 b_1 &= t_2 b_2 \\ a_i t_i &= a_{i+1} s_{i+1}, & s_{i+1} b_i &= t_{i+1} b_{i+1} \quad (i = 2, \ldots, n-2) \\ a_{n-1} t_{n-1} &= s_n, & s_n b_{n-1} &= d; \end{aligned}$$

where $a_1, a_2, \ldots, a_{n-1}, b_1, b_2, \ldots, b_{n-1} \in S$ and $s_1, s_2, \ldots, s_n, t_1, t_2, \ldots, t_{n-1} \in U$.
Let T be a semigroup and let $\alpha, \beta : S \to T$ be homomorphisms agreeing on U; i.e.

$$\alpha \mid U = \beta \mid U$$

Now, by using zigzag equations, we have

$$\alpha(d) = \alpha(a_1 s_1) = \alpha(a_1)\alpha(s_1) = \alpha(a_1)\beta(t_1 b_1) = \alpha(a_1)\beta(t_1)\beta(b_1) = \alpha(a_1 t_1)\beta(b_1)$$
$$= \cdots = \alpha(a_{n-1} t_{n-1})\beta(b_{n-1}) = \alpha(s_n)\beta(b_{n-1}) = \beta(s_n b_{n-1}) = \beta(d)$$

$\Rightarrow d \in Dom(U,S)$.
To prove the converse, we first show that for a commutative monoid, the equivalence relation τ is a congruence; i.e.

$$(a,b)\tau(c,d)\tau = (ac, bd)\tau.$$

For this, we have to show that τ is compatible; i.e.
if $(a,b)\tau = (c,d)\tau$ and $(a',b')\tau = (c',d')\tau$, then $((a,b)(a',b'))\tau = ((c,d)(c',d'))\tau$.
Since $a \otimes b = c \otimes d$, by Result 1.2, we have

$$\begin{aligned} a &= a_1 s_1, & s_1 b &= t_1 b_1 \\ a_1 t_1 &= a_2 s_2, & s_2 b_1 &= t_2 b_2 \\ a_i t_i &= a_{i+1} s_{i+1}, & s_{i+1} b_i &= t_{i+1} b_{i+1} \quad (i = 2, \ldots, n-2) \\ a_{n-1} t_{n-1} &= c s_n, & s_n b_{n-1} &= d; \qquad (A) \end{aligned}$$

for some $a_1, a_2, \ldots, a_{n-1}, b_1, b_2, \ldots, b_{n-1} \in S$ and $s_1, s_2, \ldots, s_n, t_1, t_2, \ldots, t_{n-1} \in U$.

Similarly, as $a' \otimes b' = c' \otimes d'$, we have

$$\begin{aligned} a' &= a'_1 s'_1, & s'_1 b' &= t'_1 b'_1 \\ a'_1 t'_1 &= a'_2 s'_2, & s'_2 b'_1 &= t'_2 b'_2 \\ a'_i t'_i &= a'_{i+1} s'_{i+1}, & s'_{i+1} b'_i &= t'_{i+1} b'_{i+1} \quad (i = 2, \ldots, n-2) \\ a'_{n-1} t'_{n-1} &= c' s'_n, & s'_n b'_{n-1} &= d'; \qquad (B) \end{aligned}$$

for some $a'_1, a'_2, \ldots, a'_{n-1}, b'_1, b'_2, \ldots, b'_{n-1} \in S$ and $s'_1, s'_2, \ldots, s'_n, t'_1, t'_2, \ldots, t'_{n-1} \in U$.

Now, from equations (A) and (B), we have

$$\begin{aligned} aa' &= (a_1 s_1)(a'_1 s'_1), & (s_1 b)(s'_1 b') &= (t_1 b_1)(t'_1 b'_1) \\ (a_1 t_1)(a'_1 t'_1) &= (a_2 s_2)(a'_2 s'_2), & (s_2 b_1)(s'_2 b'_1) &= (t_2 b_2)(t'_2 b'_2) \\ (a_i t_i)(a'_i t'_i) &= (a_{i+1} s_{i+1})(a'_{i+1} s'_{i+1}), & (s_{i+1} b_i)(s'_{i+1} b'_i) &= (t_{i+1} b_{i+1})(t'_{i+1} b'_{i+1}) \\ & & (i = 2, \ldots, n-2) & \\ (a_{n-1} t_{n-1})(a'_{n-1} t'_{n-1}) &= (c s_n)(c' s'_n), & (s_n b_{n-1})(s'_n b'_{n-1}) &= dd'. \end{aligned}$$

Since, in the above system of equalities all members belong to S, so, by using commutativity of S, we have

$$\begin{aligned} aa' &= (a_1 a'_1)(s_1 s'_1), & (s_1 s'_1)(bb') &= (t_1 t'_1)(b_1 b'_1) \\ (a_1 a'_1)(t_1 t'_1) &= (a_2 a'_2)(s_2 s'_2), & (s_2 s'_2)(b_1 b'_1) &= (t_2 t'_2)(b_2 b'_2) \\ (a_i a'_i)(t_i t'_i) &= (a_{i+1} a'_{i+1})(s_{i+1} s'_{i+1}), & (s_{i+1} s'_{i+1})(b_i b'_i) &= (t_{i+1} t'_{i+1})(b_{i+1} b'_{i+1}) \\ & & (i = 2, \ldots, n-2) & \\ (a_{n-1} a'_{n-1})(t_{n-1} t'_{n-1}) &= (cc')(s_n s'_n), & (s_n s'_n)(b_{n-1} b'_{n-1}) &= dd'; \end{aligned}$$

where $a_1 a'_1, a_2 a'_2, \ldots, a_{n-1} a'_{n-1}, b_1 b'_1, b_2 b'_2, \ldots, b_{n-1} b'_{n-1} \in S$ and $s_1 s'_1, s_2 s'_2, \ldots, s_n s'_n, t_1 t'_1, t_2 t'_2, \ldots,$ $t_{n-1} t'_{n-1} \in U$.

Thus, by Result 1.2, we have

$aa' \otimes bb' = cc' \otimes dd' \Rightarrow (aa', bb')\tau = (cc', dd')\tau \Rightarrow ((a, b)(a', b'))\tau = ((c, d)(c', d'))\tau \Rightarrow \tau$ is a congruence.

Now define $\alpha : S \to S \times A$ and $\beta : S \to S \times A$
by

$$\alpha(s) = (s, s \otimes 1),\ \beta(s) = (s, 1 \otimes s).$$

Then α, β are, clearly, semigroup morphisms.

Since, $u \otimes 1 = 1 \otimes u$, we have

$\alpha(u) = \beta(u)$, for all $u \in U$.

Therefore $\alpha(d) = \beta(d)$

$\Rightarrow (d, d \otimes 1) = (d, 1 \otimes d)$

$\Rightarrow d \otimes 1 = 1 \otimes d$.

This completes the proof of the theorem. □

Thus we have the following:

Theorem 3.2. *If U is a submonoid of a commutative monoid S, then d is in $Dom(U, S)$ if and only if either $d \in U$ or there exists a zigzag in S over U with value d.* □

It may easily be verified that the arguments employed by Howie[4] in proving Theorems 8.3.4 to 8.3.5 work through to complete the proof for the following Isbell's Zigzag Theorem for the category of all commutative semigroups.

Theorem 3.3. *Let U be a subsemigroup of a commutative semigroup S. Then $d \in Dom(U, S)$ if and only if either $d \in U$ or there exists a zigzag in S over U with value d.* □

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