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A STUDY ON THE SYMMETRIC NUMERICAL SEMIGROUPS

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Abstract: In this paper, we will give some results about the numerical semigroups such that $S_k =< 5, 5k+4 >$ where $k \geq 1,\ k \in \mathbb{Z}$. Also, we will obtain Arf closure of these symmetric numerical semigroups.

Keywords: Symmetric numerical semigroup, Arf closure, genus.

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1. INTRODUCTİON

Let $\mathbb{N} = \ 0,1,2,...,n,...$ and $\mathbb{Z}$ be integer set. S is called a numerical semigroup if

(i) $a_1 + a_2 \in S$, for $a_1, a_2 \in S$

(ii) gcd S $=1$

(iii) $0 \in S$

where $S \subseteq \mathbb{N}$ (Here, gcd S $=$ greatest common divisor the elements of S).

A numerical semigroup S can be written that

$$S =< a_1, a_2, ..., a_n >= \left\{ \sum_{i=1}^{n} k_i a_i : k_i \in \mathbb{N} \right\}$$ (for detail see [4]).

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$U \subset \mathbb{N}$ is minimal system of generators of S if $<U>=S$ and there isn't any subset $V \subset U$ such that $<V>=S$ .Also, $m(S)=\min\ x \in S : x > 0$ is called as multiplicity of S (See [3]). Let S be a numerical semigroup, then $F(S)=\max\ \mathbb{Z}\backslash S$ is called as Frobenius number of S . $n(S)=Card\ \ 0,1,2,...,F(S)\ \cap S$ is called as the determine number of S (see [5]).

If S is a numerical semigroup such that $S=<a_1,a_2,...,a_n>$, then we observe that $S=<a_1,a_2,...,a_n>=\ s_0=0,s_1,s_2,...,s_{n-1},s_n=F(S)+1,\rightarrow ...$, where $s_i < s_{i+1}$, $n=n(S)$ and the arrow means that every integer greater than $F(S)+1$ belongs to S for $i=1,2,...,n=n(S)$ (see [6]).

If $b \in \mathbb{N}$ and $b \notin S$, then b is called gap of S . We denote the set of gaps of S , by $H(S)$, i.e, $H(S)=\mathbb{N}\backslash S$. The $G(S)=\#(H(S))$ is called the genus of S . It known that $G(S)+n(S)=F(S)+1$ (see [4]) .

S is called symmetric numerical semigroup if $F(S)-t$ belongs to S , for $t \in \mathbb{Z}\backslash S$. It is know the numerical semigroup $S=<a_1,a_2>$ is symmetric and $F(S)=a_1a_2-a_1-a_2$. In this case, we write $n(S)=\dfrac{F(S)+1}{2}$ (see [1]).

A numerical semigroup S is called Arf if $a_1+a_2-a_3 \in S$, for all $a_1,a_2,a_3 \in S$ such that $a_1 \geq a_2 \geq a_3$. The smallest Arf numerical semigroup containing a numerical semigroup S is called the Arf closure of S , and it is denoted by $Arf(S)$ (for detail see [2, 3]). If S is a numerical semigroup such that $S=<a_1,a_2,...,a_n>$, then $L(S)=\langle a_1,a_2-a_1,a_3-v_1,...,a_n-v_1\rangle$ is called Lipman numerical semigroup of S , and it is known that
$L_0(S)=S \subseteq L_1(S)=L(L_0(S)) \subseteq L_2=L(L_1(S)) \subseteq ... \subseteq L_m=L(L_{m-1}(S)) \subseteq ... \subseteq \mathbb{N}$ (see [7]).

2. MAIN RESULTS

Theorem 1. Let $S_k=<5,5k+4>$ be numerical semigroups , where $k \geq 1$, $k \in \mathbb{Z}$. Then, we have

(a) $F(S_k)=20k+11$

(b) $n(S_k)=10k+6$

(c) $G(S_k) = 10k + 6$.

Proof. Let $S_k = < 5, 5k+4 >$ be numerical semigroups , where $k \geq 1,\ k \in \mathbb{Z}$. Then, S_k is symmetric and we find that

(a) $F(S_k) = 5(5k+4) - 5 - 5k - 4 = 20k + 11$.

(b) $n(S_k) = \dfrac{F(S_k)+1}{2} = \dfrac{20k+11+1}{2} = 10k + 6$.

(c) $G(S_k) = 20k + 11 + 1 - 10k - 6 = 10k + 6$ from $G(S_k) = F(S_k) + 1 - n(S_k)$.

Theorem 2. Let $S_k = < 5, 5k+4 >$ be numerical semigroups , where $k \geq 1,\ k \in \mathbb{Z}$. Then, $Arf(S_k) = \ 0, 5, 10, 15, \ldots, 5k, 5k+4, \rightarrow \ldots$.

Proof. Let $S_k = < 5, 5k+4 >$ be numerical semigroups , where $k \geq 1,\ k \in \mathbb{Z}$. Then, we have $L_i(S_k) = < 5, 5k + (4-5i) >$ for $i = 0, 1, 2, \ldots, k-2$. In this case,

If $5 < 5k + (4-5i)$ then $m_i = 5$.

If $5 > 5k + (4-5i)$ then $m_i = 4$. So, we write $L_{k-1}(S_k) = < 5, 6 >,\ m_{k-1} = 5$

and $L_k(S_k) = < 5, 1 > = < 1 > = \mathbb{N},\ m_k = 1$.

Thus, we obtain $Arf(S_k) = \ 0, 5, 10, 15, \ldots, 5k, 5k+4 \rightarrow \ldots$.

Corollary 3 . Let $S_k = < 5, 5k+4 >$ be numerical semigroups , where $k \geq 1,\ k \in \mathbb{Z}$. Then, we have

(a) $F(Arf(S_k)) = 5k + 3$

(b) $n(Arf(S_k)) = k + 1$

(c) $G(Arf(S_k)) = 4k + 3$.

Proof. Let $S_k = < 5, 5k+4 >$ be numerical semigroups , where $k \geq 1,\ k \in \mathbb{Z}$. Then, we write that $F(Arf(S_k)) = 5k + 3$ from Theorem 2. On the other hand, we find that

$n(Arf(S_k)) = \#(\ 0, 1, 2, \ldots, 5k+3\ \cap Arf(S)) = \#(\ 0, 5, 10, \ldots, 5k\ ) = k + 1$ and we obtain

$G(Arf(S_k)) = 5k + 3 + 1 - k - 1 = 4k + 3$ since $G(Arf(S_k)) = F(Arf(S_k)) + 1 - n(Arf(S_k))$.

Corollary 4 . Let $S_k = < 5, 5k+4 >$ be numerical semigroups , where $k \geq 1,\ k \in \mathbb{Z}$. Then, we have

(a) $F(S_k) = 4F(Arf(S_k)) - 1$

(b) $n(S_k)=10n(Arf(S_k))-4$

(c) $G(S_k)=3G(Arf(S_k))-(2k+3)$.

Proof. Let $S_k=<5,5k+4>$ be numerical semigroups , where $k\geq 1,\ k\in\mathbb{Z}$. We write that (a) $4F(Arf(S_k))-1=4(5k+3)-1=20k+11=F(S_k)$. However, we find that

(b) $10n(Arf(S_k))-4=10(k+1)-4=10k+6=n(S_k)$,

(c) $3G(Arf(S_k))-(2k+3)=3(4k+3)-2k-3=10k+6=G(S_k)$.

Corollary 5. Let $S_k=<5,5k+4>$ be numerical semigroups , where $k\geq 1,\ k\in\mathbb{Z}$. Then, it satisfies following conditions:

(a) $F(S_{k+1})=F(S_k)+20$

(b) $n(S_{k+1})=n(S_k)+10$

(c) $G(S_{k+1})=G(S_k)+10$.

Corollary 6. Let $S_k=<5,5k+4>$ be numerical semigroups , where $k\geq 1,\ k\in\mathbb{Z}$. Then, it satisfies following conditions:

(a) $F(Arf(S_{k+1}))=F(Arf(S_k))+5$

(b) $n(Arf(S_{k+1}))=n(Arf(S_k))+1$

(c) $G(Arf(S_{k+1}))=G(Arf(S_k))+4$.

Example 7. We put $k=1$ in $S_k=<5,5k+4>$ symmetric numerical semigroups. Then we have $S_1=<5,9>=\ 0,5,9,10,14,15,18,19,20,23,24,25,27,28,29,30,32,\rightarrow...$. In this case, we obtain

$F(S_1)=31,\ n(S_1)=16,\ H(S_1)=\ 1,2,3,4,6,7,8,11,12,13,16,17,21,22,26,31$, $G(S_1)=16$,

$Arf(S_1)=\ 0,5,9,\rightarrow...$, $F(Arf(S_1))=8,\ n(Arf(S_1))=2,\ H(Arf(S_1))=\ 1,2,3,4,6,7,8$ and $G(Arf(S_1))=7$. Thus, we find that

$4F(Arf(S_1))-1=4.8-1=31=F(S_1)$, $10n(Arf(S_1))-4=10.2-4=16=n(S_1)$

and $3G(Arf(S_1))-(2+3)=3G(Arf(S_1))-5=3.7-5=16=G(S_1)$.

If $k=2$ then we write

$S_2=<5,14>=\ 0,5,10,14,15,19,20,24,25,28,29,30,33,34,35,38,39,40,42,43,44,45,47,48,49,50,52,\rightarrow...$.

Thus, we have $F(S_2)=51,\ n(S_2)=26,\ G(S_2)=26$, $Arf(S_2)=\ 0,5,10,14,\rightarrow...$,

$F(Arf(S_2))=13$, $n(Arf(S_2))=3$ and $G(Arf(S_2))=11$.

So, we write that $F(S_1)+20=31+20=51=F(S_2)$,
$n(S_1)+10=16+10=26=n(S_2)$ and $G(S_1)+10=16+10=26=G(S_2)$. Also, we obtain that $F(Arf(S_1))+5=8+5=13=F(Arf(S_2))$, $n(Arf(S_1))+1=2+1=3=n(Arf(S_2))$ and $G(Arf(S_1))+4=7+4=11=G(Arf(S_2))$.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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