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ON THE MODULE OF FIRST AND SECOND ORDER DIFFERENTIALS OF $R \otimes_k S$

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Abstract: Let R and S be k-algebras with characteristic zero. Let $\Omega_k^1(R \otimes_k S)$ and $\Omega_k^2(R \otimes_k S)$ are first and second order universal differential modules over $R \otimes_k S$, respectively. The main result of this paper asserts that in which cases $\Omega_k^1(R \otimes_k S)$ and $\Omega_k^2(R \otimes_k S)$ can be free modules by using symmetric derivation.

Keywords: higher order differentials; symmetric derivations; regular rings.

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1. INTRODUCTION

Let R be k-algebra. The module of Kahler differentials of R is defined by H. Osborn [1]. J. Johnson has given differential module structures on particular modules of Kahler differentials [2]. In [3] the author give fundamental theories for the computation of high order derivations. Hart has studied on higher derivations and universal differential operators [10]. Olgun and Erdoğan study universal modules on $R \otimes_k S$ and examine the homological dimension of $\Omega_k^n(R \otimes_k S)$ in [4]. In [5], author show that regularity of any affine local k-algebra is equivalent to freeness of $\Omega_k^2(R)$.

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In [11], the author states that projective dimension of $\Omega_k^n(S)\oplus S$ is one or smaller than one where S is hypersurface. On the other hand, Barajas and Duarte have studied the module of differentials of order n using the high order Jacobian matrix and they have proved projective dimension of $\Omega_k^n(R)$ is one or smaller than one in the case of hypersurfaces [12].

Throughout this paper, R is commutative algebra over an algebraically closed field k with characteristic zero. We will examine in which cases first and second order universal differential modules of $R \otimes_k S$ can be free $R \otimes_k S$-modules using symmetric derivation. We will give a relationship between regularity of rings and projectivity of $\Omega_k^1(R \otimes_k S)$ and $\Omega_k^2(R \otimes_k S)$.

2. PRELIMINARIES

Let R and S be k-algebras. $R \otimes_k S$ is a commutative ring and with the multiplication of $(\sum_i r_i \otimes s_i)(\sum_j k_j \otimes l_j) = \sum_{i,j} r_i k_j \otimes s_i l_j$. In this section, we will give some conclusions about universal differential modules on $R \otimes_k S$ and certain properties of symmetric power modules.

Proposition 2.1. [6] Let R and S be affine k-algebras. If R and S are integral domain, then $R \otimes_k S$ is integral domain.

Proposition 2.2. [6] Let $f: R \longrightarrow R'$ and $g: S \longrightarrow S'$ be homomorphism of k-algebras. Then $f \otimes g: R \otimes_k S \longrightarrow R' \otimes_k S'$ is a homomorphism of k-algebras.

Theorem 2.1. [3] Let R and S be k-algebras. Then there exists a $R \otimes_k S$-module isomorphism:

$$\Omega_k^n(R \otimes_k S) \simeq \Omega_k^n(R) \otimes_k S \oplus \Omega_k^n(S) \otimes_k R \oplus U$$

where U is a submodule of $\Omega_n(R \otimes_k S)$ satisfied the universal mapping property.

Corollary 2.1. [3] For $n = 1$, there exists a $R \otimes_k S$-module isomorphism

$$\Omega_k^1(R \otimes_k S) \simeq \Omega_k^1(R) \otimes_k S \oplus R \otimes_k \Omega_k^1(S)$$

Theorem 2.2. [7] Let R and S be k-algebras. Then there exists a $R \otimes_k S$-module isomorphism

$$\Omega_k^2(R \otimes_k S) \simeq \Omega_k^2(R) \otimes_k S \oplus \Omega_k^2(S) \otimes_k R \oplus \Omega_k^1(R) \otimes_k \Omega_k^1(S)$$

Definition 2.1. Let R be a commutative ring with an unit element. Let M and N be R-modules. If a R-multilinear map $f: M^n \longrightarrow N$ is unchanged under all permutations of the arguments, then it is called symmetric.

Definition 2.2. Let M be a R-module. A universal symmetric R-multilinear map $f: M^n \longrightarrow S^n(M)$ is defined by $f(x_1, x_2, \dots, x_n) = x_1 x_2 \dots x_n$. Then $S^n(M)$ is called the n-th order symmetric power of M. $S^n(M)$ can be constructed as factor modules of $\otimes^n M$ by submodule generated by all elements of the forms

$$x_1 \otimes x_2 \otimes \dots . \otimes y \otimes z \otimes \dots \otimes x_n - x_1 \otimes x_2 \otimes \dots . \otimes z \otimes y \otimes \dots \otimes x_n$$

Example 2.1. Let M be a R-module. Then $S^2(M)$ is expressed

$S^2(M) = \frac{M \otimes_k M}{<x \otimes y - y \otimes x>}$ for all $x, y \in M$. In particular, $S^0(M) = R$ and $S^1(M) = M$.

Proposition 2.3. Let M be a free R-module with rank r. Then the n-order symmetric power module $S^n(M)$ is free R-module with rank $\binom{r+n-1}{r-1}$.

Lemma 2.1. [8] Let M and N be R-modules. Let $\theta: M^n \longrightarrow N$ be a multilinear map. Then there exists a unique R-module homomorphism $f: S^n(M) \longrightarrow N$ such that the following diagram is commutative:

$$\begin{array}{ll} M^n & \longrightarrow N \\ \downarrow & \nearrow \\ S^n(M) & \end{array}$$

Proposition 2.3. [8] Let T be R-algebra and M be R-module. Then there exists a R-module isomorphism:

$$S^n(M) \otimes_R T \simeq S^n(M \otimes_R T)$$

Now, we give definition of symmetric derivation of first order Kahler differential module of R over k.

Definition 2.3. [1] Let R be any k-algebra. $R \longrightarrow \Omega_k^1(R)$ be a first order Kahler derivation of R and $S(\Omega_k^1(R))$ be the symmetric algebra $\oplus S^p(\Omega_k^1(R))$ generated over R by $\Omega_k^1(R)$. A symmetric derivation is any linear map D of $S(\Omega_k^1(R))$ into itself such that

i. $D(S^p(\Omega_k^1(R)) \subseteq S^{p+1}(\Omega_k^1(R))$
ii. D is a first order derivation over k
iii. The restriction of D to R ($R \simeq S^0\Omega_k^1(R)$) is Kahler derivation $d_1: R \longrightarrow \Omega_k^1(R)$.

In [5], he has generalized this definition to q-th order Kahler differential module of R.

Theorem 2.3. [1] Let R be an affine k-algebra. Then there exists a short exact sequence of R-modules

$$0 \longrightarrow S^2(\Omega_k^1(R)) \longrightarrow \Omega_k^2(R) \xrightarrow{\theta} \Omega_k^1(R) \longrightarrow 0$$

such that $\theta(d_2(\mathrm{f})) = d_1(\mathrm{f})$ and $\ker\theta \simeq S^2(\Omega_k^1(R))$.

Proposition 2.4. [9] Let M and N be R-modules. Then there is a natural isomorphism

$$S^p(\mathrm{M}\oplus\mathrm{N}) \simeq \oplus_{m+n=p} S^m(M) \otimes_R S^n(N)$$

Example 2.2. Let M and N be R-modules. Then we obtain the following isomorphism:

$$S^2(\mathrm{M}\oplus\mathrm{N}) \simeq \oplus_{m+n=2} S^m(M) \otimes_R S^n(N)$$

$$S^2(\mathrm{M}\oplus\mathrm{N}) \simeq R \otimes_R S^2(N) \oplus S^1(M) \otimes_R S^1(N) \oplus S^2(M) \otimes_R N$$

$$S^2(\mathrm{M}\oplus\mathrm{N}) \simeq R \otimes_R S^2(N) \oplus M \otimes_R N \oplus S^2(M) \otimes_R N$$

Corollary 2.2. Let R and S be affine k-algebras. Suppose that M is a R-module and N is a S-module. Then there exists a natural isomorphism:

$$S^p(\mathrm{M}\oplus\mathrm{N}) \simeq \oplus_{m+n=p} S^m(M) \otimes_k S^n(N)$$

Proof. Since M is a R-module, M is a k-module. Similarly, N is a k-module when N is a S-module. Then we have a natural isomorphism from Proposition 2.4

$$S^p(\mathrm{M}\oplus\mathrm{N}) \simeq \oplus_{m+n=p} S^m(M) \otimes_k S^n(N)$$

Thus, the proof is completed.

3. MAIN RESULTS

In this section, we will show in which cases $\Omega_k^1(R \otimes_k S)$ and $\Omega_k^2(R \otimes_k S)$ are free $R \otimes_k S$-module by using symmetric power modules. We give the relationships between the freeness of $\Omega_k^n(R \otimes_k S)$ (for $\mathrm{n} = 1$ and $\mathrm{n} = 2$) and regularity of the k-algebras R and S.

Theorem 3.1. Let R and S be affine local k-algebras. $S(\Omega_k^1(R))$ has at least one symmetric derivation. If $\Omega_k^1(R)$ and $\Omega_k^1(S)$ are free modules, then $\Omega_k^1(R \otimes_k S)$ is a free module.

Before proof, we need some information.

Lemma 3.1. [5] Let R be an affine domain with dimension s. $\Omega_k^q(R)$ is a free R-module if and only if $S^2(\Omega_k^q(R))$ is a free R-module.

Lemma 3.2. Let R and S be affine local k-algebras, $S(\Omega_k^1(R))$ and $S(\Omega_k^1(S))$ have at least one symmetric derivation. Then there exists the following isomorphism:

$$S^2(\Omega_k^1(R \otimes_k S) \simeq R \otimes_k S \otimes_k [R \otimes_k S^2\left(\Omega_k^1(S)\right) \oplus \Omega_k^1(R) \otimes_k \Omega_k^1(S) \oplus S \otimes_k S^2\left(\Omega_k^1(R)\right)]$$

We can try to write the symmetric power module $S^2(\Omega_k^1(R \otimes_k S))$ using the isomorphism in Corollary 2.1. Then we have the following isomorphism

$$S^2(\Omega_k^1(R \otimes_k S)) \simeq S^2(\Omega_k^1(R) \otimes_k S \oplus \Omega_k^1(S) \otimes_k R) \tag{2.1}$$

We have

$$S^2(\mathrm{M} \oplus \mathrm{N}) \simeq \oplus_{m+n=2} S^m(M) \otimes_k S^n(N)$$

If we use this isomorphism in (2.1), then we obtain that

$$S^2\left(\Omega_k^1(R \otimes_k S)\right) \simeq R \otimes_k S \otimes_k S^2\left(R \otimes_k \Omega_k^1(S)\right) \oplus \Omega_k^1(R) \otimes_k \mathrm{S} \otimes_k \Omega_k^1(S)$$
$$\otimes_k \mathrm{R} \oplus R \otimes_k S \otimes_k S^2\left(S \otimes_k \Omega_k^1(R)\right)$$

Since R is a k-algebra and $\Omega_k^1(S)$ is a k-module, then by Proposition 2.3. we have the following isomorphism:

$$S^2\left(R \otimes_k \Omega_k^1(S)\right) \simeq R \otimes_k S^2(\Omega_k^1(S))$$

Similarly, we have $S^2\left(S \otimes_k \Omega_k^1(R)\right) \simeq S \otimes_k S^2(\Omega_k^1(R))$. If we use these isomorphisms, we obtain that

$$S^2\left(\Omega_k^1(R \otimes_k S)\right) \simeq R \otimes_k S \otimes_k R \otimes_k S^2\left(\Omega_k^1(S)\right) \oplus R \otimes_k S \otimes_k S \otimes_k S^2\left(\Omega_k^1(R)\right)$$
$$\oplus R \otimes_k S \otimes_k \Omega_k^1(R) \otimes_k \Omega_k^1(S)$$

Now we can prove Theorem 3.1.

Proof of Theorem 3.1. $\Omega_k^1(R)$ and $\Omega_k^1(S)$ are free modules if and only if $S^2\left(\Omega_k^1(R)\right)$ and $S^2\left(\Omega_k^1(S)\right)$ are free modules by Lemma 3.1. By Lemma 3.2, we have the following isomorphism

$$S^2\left(\Omega_k^1(R \otimes_k S)\right) \simeq R \otimes_k S \otimes_k R \otimes_k S^2\left(\Omega_k^1(S)\right) \oplus R \otimes_k S \otimes_k S \otimes_k S^2\left(\Omega_k^1(R)\right)$$

$$\oplus R \otimes_k S \otimes_k \Omega_k^1(R) \otimes_k \Omega_k^1(S)$$

Thus, we can write the symmetric power module $S^2\left(\Omega_k^1(R \otimes_k S)\right)$ as a direct sum of free modules. Then we obtain that $S^2\left(\Omega_k^1(R \otimes_k S)\right)$ is a free $R \otimes_k S$-module. In Lemma 3.1., we say $\Omega_k^1(R \otimes_k S)$ is a free $R \otimes_k S$-module.

Corollary 3.1. Let R and S be affine k-algebras. If $\Omega_k^1(R)$ and $\Omega_k^1(S)$ are projective modules, then $\Omega_k^1(R \otimes_k S)$ is a projective module.

Corollary 3.2. Let R and S be affine regular k-algebras. Then $\Omega_k^1(R \otimes_k S)$ is a projective module.

Theorem 3.2. Let R and S be affine local k- algebras. $S(\Omega_k^1(R))$ has at least one symmetric derivation. $\Omega_k^2(R)$, $\Omega_k^2(S)$ and $S^2(\Omega_k^1(R) \otimes_k \Omega_k^1(S))$ are free modules, then $\Omega_k^2(R \otimes_k S)$ is free module.

Lemma 3.3. [5] Let R affine local k- algebra. $S(\Omega_k^1(R))$ has at least one symmetric derivation. $\Omega_k^1(R)$ is a free R-module if and only if $\Omega_k^2(R)$ is a free R-module.

Lemma 3.4. Let R and S be affine local k- algebras and $S(\Omega_k^2(R))$ and $S(\Omega_k^2(S))$ have at least one symmetric derivation.

$$S^2(\Omega_k^2(R \otimes_k S)) \simeq R \otimes_k S \otimes_k R \otimes_k S \otimes_k S^2\Omega_k^2(S) \otimes_k R \oplus R \otimes_k S \otimes_k R \otimes_k S \otimes_k$$

$$S^2\Omega_k^2(R) \otimes_k S \oplus R \otimes_k S \otimes_k \Omega_k^2(R) \otimes_k S \otimes_k \Omega_k^2(S) \otimes_k R \oplus$$

$$R \otimes_k S \otimes_k S^2(\Omega_k^1(R)) \otimes_k \Omega_k^1(S)) \oplus \Omega_k^1(R) \otimes_k \Omega_k^1(S) \otimes_k$$

$$\Omega_k^2(R) \otimes_k S \otimes_k \Omega_k^2(S) \otimes_k R.$$

Proof. We have the following isomorphism

$$\Omega_k^2(R \otimes_k S) \simeq \Omega_k^2(R) \otimes_k S \oplus \Omega_k^2(S) \otimes_k R \oplus \Omega_k^1(R) \otimes_k \Omega_k^1(S)$$

Then we have obtained that

$$S^2(\Omega_k^2(R \otimes_k S)) \simeq S^2(\Omega_k^2(R) \otimes_k S \oplus \Omega_k^2(S) \otimes_k R \oplus \Omega_k^1(R) \otimes_k \Omega_k^1(S))$$

Let $\mathrm{M} := \Omega_k^1(R) \otimes_k \Omega_k^1(S)$ and $\mathrm{N} := \Omega_k^2(R) \otimes_k S \oplus \Omega_k^2(S) \otimes_k R$.

After that, we will try to write $S^2(\Omega_k^2(R \otimes_k S))$ as a direct sum of modules.

$$S^2(\Omega_k^2(R \otimes_k S)) \simeq R \otimes_k S \otimes_k S^2(N) \oplus R \otimes_k S \otimes_k S^2(\Omega_k^1(R)) \otimes_k \Omega_k^1(S) \oplus M \otimes_k N$$

$$S^2(N) \simeq S^2(\Omega_k^2(R) \otimes_k S \oplus \Omega_k^2(S) \otimes_k R)$$

$$S^2(N) \simeq [R \otimes_k S \otimes_k S^2(\Omega_k^2(R) \otimes_k S)] \oplus [R \otimes_k S \otimes_k$$

$$S^2(\Omega_k^2(S) \otimes_k R)] \oplus [\Omega_k^2(R) \otimes_k S \otimes_k \Omega_k^2(S) \otimes_k R]$$

$$S^2(\Omega_k^2(R \otimes_k S)) \simeq R \otimes_k S \otimes_k R \otimes_k S \otimes_k S^2(\Omega_k^2(S) \otimes_k R) \oplus R \otimes_k S \otimes_k R \otimes_k S \otimes_k$$

$$S^2(\Omega_k^2(R) \otimes_k S) \oplus R \otimes_k S \otimes_k \Omega_k^2(R) \otimes_k S \otimes_k \Omega_k^2(S) \otimes_k R \oplus$$

$$R \otimes_k S \otimes_k S^2[\Omega_k^1(R)) \otimes_k \Omega_k^1(S)] \oplus \Omega_k^1(R) \otimes_k \Omega_k^1(S) \otimes_k$$

$$\Omega_k^2(R) \otimes_k S \otimes_k \Omega_k^2(S) \otimes_k R \qquad (2.2)$$

Then we have the following isomorphism:

$$S^2(\Omega_k^2(S) \otimes_k R) \simeq S^2(\Omega_k^2(S)) \otimes_k R \text{ and } S^2(\Omega_k^2(R) \otimes_k S) \simeq S^2(\Omega_k^2(R)) \otimes_k S$$

We can write again the isomorphism in (2.2) by using the isomorphism above.

$$S^2(\Omega_k^2(R \otimes_k S)) \simeq R \otimes_k S \otimes_k R \otimes_k S \otimes_k S^2(\Omega_k^2(S)) \otimes_k R \oplus R \otimes_k S \otimes_k R \otimes_k S \otimes_k$$

$$S^2(\Omega_k^2(R)) \otimes_k S \oplus R \otimes_k S \otimes_k \Omega_k^2(R) \otimes_k S \otimes_k \Omega_k^2(S) \otimes_k R \oplus$$

$$R \otimes_k S \otimes_k S^2[\Omega_k^1(R)) \otimes_k \Omega_k^1(S)] \oplus \Omega_k^1(R) \otimes_k \Omega_k^1(S) \otimes_k$$

$$\Omega_k^2(R) \otimes_k S \otimes_k \Omega_k^2(S) \otimes_k R.$$

Proof of Theorem 3.2. Suppose that $\Omega_k^2(R)$, $\Omega_k^2(S)$ and $S^2(\Omega_k^1(R)) \otimes_k \Omega_k^1(S))$ are free modules. If $\Omega_k^2(R)$ and $\Omega_k^2(S)$ are free modules, then $\Omega_k^1(R)$ and $\Omega_k^1(S)$ are free modules [Lemma 3.3.]. So $S^2(\Omega_k^2(R))$ and $S^2(\Omega_k^2(S))$ are free modules by Lemma 3.1. Using Lemma 3.4., since we can write the symmetric power module $S^2(\Omega_k^2(R \otimes_k S))$ as direct sums of free modules, $S^2(\Omega_k^2(R \otimes_k S)$ is a free $R \otimes_k S$-modules. Thus we can conclude that $\Omega_k^2(R \otimes_k S)$ is a free module by Lemma 3.1.

Corollary 3.3. Let R and S be affine k-algebras. $S(\Omega_k^1(R))$ has at least one symmetric derivation. $\Omega_k^2(R)$, $\Omega_k^2(S)$ and $S^2(\Omega_k^1(R) \otimes_k \Omega_k^1(S))$are projective modules, then $\Omega_k^2(R \otimes_k S)$ is projective module.

Corollary 3.4. Suppose that $S(\Omega_k^1(R))$ has at least one symmetric derivation. Let $S^2(\Omega_k^1(R) \otimes_k \Omega_k^1(S))$ be a projective module. If R and S are affine regular k-algebras, then $\Omega_k^2(R \otimes_k S)$ is projective module.

Proof. Suppose that $S^2(\Omega_k^1(R) \otimes_k \Omega_k^1(S))$ is a projective module. R and S are affine regular k-algebras if and only if $\Omega_k^2(R)$ and $\Omega_k^2(S)$ are projective modules (this was proved by Olgun in [5]). By Corollary 3.3., $\Omega_k^2(R \otimes_k S)$ is a projective module.

Example 3.1. Let $R = k[x]$ and $S = k[y, z]$ be polynomial algebras. Since $R \otimes_k S \simeq k[x, y, z]$, then $R \otimes_k S$ is a polynomial algebra with dimension 3. $\Omega_k^1(R) =< \{d_1(x)\} >$ is a free R-module with rank 1 and $\Omega_k^1(S) =< \{d_1(y), d_1(z)\}>$ is a free S-module with rank 2. Then $S^2\left(\Omega_k^1(R)\right) =< \{d_1(x) \vee d_1(x)\} >$ and

$S^2\left(\Omega_k^1(S)\right) =< \{d_1(y) \vee d_1(y), d_1(y) \vee d_1(z), d_1(z) \vee d_1(z)\} >$ are free modules.

Thus, the second order symmetric power module of $\Omega_k^1(R \otimes_k S)$ can be written as a direct sum of free modules (Lemma 3.2.). Therefore, $\Omega_k^1(R \otimes_k S)$ is a free $R \otimes_k S$-module (Theorem 3.1). In other way, since $R \otimes_k S \simeq k[x, y, z]$ is a regular ring, then the first universal differential module of $R \otimes_k S$ will be a free module.

$\Omega_k^2(R) =< \{d_2(x), d_2(x^2)\} >$ is a free R-module with rank 2 and

$\Omega_k^2(S) = \{d_2(y), d_2(y^2), d_2(yz), d_2(z), d_2(z^2)\}$ is a free S-module with rank 5.

$S^2\left(\Omega_k^2(R)\right)$ is a free module generated by the set

$\{d_2(x) \vee d_2(x), d_2(x) \vee d_2(x^2), d_2(x^2) \vee d_2(x^2)\}$ and

$S^2\left(\Omega_k^2(S)\right)$ is a free module generated by the following set

$\{d_2(y) \vee d_2(y), d_2(y) \vee d_2(y^2), d_2(y) \vee d_2(yz), d_2(y) \vee d_2(z), d_2(y) \vee d_2(z^2), d_2(y^2) \vee d_2(y^2), d_2(y^2) \vee d_2(yz), d_2(y^2) \vee d_2(z), d_2(y^2) \vee d_2(z^2), d_2(yz) \vee d_2(yz), d_2(yz) \vee d_2(z), d_2(yz) \vee d_2(z^2), d_2(z) \vee d_2(z), d_2(z) \vee d_2(z^2), d_2(z^2) \vee d_2(z^2)\}$.

Thus, $S^2(\Omega_k^2(R \otimes_k S))$ can be written as a direct sum of free modules (Lemma 3.4.). Then $\Omega_k^2(R \otimes_k S)$ is a free $R \otimes_k S$-module by Theorem 3.2.

Example 3.2. Let $R = k[z]$ and $= {k[x,y]}/{< y^2 - x^3 >}$.

Then $R \otimes_k S \simeq {k[x,y,z]}/{< y^2 - x^3 >}$ [6]. The first order universal differential module $\Omega_k^1(R \otimes_k S)$ is not free module. By Theorem 3.1. $\Omega_k^1(R)$ or $\Omega_k^1(S)$ can not be free module. We know R is a regular ring, so $\Omega_k^1(R)$ is a free module. Thus $\Omega_k^1(S)$ can not be a free module. Similarly, by examining the second order universal differential module $\Omega_k^2(R \otimes_k S)$, we can show that $\Omega_k^2(S)$ is not free module.

Conflict of Interests

The authors declare that there is no conflict of interests.

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