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ON χ-INJECTIVE MODULES

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Abstract. In this paper, we introduce the notion of χ-injective modules where χ denotes a collection of right ideals of a ring R. We establish various important properties of this module.

Keywords: χ-injective module, essential ideal, direct summand, divisible module..

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1. Introduction

The notion of injective modules was first introduced by Baer in 1940 in [2] in the form of divisible abelian groups. A right R-module M is said to be injective if it satisfies Baer's criteria of injectivity: every homomorphism from any right ideal I of R to M can be extended to whole of R. Since then many researchers have embarked on to determine a class of ideals of a ring R such that an R-module M is injective if and only if it satisfies Baer's criteria of injectivity for such a class. For instance, Smith [11] showed that if R is a commutative Noetherian ring, then the collection of all prime ideals of R is such a class. Later on, Vamos [12] termed such a class

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as a test set for injectivity of a module. Beachy et. al. [3] finally showed that for a piecewise Noetherian ring a set of prime ideals is a test set if and only if it contains all essential prime ideals. In the same spirit, in our present work, we introduce the notion of χ-injective module. Let R be a ring and χ be a collection of right ideals of R. A right R module M is said to be χ-injective if for every ideal $I \in \chi$, every homomorphism $f : I \to M$ can be extended to whole of R. Unlike the authors listed above, we study the properties of such a module rather than emphasizing on the collection χ.

We have also related various other notions like pure-exact sequence, multiplication module with the notion of χ-injective module in [8] and [9].

2. Preliminaries

Definition 1.1. An essential (large) submodule of a module B is any submodule A which has non-zero intersection with every non-zero submodule of B. We write $A \leq_e B$ to denote the situation. Moreover we say that B is an essential extension of A.

Definition 1.2. A ring R is said to be Baer if the left annihilator of any subset of R is generated as a left ideal by an idempotent of R.

Definition 1.3. For a ring R a right R module M is called semisimple (or completely reducible) if it is a direct sum of simple modules. Thus, a ring R is said to be left (right) semisimple if it is semisimple as a left (right) R module.

Definition 1.4. A short exact sequence is an exact sequence of the form $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$. A short exact sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is split if $\exists$ a homomorphism $j : C \to B$ with $gj = 1_C$.

Definition 1.5. A ring R is said to be von-Neumann regular if for each $r \in R$, $\exists\ r^{'} \in R$ with $rr^{'}r = r$.

Definition 1.6. A right R module P is said to be projective if whenever p is a surjective homomorphism from A to B and h is any homomorphism from P to B, there exists another homomorphism g from P to A such that $pg = h$.

Definition 1.7. An R module M is said to be divisible if for any $u \in M$ and $a \in R$ such that $ann_r(a) \subseteq ann(u)$, u is divisible by a, i.e. $\exists v \in M$ such that $u = va$, where $ann_r(a)$ denotes the right annihilator of the element a.

Any other terminology or result relevant to the present work can be found in [4], [5],[6], [7] and [10].

3. Main results

Definition 3.1. Let M be a right R module and χ be a collection of right ideals of R. Then M is said to be χ-injective if every homomorphism $f : I \to M$, $I \in \chi$ can be extended to whole of R.

Example 3.1. Let M be a right R module where R is a commutative Noetherian ring. If we let χ to be collection of all prime ideals of R, then by [11] it follows that M is χ-injective.

Theorem 3.1. Let M be a right R module and χ be a collection of right ideals of R. Then the following are equivalent:

(1) M is χ-injective.
(2) for any $I \in \chi$ and for every homomorphism $f : I \to M$, there exists $m \in M$ such that $f(a) = ma$.

Proof.(1) $\Longrightarrow$ (2) Let i be a natural embedding from I to R and $f : I \to M$ be any homomorphism such that there exists another homomorphism $\varphi : R \to M$ such that $f = \varphi i$. As f, φ are module homomorphisms, for $a \in I$, we have,

$$\begin{aligned} f(a) &= \varphi(a) \\ &= \varphi(1 \cdot a) \\ &= \varphi(1)a \\ &= ma \end{aligned}$$

where $\varphi(1) = m$ for $m \in M$. Thus, (2) follows.

(2) $\Longrightarrow$ (1). Let for a right ideal $I \in \chi$ and for a homomorphism $f : I \to M$, there exists $m \in M$

such that $f(a) = ma$. If we define $\varphi : R \to M$ by $\varphi(a) = ma$ for $a \in R$, then clearly φ is a module homomorphism and $\varphi_{|_I} = f$. This shows that M is χ -injective.

From the definition of χ-injective modules, it is clear that an injective module is χ-injective. However a χ-injective module need not be injective. We consider the following example.

Example 3.1. We recall from [1] that a right ideal of a ring R is said to be pure if and only if for every $x \in I$, $\exists y \in I$ such that $x = xy$. We consider the ring of integers $\mathbb{Z}$ as a module over itself. If χ denotes the collection of all non-zero proper pure ideals, then $\mathbb{Z}$ as a module over itself is χ-injective. Infact $\mathbb{Z}$ does not possess any non-zero proper pure ideal in this case as $\mathbb{Z}$ is free from non-zero one sided zero divisors. But $\mathbb{Z}_{\mathbb{Z}}$ is not injective.

We now establish a condition under which a χ-injective module is injective.

Theorem 3.2. Let Q be a χ-injective module, where χ is the collection of all essential right ideals of R. Let M,N be right R modules. Then Q is injective if $M \leq_e N$ and any homomorphism $\varphi : M \to Q$ can be extended to N.

Proof. Let a module Q be χ-injective, and let us consider the following diagram, where $M \leq_e N$

$$\begin{array}{l} O \to M \to N \\ \quad\;\; \downarrow \\ \quad\;\; Q^{\varphi} \end{array}$$

We now consider a set κ of extensions,i.e. the set of all pairs (C,h) where $M \leq C \leq_e N$ and $h : C \to Q$ such that $h|_C = \varphi$. Then clearly $\kappa \neq \phi$ as $(M, \varphi) \in \kappa$. We now introduce an ordering relation by setting $(C_1, h_1) \leq (C_2, h_2)$ if and only if $C_1 \subseteq C_2$ and h_2 extends h_1. This can be easily verified to be a partial ordering on κ. Every non-empty increasing chain $\{(C_i, h_i) | i \in I\}$ in κ has a upper bound $(C^{'}, h^{'})$, where $C^{'} = \bigcup_{i \in I} C_i$ and $h^{'}|_{C'} = h_i$. Thus, in view of Zorn' lemma, $\exists$ a maximal element (C^*, h^*) in κ. By construction, $M \leq C^* \leq_e N$. We need to show $C^* = N$ i.e. $N \subseteq C^*$

Suppose $\exists$ a non-zero $b \in N$ such that $b \notin C^*$. We set $I = \{a \in R : ba \in C^*\}$. Then I is an essential right ideal of R and hence $I \in \chi$. Thus $\exists$ a homomorphism $f : I \to R$ defined by $f(a) = h^*(ba)$. By assumption, $\exists q \in Q$ such that $f(a) = qa = h^*(ba)$(as Q is χ-injective) $\forall a \in I$. Then we can define a homomorphism $g : C^* + bR \to Q$ by setting $g(c + ba) = h^*(c) + qa$ $\forall c \in C^*$ and $a \in R$. It extends to a homomorphism h^* and is well defined. For suppose,

$c_1+ba_1=c_2+ba_2$ for $c_1,c_2\in C^*$ and $a_1,a_2\in R$. Then $a_1-a_2\in I$ and hence $f(a_1-a_2)=f(a_1)-f(a_2)=qa_1-qa_2$. On the other hand $f(a_1)-f(a_2)=h^*(ba_1)-h^*(ba_2)=h^*(ba_1-ba_2)=h^*(c_2-c_1)=h^*(c_2)-h^*(c_1)$. Hence we have $h^*(c_2)-h^*(c_1)=qa_1-qa_2$. Thus $g(c_1+ba_1)=h^*(c_1)+qa_1=h^*(c_2)+qa_2=g(c_2+ba_2)$, as required i.e. the function is well-defined.

Thus we have $(C^*,h^*)\leq(C^*+bR,g)$ i.e. we have obtained a contradiction regarding the maximality of (C^*,h^*). This completes the proof.

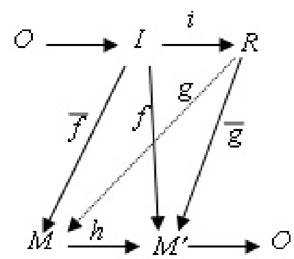
Remark 3.1. At this point, we note that in Theoem 3.2, the condition of essentiality is a sufficient condition for a χ-injective module to be injective. However, the condition is not necessary. For instance, let us consider the $\mathbb{Z}/<2>$ as a module over $\mathbb{Z}/<6>$. $\mathbb{Z}/<6>$ has two non-trivial ideals, $<3>=\{0,3\}\simeq\mathbb{Z}/<2>$ and $<2>=\{0,2,4\}\simeq\mathbb{Z}/<3>$. Since there is no non-zero homomorphism from $\mathbb{Z}/<3>$ to $\mathbb{Z}/<2>$ so the only ideal at stake is $\mathbb{Z}/<2>$. The homomorphism f from $\mathbb{Z}/<2>$ to itself is determined by $f(1)$. Also, the inclusion map $i:\mathbb{Z}/<2>\to\mathbb{Z}/<6>$ can be defined as $f(1)=3$. Thus if $\tilde{f}:\mathbb{Z}/<6>\to\mathbb{Z}/<2>$ then we have $\tilde{f}\circ i(1)=\tilde{f}(3)=3\tilde{f}(1)=\tilde{f}(1)$. Thus if we define $\tilde{f}(1)=f(1)$ then $\tilde{f}$ is an extension of f. Consequently $\mathbb{Z}/<2>$ is injective over $\mathbb{Z}/<6>$ but none of $<3>$ or $<2>$ is essential in $\mathbb{Z}/<6>$.

Theorem 3.3. Let R be a semisimple ring. Let M be a χ-injective right R module. Then the following hold:

(1) any submodule K of M is χ-injective.
(2) the homomorphic image of M is χ-injective module.
(3) the quotient of M is χ-injective module.

Proof.(1) Since R is semisimple, we have K, M and M/K all are projective. So, the short exact sequence $0\longrightarrow K\longrightarrow M\longrightarrow M/K\longrightarrow 0$ splits. Thus, if $i:K\to M$ be the inclusion, then there exists a homomorphism $k:M\to K$ such that $ki=id_K$. Let $I\in\chi$ and $f:I\to K$ a homomorphism. Then the composite $if:I\to M$ extends to a homomorphism $g:R\to M$ as M is χ-injective.If we take $h:R\to K$ to be the composite kg, then the restriction of h on I is equal to the composite k(restriction of g on I)=$kif=f$. So. h is an extension of f. This proves that K is χ-injective.

(2) Let $M^{'}$ be the homomorphic image of M and we consider the following diagram



Using the χ-injectivity of M, we get

$$gi = \bar{f} \qquad (1)$$

Again R being semisimple and I as a module over R is projective. Thus, we have

$$h\bar{f} = f \qquad (2)$$

By similar arguments, we have

$$hg = \bar{g} \qquad (3)$$

Then, (1) gives;

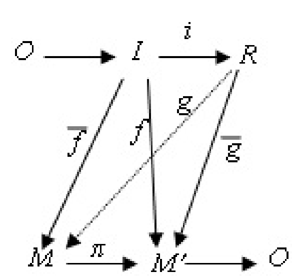
$$(gi) = h\bar{f}$$

$$(hg)i = h\bar{f}$$

$$\bar{g}i = f \text{ (using (2) and (3))}$$

as required.

(3) Let M be χ-injective and K be a submodule of M. Then to show that $M' = \frac{M}{K}$ is also χ-injective. We consider the following diagram



where π denotes the natural homomorphism. Using the fact that,

$$gi = \bar{f}$$

and that the homomorphic image of a χ-injective module is χ-injective, we get the desired result.

Theorem 3.4. The direct sum of two χ-injective modules is again χ-injective.

Proof. Let M_1 and M_2 be χ-injective modules. Then to show that $M_1 \bigoplus M_2$ is χ-injective. Since M_1 and M_2 are χ-injective, so given $I \in \chi$ and homomorphisms $f_1 : I \to M_1$ and $f_2 : I \to M_2$, we have extensions $g_1 : R \to M_1$ and $g_2 : R \to M_2$ respectively. Again any homomorphism $f : I \to M_1 \bigoplus M_2$ can be written as $f = (f_1, f_2)$, where $f_1 = p_1 f$ and $f_2 = p_2 f$, p_1 and p_2 being the projections of M_1 and M_2 to $M_1 \bigoplus M_2$ respectively. Now if we take $(g_1, g_2) : R \to M_1 \bigoplus M_2$, then (g_1, g_2) extends $f = (f_1, f_2)$. Consequently, $M_1 \bigoplus M_2$ is χ-injective.

Theorem 3.5. Let R be a right Noetherian Von-Neumann regular ring. Then a right R module I is χ-injective if and only if it is divisible, χ being the collection of all right ideals of R of the type $\{aR : a \in R\}$.

Proof. Let I be divisible. Let $f : aR \to I$ be a homomorphism where $aR \in \chi$. Let

$$u = f(a) \in I$$

. Then by definition

$$x \in ann_r(a)$$

$$ax = 0$$

$$f(ax) = 0$$

$$f(a)x = 0$$

$$ux = 0$$

$$x \in ann(u)$$

Then, by definition $u = va$ for some $v \in I$ Then, if we take $g : R_R \to I$ defined by $g(1) = v$ Then;

$$g(1)a = va$$

$$g(a) = va$$

$$= u$$

$$= f(a) \ , \forall a \in aR$$

i.e.

$$g_{|_{aR}} = f$$

or that f extends R_R. Consequently I is χ-injective.

Conversely, let I be χ-injective. Then any homomorphism $f : aR \to I$, $aR \in \chi$ extends to R_R. We now show that I is divisible, i.e. $ann^I(ann_r(a)) = Ia$, where $ann^I(ann_r(a))$ denotes the annihilator of $ann_r(a)$ taken in I. We first show that

$$Ia \subseteq ann^I(ann_r(a))$$

Let $x \in Ia$. Then $x = ra$ for some $r \in I$. Then, we have,

$$ann_r(a) \cdot a = 0$$

$$r \cdot ann_r(a) \cdot a = 0$$

$$ann_r(a) \cdot ra = 0$$

$$ann_r(a) \cdot x = 0$$

$$x \in ann(ann_r(a))$$

Since $x \in I$, we have

$$x \in ann^I(ann_r(a))$$

i.e.

$$Ia \subseteq ann^I(ann_r(a)).$$

Now to show that

$$ann^I(ann_r(a)) \subseteq Ia$$

Let

$$x \in ann^I(ann_r(a)).$$

As I is χ-injective, the homomorphism $f : aR \to I$ extends $g : R_R \to I$. Then $f(aR) = xR$ is a well-defined homomorphism as for

$$ar = as$$

$$a(r-s) = 0$$

$$r - s \in ann_r(a)$$

$$x(r-s) = 0$$

$$xr = xs$$

Again,

$$x = f(a) = g(a) = g(1)a = va$$

for some $g(1) = v$ Thus

$$ann^I(ann_r(a)) \subseteq Ia$$

consequently,

$$ann^I(ann_r(a)) = Ia$$

or that, I is divisible.

Corollary 3.1 Over a Baer ring R, a right R module is χ-injective if and only if it is divisible.

Proof. We need to establish that a right Noetherian von Neumann regular is Baer, since in that case the result will follow from proposition 5. If R is right Noetherian, it follows that the ideals of R are finitely generated [6]. Also if R is von-Neumann regular, every finitely generated ideal is principal and is generated by an idempotent [10]. Thus, we may conclude that R is Baer.

Conflict of Interests

The authors declare that there is no conflict of interests.

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