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**ON THE STRUCTURE THEORY OF GRADED BURNSIDE RINGS**

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**Abstract.** Let $G$ denote a finite group and let $S$ be a finite $G$-set. It is well known that the Burnside ring $\Omega(G)$ of $G$ has its elements as the formal differences of isomorphism classes of finite G-sets. In [8], the category $(G, S, \Omega(G))$-gr, which consists of $\Omega(G)$-modules graded by $S$ as objects and the degree preserving $\Omega(G)$-linear maps as morphisms, was introduced. Using this category as a springboard, some interesting results in the structure theory of graded Burnside rings are brandished.

**Keywords**: Burnside ring; matrix ring; modules; categories; smash product; adjoint; Morita equivalence.

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## 1. Introduction

It is well known that module theory is usually adopted when we study the structure theory of rings or the Clifford theory for graded rings (see [1], [2], [5], [6], [10], and several others). When we consider a ring $R$ graded by a group $G$, we usually work with $G$-graded modules. A typical example of this arises when we consider $G$-graded modules over a group algebra $kG$ as modules over $kH$ by means of restriction of scalars, where $H$ is a subgroup of $G$ (see for instance [2], [5]). In this paper, we review the concept of graded Burnside module as in [8] and investigate the structure of graded Burnside ring. For the

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balance of this section, we give a brief description of the Burnside ring and describe the organisation of this paper.

Now let $G$ be a finite group. Recall that a $G$-set is a set $S$ on which $G$ acts from the left by permutation; that is we have a map $f : G \times S \to S : (g, s) \to gs$ such that $es = s$ ($e$ is the identity element of $G$) and $g(hs) = (gh)s$ for all $g, h$ in $G$ and $s$ in $S$. Furthermore $G$-sets form a category $G^{\wedge}$ with an obvious notion of morphisms. For any two $G$-sets $S_1$ and $S_2$, there exist the sum $S_1 \cup S_2$ (disjoint union) and product $S_1 \times S_2$ (cartesian product with diagonal action). In this way the isomorphism classes of $G$-sets form a semi ring $\Omega^+(G)$. The Grothendieck ring of $\Omega^+(G)$ is called the Burnside ring of $G$, denoted by $\Omega(G)$. Also recall that for two finite groups $G$ and $H$, any group homomorphism $\mu : H \to G$, gives rise to a functor $\mu^* : \mathfrak{G}^{set} \to \mathfrak{H}^{set}$, from the category $\mathfrak{G}^{set}$ of $G$-sets into the category $\mathfrak{H}^{set}$ of $H$-sets, obtained by restricting the action of $G$ on a $G$-set $S$ (via $\mu$) to $H$. Obviously $\mu^*$ commutes with sums and products and so defines a ring homomorphism $\mu^* : \Omega(G) \to \Omega(H)$, from the Burnside ring $\Omega(G)$ of $G$ into the Burnside ring $\Omega(H)$ of $H$. Furthermore $\mu^*$ has a left adjoint $\mu_* : \mathfrak{H}^{set} \to \mathfrak{G}^{set}$, from $H$-sets to $G$-sets, which commutes with sums and thus defines an additive map $\mu_* : \Omega(H) \to \Omega(G)$, which by adjointness, turns out to be an $\Omega(G)$-module homomorphism (see [9]). By the Frobenius reciprocity law $\Omega(H)$ is considered as a $\Omega(G)$-module via $\mu^*$. In this paper, we denote such modules by $M$.

In section 2 we start with the basic theory of $(G, S, \Omega(G))$-gr, which is the category consisting of the $\Omega(G)$-module $M$ graded by a $G$-set $S$ as objects and the degree preserving $\Omega(G)$-linear maps as morphisms. Theorem 2.8 is the first significant result in this section, namely that the category $(G, S, \Omega(G))$-gr is a Grothendieck category. In section 3, we introduce the notion of smash product for finite $G$-set denoted by $\Omega(G)\#S$. This notion has recently been used extensively to obtain duality results and category equivalences that provide a neat approach to categorical properties of modules studied in the theory of group actions as well as group gradings. We demonstrate here a sort of generalization of the notion of smash product from groups to finite $G$-sets. In theorem 3.2, we demonstrate that the category $(G, S, \Omega(G))$-gr is isomorphic to the category $\Omega(G)\#S$-mod. This implies that we can always consider $(G, S, \Omega(G))$-gr as a module category isomorphic to $\Omega(G)\#S$-module. We conclude the study of the smash product in section 4 by considering a characterization of smash products in terms of matrix rings and derive a glossary of some interesting corollaries; for instance, we demonstrate that $\Omega(G)\#G/H$ is Morita anti-equivalent to the ring $(\Omega(G)\#G) * H$, the skew group ring of $H$ over the ring $\Omega(G)\#G$. In section 5, we extend the applicability of some functors considered by Dade in [2] for homogenous $G$-sets and a ring $R$, to arbitrary $G$-sets and the Burnside ring $\Omega(G)$. Given a morphism $\phi : S \to S'$ of $G$ sets we associate the functor $T_\phi : (G, S, \Omega(G))$-gr $\longrightarrow (G, S', \Omega(G))$-gr. $T_\phi$ has an exact right adjoint denoted by $U^\phi$. Furthermore $U^\phi$ is a left adjoint for $T_\phi$ if for every $s' \in S'$ the set $\phi^{-1}(s')$ is finite. With this simple observation, we can infer that if $S$ is finite and $Q \in (G, S, \Omega(G))$-gr is an injective object then $Q$ is injective in $\Omega(G)$-module. If $S$ is a $G$-set and $B$ is a subset of $S$ fixed by some subgroup $H$ of $G$, then we may define a functor

$$T^B : (G, S, \Omega(G))\text{-gr} \longrightarrow (H, B, \Omega(G)^{(H)})\text{-gr},$$

where $(\Omega(G))^{(H)} = \bigoplus_{h\in H} \Omega(G)_h$, by setting $T^B(M) = \bigoplus_{s\in B} M_s$. In theorem 5.6, we demonstrate that $T^B$ has a left adjoint, denoted by $U^B$; which is an extension of the case considered by Dade in [2]. The same result tells us that $T^B$ has also a right adjoint, and it will be denoted by $U_B$. We say that $U^B$ and $U_B$ are the induction and the coinduction functor, respectively. Particular cases of these functors have been used in many literature (see for instance [6]) and it is well known (see [5]) that for strongly graded rings induction and coinduction are isomorphic. The case where $B = \{s\}$ and $H = G_s$ is the stabilizer subgroup of $s$ lead to the functors:

$$T^s : (G, S, \Omega(G))\text{-gr} \longrightarrow \Omega(G)^{\{G_s\}}\text{-mod}, \quad T^s(M) = M_s.$$

$$U^s : \Omega(G)^{\{G_s\}}\text{-mod} \longrightarrow (G, S, \Omega(G))\text{-gr}, \quad U^s(N) = \Omega(G) \bigotimes_{\Omega(G)^{(G_s)}} N.$$

If $\Omega(G)$ is strongly $G$-graded and $S$ is a transitive $G$-set then $(G, S, \Omega(G))$-gr is equivalent to $(\Omega(G))^{(G_s)}$-mod and the equivalence is given by $T^s$ and $U^s$ (or $U_s$). Furthermore, for any subgroup $H$ of $G$, we have that $(\Omega(G))^{(H)}$-mod is equivalent to $(G, G/H, \Omega(G))$-gr. For a detailed treatment of the application of the induction and coinduction we refer the reader to [6]. For the balance of this paper, $G$ is a multiplicative group with identity element 1. $\Omega(G)$ is said to be $G$-graded if $\Omega(G) = \bigoplus_{g\in G} \Omega(G)_g$, where each $\Omega(G)_g$ is an additive subgroup of $\Omega(G)$ and $\Omega(G)_g\Omega(G)_h \subset \Omega(G)_{gh}$ for all $g, h \in G$; when $\Omega(G)_g\Omega(G)_h = \Omega(G)_{gh}$ we say that $\Omega(G)$ is strongly graded. We shall write $\Omega(G)$-mod for the category of $\Omega(G)$-modules and $\Omega(G)$-gr for the category of graded $\Omega(G)$-modules.

## 2. Preliminaries

We begin this section with the following definition.

**Definition 2.1.** *An $\Omega(G)$-module $M$ such that $M = \bigoplus_{s\in S} M_s$, and for all $g \in G$ and $s \in S$ we have that $\Omega(G)_g M_s \subset M_{gs}$, where each $M_s$ is an additive subgroup of $M$, is said to be a (left) graded $\Omega(G)$-module of type $S$.*

Suppose $M = \bigoplus_{s\in S} M_s$ and $N = \bigoplus_{s\in S} N_s$ are graded $\Omega(G)$-modules of type $S$, then a morphism $f : M \to N$ is a $\Omega(G)$-linear map such that $f(M_s) \subset N_s$ for all $s \in S$. The category $(G, S, \Omega(G))$-gr consists of objects as the graded $\Omega(G)$-modules of type $S$ and morphisms as the $\Omega(G)$-linear maps just defined. Now consider $M = \bigoplus_{s\in S} M_s$ in $(G, S, \Omega(G))$-gr and an $m$ $(\neq 0)$ in $M$. Then $m$ has a unique decomposition $m = \sum_{s\in S} m_s$, with $m_s \in M_s$, and the $m_s(\neq 0)$ are called the homogenous components of $m$. A submodule $N$ of $M$ is called a graded submodule (of type S) if for any element $n \in N$, each homogenous component of $n$ also belongs to $N$; this means that $N = \bigoplus_{s\in S}(N \cap M_s)$. Observe that if $N$ is a graded submodule of $M$, then $M/N$ is a graded module of type $S$ by setting $(M/N)_s = (M_s + N)/N$ for all $s \in S$. In the category $(G, S, \Omega(G))$-gr it is easy to see that direct sums and products exist (indeed if $(M_i)_{i\in I}$ is a family of objects of $(G, S, \Omega(G))$-gr , then the module $\bigoplus_{s\in S}(\bigoplus_{i\in I}(M_i)_s)$ is a direct sum, and $\bigoplus_{s\in S}(\prod_{i\in I}(M_i)_s)$ is a direct product of the family in the category $(G, S, \Omega(G))$-gr, respectively). For $M, N$ in $(G, S, \Omega(G))$-gr the set $Hom_{(G,S,\Omega(G))-gr}(M, N)$ is a subgroup of $Hom_{\Omega(G)}(M, N)$. Because for any morphism $f$ in $Hom_{(G,S,\Omega(G))-gr}(M, N)$ both $ker(f)$ and $coker(f)$ are objects of

$(G, S, \Omega(G))$-gr, we have that $(G, S, \Omega(G))$-gr is an abelian category. When $G = S$ with the natural left action of $G$ on itself, then the category $(G, G, \Omega(G))$-gr coincides with the category $\Omega(G)$-gr. For a subgroup $H$ of $G$, and $S = G/H$, where $G$ acts by left translation, we write $(G/H, \Omega(G))$-gr for the category $(G, G/H, \Omega(G))$-gr. Furthermore, when $H$ is normal then $(G/H, \Omega(G))$-gr may be identified with $\Omega(G)$-$\text{gr}_{G/H}$, the category of all graded left $\Omega(G)$-modules of type $G/H$ (here $\Omega(G)$ is considered as a $G/H$-graded ring). If $S$ is a singleton with $G$ acting trivially on it, then the category $(G, S, \Omega(G))$-gr coincides with $\Omega(G)$-mod. The elementary properties of $(G, S, \Omega(G))$-gr considered in this section are similar to those of $R$-gr (where $R$ is an arbitrary ring) and the proofs are similar to those in [5]. Therefore we shall omit proofs to most results in this section.

**Proposition 2.2.** *Suppose $M, N, P \in (G, S, \Omega(G))$-gr is fixed to satisfy the following commutative diagram of $\Omega(G)$-linear maps*

$$\begin{array}{ccc} M & \xrightarrow{\gamma} & N \\ {\scriptstyle\alpha}\searrow & & \swarrow{\scriptstyle\beta} \\ & P & \end{array}$$

*If $\beta$, resp. $\gamma$ is a morphism in $(G, S, \Omega(G))$-gr, then there exists a morphism $\gamma'$, resp. $\beta'$ in this category, such that $\alpha = \beta \circ \gamma'$ resp $\alpha = \beta' \circ \gamma$. More especially, we have that if $M \in (G, S, \Omega(G))$-gr is projective, resp. injective in $\Omega(G)$-mod, then $M$ is projective, resp. injective, in $(G, S, \Omega(G))$-gr.*

**Proof.** Follow similar argument of the proof of lemma 1.2.1 of [5].

**Proposition 2.3.** *Let $G$ be a finite group and let $S$ be a $G$-set such that each stabilizer $G_s$ $(s \in S)$ is normal (this condition holds if for example $S$ is a free $G$-set or if $G$ is abelian) in $G$. If $N$ is a graded submodule of $M \in (G, S, \Omega(G))$-gr, then $N$ is essential in $M$ as an object of $(G, S, \Omega(G))$-gr, if and only if $N$ is essential in $M$ as an object of $\Omega(G)$-module.*

**Proof.** Because we have that $(G, S, \Omega(G))$-gr is equivalent to a product of categories of the type $(G/G_s, \Omega(G))$-gr (see [5]), it suffices to restrict our attention to the case where $S = G/H$, for a normal subgroup $H$ of $G$. In which case $(G/H, \Omega(G))$-gr is a category of graded modules in the classical sense, and so we invoke lemma 1.2.8 of [5] to complete the proof.

For a morphism of $G$-sets $\rho : S \to S'$, we may associate a canonical covariant functor $T_\rho : (G, S, \Omega(G))$-gr $\to (G, S', \Omega(G))$-gr, defined as follows: $T_\rho(M)$ is the $\Omega(G)$-module $M$ with $S'$-gradation given by $M_{s'} = \bigoplus\{M_s \mid s \in S,\ \rho(s) = s' \text{ for } s' \in S'\}$, where we set $M_{s'} = 0$ if $s' \notin \rho(S)$. Clearly for any $\lambda_g \in \Omega(G)_g$, we have

$$\lambda_g M_{s'} \;=\; \lambda_g(\bigoplus\{M_s \mid s \in \rho^{-1}(s')\}) \;\subset\; \bigoplus\{M_{gs} \mid s \in \rho^1(s')\} \;=\; M_{gs'}.$$

That $M = \bigoplus_{s' \in S'} M_{s'}$, is obvious. For a morphism $f \in Hom_{(G,S,\Omega(G))-gr}(M, N)$, we set $T_\rho(f) = f$. Note that $T_\rho$ is an exact functor.

**Proposition 2.4.** *If $S \xrightarrow{\rho} S' \xrightarrow{\rho'} S''$ is a morphism of $G$-sets, then*
*(1) $T_{\rho' \circ \rho} = T_{\rho'} \circ T_\rho$.*
*(2) If $\rho$ is an isomorphism, then $T_\rho \circ T_{\rho^{-1}} = T_{\rho^{-1}} \circ T_\rho = Id$; more especially $T_\rho$ is an isomorphism of categories.*

**Proof.** Easy verifications.

Let $S$ be a $G$-$G'$-set, where $G$ and $G'$ are groups. For $h \in G'$ we may define a morphism of $G$-sets: $\rho_h : S \to S, s \to xh$ for $s \in S$. Because we have that $\rho$ is an isomorphism of $G$-sets, the functor $T_h = T_{\rho h}$ is an isomorphism of categories. It is a folklore to verify the following:

1. $T_1 = Id$.
2. $T_h \circ T_{h'} = T_{h'h}$ for all $h, h' \in G'$.
3. $T_h \circ T_{h^{-1}} = Id$ for all $h \in G'$.

If $M \in (G, S, \Omega(G))$-gr, then denote $T_h(M)$ by $M(h)$ for each $h \in G'$, so $M(h) = M$ as $\Omega(G)$-modules and for each $s \in S$ we have $M(h)_s = M_{sh}$. We call the object $M(h)$ the $h$-suspension of $M$. For some specific examples see [5]. For $S$ a $G$-$G'$-set, we say that $\Omega(G)$-linear map $f : M \to N$, where $M, N \in (G, S, \Omega(G))$-gr is a graded morphism of degree $h \in G'$ if $f(M_s) \subset N_{sh}$ for all $s \in S$. Graded morphisms of degree $h \in G'$ form an additive subgroup $HOM_{\Omega(G)}(M, N)_h$, of $Hom_{\Omega(G)}(M, N)$. We denote by $HOM_{\Omega(G)}(M, N)$ the subgroup of $Hom_{\Omega(G)}(M, N)$ generated by all $HOM_{\Omega(G)}(M, N)_h$, $(h \in G')$. It is clear that $HOM_{\Omega(G)}(M,N) = \bigoplus_{h\in G'} HOM_{\Omega(G)}(M,N)$, is a graded abelian group of type $G'$. If $f : M \to N$ and $e : N \to P$ are graded morphisms of degree $h, h' \in G'$ respectively then $e \circ f$ has degree $hh'$. Consequently, we may view $END_{\Omega(G)}(M) = HOM_{\Omega(G)}(M,M)$ as a graded ring of type $G'$ if we define the multiplication by $f \cdot e = e \circ f$.

**Proposition 2.5.** *If $\Omega(G)$ is $G$ graded then for a family of $G$-sets $(S_i)_{i\in I}$, the categories $(G,\ \coprod_{i\in I} S_i,\ \Omega(G))$-gr and $\prod_{i\in I}(G, S_i, \Omega(G))$-gr are equivalent.*

**Proof.** We assume that the sets $S_i$ are mutually disjoint; in which case, $S = \cup_{i\in I} S_i$ is a direct sum. Let $T : (G, S, \Omega(G))$-gr $\to \prod_{i\in I}(G, S_i, \Omega(G))$-gr be a functor defined by $T(M) = (M_i)_{i\in I}$, where $M_i = \bigoplus_{s\in S_i} M_s$, and for a morphism in $(G, S, \Omega(G))$-gr, $\ f : M \to N$ say, we put $T(f) = (f_i)_{i\in I}$, where $f_i$ is the restriction of $f$ to $M_i$. Now let $U : \prod_{i\in I}(G, S_i, \Omega(G))$-gr $\to (G, S, \Omega(G))$-gr, be a functor defined by $U((M_i)_{i\in I}) = \bigoplus_{i\in I} M_i = M$ as $\Omega(G)$-modules, and for $s \in S,\ M_s = (M_i)_s$, where $i$ is such that $S_i$ is the unique one containing $s$ so that $M$ is clearly a graded $\Omega(G)$-module of type $S$. Furthermore, if $(f_i)_{i\in I}$ is a morphism in $\prod_{i\in I}(G, S_i, \Omega(G))$-gr, then $U((f_i)_{i\in I}) = f$, where $f = \bigoplus_{i\in I} f_i$. It is obvious that $U \circ T$ is the identity of $(G, S, \Omega(G))$-gr and $T \circ U$ is the identity of $\prod_{i\in I}(G, S_i, \Omega(G))$-gr.

**Corollary 2.6.** *Let $\Omega(G) = \bigoplus_{g\in G} \Omega(G)_g$ be a $G$ graded and let $S$ be a $G$-set. Then $(G, S, \Omega(G))$-gr is equivalent to the product of the $(G/G_s, \Omega(G))$-gr, where $s$ varies over a set of representatives for the $G$-orbits in $S$. More especially, if $S$ is a free $G$-set, then $(G, S, \Omega(G))$-gr is a product of copies of $\Omega(G)$-gr over the $G$-orbits in $S$.*

**Proof.** One has $S = \cup_s G_s$ and $G_s \simeq G/G_s$, so the assertion follows from the foregoing proposition.

To state the next theorem we need the following definition.

**Definition 2.7.** *Let $\Omega(G) = \bigoplus_{g\in G} \Omega(G)_g$ be $G$-graded and $S$ a $G$-set. For each $s \in S$ the $s$-suspension $\Omega(G)[s]$ of $\Omega(G)$ is defined to be the object of $(G, S, \Omega(G))$-gr which coincides with $\Omega(G)$ as an $\Omega(G)$-module, but with gradation defined by $\Omega(G)[s]_t = \bigoplus\{\Omega(G)_g \mid g \in G, gs = t,\ t \in S\}$.*

**Theorem 2.8.** *The object* $V = \bigoplus_{s\in S} \Omega(G)[s]$ *is a projective generator of the category* $(G, S, \Omega(G))$*-gr and it is free of rank* $|S|$ *as an* $\Omega(G)$*-module. In particular,* $(G, S, \Omega(G))$*-gr is a Grothendieck category.*

**Proof.** Obviously $V$ is a free $\Omega(G)$-module, and so we can fix a canonical basis $\{e_s \mid s \in S\}$. Let $N \subset M$ in $(G, S, \Omega(G))$-gr be such that $N \neq M$; say $M = \bigoplus_{s\in S} M_s,\ N = \bigoplus_{s\in S} N_s$. Then there exist $s \in S,\ m_s \in M_s$ such that $m_s \notin N_s$. Now we may define a morphism in $(G, S, \Omega(G))$-gr by setting $f(e_s) = m_s, f(e_s) = 0$ for $s' \neq s$ in $S$, and clearly $Im(f) \not\subset N$. Consequently, $V$ is a generator for $(G, S, \Omega(G))$-gr and a projective object of $(G, S, \Omega(G))$-gr by Proposition (2.2).

**Corollary 2.9.** *Let* $M \in (G, S, \Omega(G))$*-gr. Then* $M$ *is a projective object in* $(G, S, \Omega(G))$*-gr if and only if* $M$ *is a projective left* $\Omega(G)$*-module.*

**Proof.** Proposition (2.2), concludes the "if" part, and the converse follows from Theorem (2.8) by observing that any projective object $M$ in $(G, S, \Omega(G))$-gr is isomorphic to a direct summand of some direct sum of copies of $V$ which is a projective generator in $(G, S, \Omega(G))$-gr.

Let $gr - p\ dim(M)$ denote the projective dimension of an $M$ in $(G, S, \Omega(G))$-gr. Then corollary (2.9) yields:

**Corollary 2.10.** *gr-p* $dim(M) = p\ dim(M)$ *for any* $M$ *in* $(G, S, \Omega(G))$*-gr.*

## 3. Smash Product

Let $\Omega(G) = \bigoplus_{g\in G} \Omega(G)_g$ be $G$-graded, and let $S$ be a finite $G$-set. We define the smash product $\Omega(G)\#S$ as the free $\Omega(G)$-module with basis $\{p_s \mid s \in S\}$ and multiplication defined by

$$(a_g p_s)(b_h p_t) := \begin{cases} (a_g b_h)p_t & \text{if } ht = s \\ 0, & \text{if } ht \neq s \end{cases} \tag{1}$$

for any $g, h \in G,\ a_g \in \Omega(G)_g, b_h \in \Omega(G)_h$, and $s, t \in S$. We may extend this by $\mathbb{Z}$-bilinearity to a product on all of $\Omega(G)\#S = \bigoplus\{\Omega(G)_g p_s \mid g \in G, s \in S\}$.

**Proposition 3.1.** *The multiplication defined by equation (1) above turns $\Omega(G)\#S$ into a ring with identity $1 = \sum_{s\in S} p_s$ and $\{p_s \mid s \in S\}$ is a set of orthogonal idempotents. The following properties also hold:*

*(i) The map $\theta : \Omega(G) \to \Omega(G)\#S,\ a \to a \cdot 1 = \sum_{s\in S} ap_s$ is an injective ring morphism.*

*(ii) For all $a_g \in \Omega(G)_g, s \in S$ one has that: $p_s a_g = a_g p_{g^{-1}s}$, where $p_s a_g$ is the product $P_s\theta(a_g)$ in the ring $\Omega(G)\#S$.*

*(iii) The set $\{p_s \mid s \in S\}$ is a basis for $\Omega(G)\#S$ as an $\Omega(G)$-module.*

*(iv) For $s \in S,\ p_s$ centralizes $\Omega(G)^{(G_s)} = \bigoplus_{t\in G_s} \Omega(G)_t$.*

*(v) $H := \cap\{G_s \mid s \in S\}$ is a normal subgroup of $G$ such that for any $b \in \Omega(G)^{(H)} = \bigoplus_{h\in H} \Omega(G)_h, s \in S$, we have $bp_s = p_s b$.*

*(vi) $Aut_G(S)$ acts on $\Omega(G)\#S$ by $(ap_s)^\alpha = ap^{-1}_{\alpha(s)}$, for $\alpha \in Aut_G(S), a \in \Omega(G)$, and $s \in S$.*

**Proof.** Associativity of equation (1) is the only difficult thing to show in the first claim. To show this, let $l, g, h \in G, a_l \in \Omega(G)_l, b_g \in B(G)_g, c_h \in \Omega(G)_h$, and let $s, t, u \in S$. Then one has that

$$[(a_l p_s)(b_g p_t)](c_h p_u) := \begin{cases} (a_l b_g c_h)p_u, & \text{if } gt = s \text{ and } hu = t \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

and

$$(a_l p_s)[(b_g p_t)(c_h p_u)] := \begin{cases} (a_l b_g c_h)p_u, & \text{if } hu = t \text{ and } ghu = s \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

If $hu = t$, then the equation $gt = s$ and $ghu = s$ are equivalent, and so associativity follows. Because $ap_s(\sum_{t\in S} p_t) = \sum_{t\in S}(ap_s)p_t = ap_s$ and $(\sum_{s\in S} p_s)bp_t = \sum_{s\in S} p_s(bp_t) = (\sum_{s\in S}(\sum_{gt=s} b_g))p_t = bp_t$, it follows that $\sum_{s\in S} p_s$ is the identity of $\Omega(G)\#S$, the claim follows since $b = \sum_{s\in S}(\sum_{gt=s} b_g)$. The assertions (i) and (ii) are obvious, and one can easily check properties (iv) and (v). We now verify (iii) and (vi).

(iii) From (ii) it is clear that $\Omega(G)_g p_s = p_{gs}\Omega(G)_g \simeq \Omega(G)_g$. So we have

$$\begin{aligned}\Omega(G)\#S &= \bigoplus\{\Omega(G)_g p_s \mid g \in G, s \in S\} = \bigoplus\{p_{gs}\Omega(G)_g \mid g \in G, s \in S\} \\ &= \bigoplus\{p_t\Omega(G)_g \mid s \in S\} \\ &= \{p_t\Omega(G) \mid t \in S\},\end{aligned}$$

with each $p_t\Omega(G) = \bigoplus\{p_t\Omega(G)_g \mid g \in G\} \simeq \bigoplus\{\Omega(G)_g \mid g \in G\} = \Omega(G)$. Hence (iii) holds.

(vi) Suppose $\alpha \in Aut_G(S)$. Then we calculate for $a_g \in \Omega(G)_g,\ b_h \in \Omega(G)_h$, and $s, t \in S$ :

$$[(a_g p_s)(b_h p_t)]^\alpha = \begin{cases} [(a_g b_h)p_t]^\alpha = (a_g b_h)p_{\alpha^{-1}(t)} & \text{if } ht = s \\ 0, & \text{if } ht \neq 0 \end{cases} \tag{4}$$

and

$$(a_g p_{\alpha^{-1}(s)}(b_h p_{\alpha^{-1}(t)}) = \begin{cases} (a_g b_h)p\alpha^{-1}(t) & \text{if } h\alpha^{-1}(t) = \alpha^{-1}(s) \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

Because we have that $ht = s$ if and only if $h\alpha^{-1}(t) = \alpha^{-1}(g)$, it follows that

$$[(a_g p_s)(b_h p_t)]^\alpha = (a_g p_s)^\alpha (b_h p_t)^\alpha.$$

So $Aut_G(S)$ acts on the ring $\Omega(G)\#S$ as described; in view of the fact that $ap_s \mapsto (ap_s)^\alpha = ap_{\alpha^{-1}(s)}$ is a bijective map.

Note that if $S = G$ and $G$ acts on $S$ by left translation, then the multiplication in $\Omega(G)\#S$ becomes $(ap_s)(bp_t) = (ab_{st^{-1}})pt$ for $a, b \in \Omega(G)$, and $s, t \in G$; in which case $\Omega(G)\#G$ coincides with the smash product defined by Cohen and Montgomery in [1] for arbitrary rings. Moreover if $S = \coprod_{i\in I} S_i$ is a finite direct sum of finite $G$-sets, then the ring $\Omega(G)\#S$ is isomorphic to the direct product $\prod_{i\in I}\Omega(G)\#S_i$. Consequently, we obtain the following result, which extends theorem 2.2 of [1] for arbitrary $G$-sets.

**Theorem 3.2.** *Let $S$ be a finite $G$-set and let $\Omega(G) = \bigoplus_{g\in G}\Omega(G)_g$ be $G$-graded. Then the category $(G, S, \Omega(G))$-gr is isomorphic to the category $\Omega(G)\#S$-mod.*

**Proof.** By proposition 3.1(i), $\Omega(G)$ may be viewed as a subring of $\Omega(G)\#S$ via the morphism $\theta$. By restriction of scalars, $M \in \Omega(G)\#S$-mod has an $\Omega(G)$-module structure. For $s \in S$, we set $M_s = p_s M$. Because we have that $1 = \sum_{s\in S} p_s$ and $\{p_s \mid s \in S\}$ is a family of orthogonal idempotents we have that $M = \bigoplus_{s\in S} M_s$. Moreover, if $a_g \in \Omega(G)_g$ then $a_g M_s = a_g p_s M = (p_{gs} a_g)M$ is an element of $p_{gs}M = M_{gs}$; implying that $M$ is an object of $(G, S, \Omega(G))$-gr. For a morphism $f : M \to N$ in $\Omega(G)\#S$-mod we have $f(M_s) = f(p_s M) = p_s f(M) \subset p_s N = N_s$. In this way we obtain a functor $()_{gr} : \Omega(G)\#S$-mod $\longrightarrow (G, S, \Omega(G))$-gr, where $(M)_{gr}$ is the graded structure of type $S$ defined on $M$ as illustrated above, and $(f)_{gr} = f$.

Conversely, take an object $M$ in $(G, S, \Omega(G))$-gr. For $a_g \in \Omega(G)_g$, and $s \in S$, put $(a_g p_s)m = a_g m_s$, where $m \in M$ is given by $m = \sum_{s\in S} m_s$. Since $1 = \sum_{s\in S} p_s$ we have that $1 \cdot m = (\sum_{s\in S} p_s)m = \sum_{s\in S} m_s = m$. If $g, h \in G, a_g \in \Omega(G)_g, b_h \in \Omega(G)_h, s, t \in S$ then we calculate

$$[(a_g p_s)(b_h p_t)]m = \begin{cases} [(a_g b_h)p_t]m = a_g b_h m_t, & \text{if } ht = s \\ 0, & \text{if } ht \neq s \end{cases} \tag{6}$$

On the other hand:

$$(a_g p_s)[(b_h p_t)m] = \begin{cases} (a_g p_s)(b_h m_t) = a_g b_h m_t, & \text{if } ht = s \\ 0, & \text{if } ht \neq s. \end{cases} \tag{7}$$

This implies that we can consider $M$ as an $\Omega(G)\#S$-module. For a morphism $f : M \to N$ in $(G, S, \Omega(G))$-gr, one has that $f[(ap_s)m] = f(am_s) = af(m_s) = ap_s f(m)$, because $f(M_s) \subset N_s$. Therefore we arrive at a functor $(-)^\# : (G, S, \Omega(G))$-gr $\longrightarrow \Omega(G)\#S$-mod, where $M^\#$ is the $\Omega(G)$-module $M$ equipped with the structure of $\Omega(G)\#S$-module defined above, and $f^\# = f$ for each morphism in the category $(G, S, \Omega(G))$-gr. If $M \in (G, S, \Omega(G))$-gr, then

$$am = (a \cdot 1)m = (\sum_{s\in S} ap_s)m = \sum_{s\in S} am_s = a(\sum_{s\in S} m_s) = am$$

holds in $M^\#$. Therefore $(-)_{gr} \circ (-)^\#$ is the identity.

Conversely, suppose we consider $M \in \Omega(G)\#S$-mod as an object of $(G, S, \Omega(G))$-gr with the grading $M \bigoplus_{s\in S} M_s$, where $M_s = P_s M$, then $(ap_s)m = am_s$ holds for $a \in \Omega(G), s \in S$, and $m \in M$. Because $m = \sum_{s\in S} p_s m$, we have $m_s = p_s m$ and so $(ap_s)m = a(p_s m)$. It follows that $(-)^{\#} \circ (-)$-gr is the identity functor.

## 4. Matrix Ring

Here we obtain a characterization of smash products in terms of matix rings and study the properties thereto. If $S$ is a finite $G$-set, then one has that $V = \bigoplus_{s\in S} \Omega(G)(s)$ is a projective generator of $(G, S, \Omega(G))$-gr by theorem 2.8. Furthermore, $(G, S, \Omega(G))$-gr is equivalent to the category of $U$-modules in view of a result of Mitchell [3], where $U = End_{(G,S,\Omega(G))-gr}(V)$. On the other hand, by theorem 3.2, the category $(G, S, \Omega(G))$-gr is equivalent to the category $\Omega(G)\#S$-mod. This implies that $U$ is Morita equivalent to the opposite ring $(\Omega(G)\#S)^{opp}$. We demonstrate a stronger result in the next theorem.

**Theorem 4.1.** *Let $\Omega(G)$ be $G$-graded and let $S = \{s_1, \ldots, s_n\}$ be a finite $G$-set. The rings $(\Omega(G)\#S)^{opp}$, $U = End_{(G,S,\Omega(G))-gr}(V)$ and $T$ are isomorphic, where*

$$T = \begin{pmatrix} \Omega(G)(s_1)_{s_1} & \Omega(G)(s_1)_{s_2} & \Omega(G)(s_1)_{s_3} & \cdots & \Omega(G)(s_1)_{s_n} \\ \Omega(G)(s_2)_{s_1} & \Omega(G)(s_2)_{s_2} & \Omega(G)(s_2)_{s_3} & \cdots & \Omega(G)(s_2)_{s_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \Omega(G)(s_n)_{s_1} & \Omega(G)(s_n)_{s_2} & \Omega(G)(s_n)_{s_3} & \cdots & \Omega(G)(s_n)_{s_n} \end{pmatrix}.$$

**Proof.** Recall that for each $s \in S$, one has that $\Omega(G)(s)$ is the object of $(G, S, \Omega(G))$-gr equal to $\Omega(G)$ as an $\Omega(G)$-module. Furthermore, it has gradation given by

$$\Omega(G)(s)_t = \bigoplus \{\Omega(G)_g \mid g \in G, gs = t\}.$$

Theorem 3.2 garantees an equivalence $\Omega(G)\#S - mod \approx (G, S, \Omega(G))$-gr. We apply this equivalence to the regular $\Omega(G)\#S$-module $\Omega(G)\#S$. As a left $\Omega(G)\#S$-module this is the direct sum of its submodules $(\Omega(G)\#S)_{p_s} = \Omega(G)p_s$, each of which is a regular left

$\Omega(G)$-module. The $S$-grading of $(\Omega(G)\#S)p_s$ is given by:

$$((\Omega(G)\#S)p_s)_t \;=\; p_t(\Omega(G)\#S)p_s \;=\; p_s \;=\; p_t\Omega(G)p_s \;=\; \sum_{g\in G} p_t\Omega(G)_g p_s$$
$$= \bigoplus\{\Omega(G)_g p_s \mid g \in G, gs = t\}.$$

This implies that $(\Omega(G)\#S)p_s \;\cong\; \Omega(G)(s)$ as objects of $(G, S, \Omega(G))$-gr. Therefore we have

$$\Omega(G)\#S \;=\; \bigoplus_{s\in S}(\Omega(G)\#S)p_s \;\cong\; \bigoplus_{s\in S}\Omega(G)(s) \;=\; V$$

as objects of $(G, S, \Omega(G))$-gr. By theorem 3.2, this means that

$$\begin{aligned} U \;=\; End_{(G,S,\Omega(G))-\mathrm{gr}}(V) \;\cong\; End_{(G,S,\Omega(G))-\mathrm{gr}}(\Omega(G)\#S) \;&\cong\; End_{\Omega(G)\#S}(\Omega(G)\#S) \\ &\cong\; (\Omega(G)\#S)^{opp}. \end{aligned}$$

Now suppose $M \in (G, S, \Omega(G))$-gr, $M = \bigoplus\limits_{i=1}^{n} M_i$, then $End_{(G,S,\Omega(G))-gr}(M) \cong$

$$\begin{pmatrix} Hom_{(G,S,\Omega(G))-gr}(M_1, M_1) & Hom_{(G,S,\Omega(G))-gr}(M_2, M_1) & \cdots & Hom_{(G,S,\Omega(G))-gr}(M_n, M_1) \\ Hom_{(G,S,\Omega(G))-gr}(M_1, M_2) & Hom_{(G,S,\Omega(G))-gr}(M_2, M_2) & \cdots & Hom_{(G,S,\Omega(G))-gr}(M_n, M_2) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ Hom_{(G,S,\Omega(G))-gr}(M_1, M_n) & Hom_{(G,S,\Omega(G))-gr}(M_2, M_n) & \cdots & Hom_{(G,S,\Omega(G))-gr}(M_n, M_n) \end{pmatrix}$$

the isomorphism sending each $u \in End_{(G,S,\Omega(G))-gr}(M)$ to the matrix $(u_{i,j})_{1\le i,j\le n}$, $u_{ij} = q_i \circ u \circ p_j$, where $q_i : M \to M_i$ denotes the canonical projections and $p_j = M_j \to M$ denotes the canonical injections. But because

$$Hom_{(G,S,\Omega(G))-gr}(\Omega(G)(s_i), \Omega(G)(s_j)) \cong \Omega(G)(s_j)_{s_i},$$

as one can easily check, the desired isomorphism between $U$ and $T$ follows.

Theorem 4.1 has many useful applications. To give one application, let $S = G/H$, where $H$ is a subgroup of $G$ of finite index. To the canonical $G$-set morphism $\tau : G \to G/H$ we associate the canonical functor $T_\tau : \Omega(G)$-gr $\to (G/H, \Omega(G))$-gr, as defined earlier. For $M \in \Omega(G)$-gr, we put $END_{\Omega(G)}(M) = \bigoplus\limits_{g\in G} END_{\Omega(G)}(M)_g$, where

$$END_{\Omega(G)}(M)_g = \{f \in HOM_{\Omega(G)}(M, M) \mid f(M_d) \subset M_{dg} \text{ for all } d \in G\}.$$

It is clear that $END_{\Omega(G)}(M)$ is a $G$-graded ring with multiplication given by $f \cdot g = g \circ f$ for $f, g \in END_{\Omega(G)}(M)$. Put $M^* = T_\tau(M)$. Then $M^*$ is an object of $(G/H, \Omega(G)$-gr which is equal to $M$ as an $\Omega(G)$-module but graded as follows: $M^* = \bigoplus_{c \in G/H} M_c$, where $M_c = \bigoplus_{g \in c} M_g$. Now we state:

**Proposition 4.2.** *If an object $M \in \Omega(G)$-gr is finitely generated or $G$ is finite, then $END_{\Omega(G)}(M)^{(H)} = END_{(G/H,\Omega(G))-gr}(M^*)$.*

**Proof.** For $g \in H$, and any $c \in G/H$, let $f \in END_{\Omega(G)}(M)_g$. Then

$$f(M_c) \;=\; f(\bigoplus_{d \in c} M_d) \;\subset\; \bigoplus_{d \in c} M_{dg} \;=\; M_c;$$

and this yields $END_{\Omega(G)}(M)^{(H)} \subset END_{(G/H,\Omega(G))_{gr}}(M^*)$.

Now let $f \in END_{(G/H,\Omega(G)-gr}(M^*)$. For the cases we have at hand here, we invoke corollary 1.2.11 of [5] which asserts that $END_{\Omega(G)}(M) = End_{\Omega(G)}(M)$; and this implies that $End_{(G/H,\Omega(G)-gr}(M^*) \subset END_{\Omega(G)}(M)$. Therefore we can decompose $f = f_{g_1} + f_{g_2} + \cdots + f_{g_n}$, where $f_{g_i}$ is a non-zero morphism of degree $g_i$. We claim that $g_1, \ldots, g_n \in H$. Indeed, because $f_{g_1} \neq 0$, there exists $c \in G/H$ (put this $c = gH$) such that $f_{g_i}(M_c) \neq 0$. Also, there exists an element $m \in M_{gh},\ h \in H,\ m \neq 0$, such that $f_{g_1}(m) \neq 0$. But $f(m) = f_{g_1}(m) + f_{g_2}(m) + \cdots + f_{g_n}(m)$, and $f_{g_i}(m) \in M_{ghg_i}$, for $i = 1, \ldots, s$. Since $ghg_i \neq ghg_j$, for $i \neq j$ and $f(m) \in f(M_c) \subset M_c$, we have $f_{g_1}(m) \in M_{gh'}$, for some $h' \in H$. Hence $ghg_1 = gh'$ and therefore $g_1 = h^{-1}h' \in H$. In a similar way $g_2, \ldots, g_n \in H$; therefore $f \in END_{\Omega(G)}(M)^{(H)}$.

Now for $H \leq G$ a subgroup of $G$ of finite index $n < \infty$, suppose $\{g_1,\ \ldots,\ g_n\}$ is a set of representatives for the left cosets of $H$ in $G$, and $V = \bigoplus_{i=1}^{n} \Omega(G)(g_i H)$; then

$$\Omega(G)(g_i H)_{g_j H} \;=\; \bigoplus \{\Omega(G)_g \mid g \in G,\ g g_i H = g_j H\} \;=\; \Omega(G)^{(g_j H g_i^{-1})} \;=\; \Omega(G)(g_i^{-1})^{(g_j H)}.$$

Therefore $\Omega(G)(g_i H) = \bigoplus_{j=1}^{n} \Omega(G)(g_i^{-1})^{(g_j H)} = T_\tau(\Omega(G)(g_i^{-1}))$, and so

$$V \;\cong\; T_\tau\left(\bigoplus_{i=1}^{n} \Omega(G)(g_i^{-1})\right)$$

is a small projective generator of $(G/H, \Omega(G))$-gr, by theorem 2.8; in which case, theorem 4.1 becomes:

**Corollary 4.3.** *The rings* $(\Omega(G)\#G/H)^{opp}$, $U = END_{(G/H,\Omega(G))-gr}(V)$ *and* $T$ *are isomorphic, where*

$$T = \begin{pmatrix} \Omega(G)^{(g_1Hg_1^{-1})} & \Omega(G)^{(g_2Hg_1^{-1})} & \dots & \Omega(G)^{(g_nHg_1^{-1})} \\ \Omega(G)^{(g_1Hg_2^{-1})} & \Omega(G)^{(g_2Hg_2^{-1})} & \dots & \Omega(G)^{(g_nHg_2^{-1})} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \Omega(G)^{(g_1Hg_n^{-1})} & \Omega(G)^{(g_2Hg_n^{-1})} & \dots & \Omega(G)^{(g_nHg_n^{-1})} \end{pmatrix}.$$

Let $G$ be a finite group, and put $K = \bigoplus_{g\in G} \Omega(G)(g)$. We have that $T_\Phi(K)$ is another small projective generator of $(G/H, \Omega(G))$-gr, and by a theorem of Mitchell [[3], p.17], $(G/H, \Omega(G))$-gr is equivalent to mod-$U'$, where $U' = End_{(G/H,\Omega(G))-gr}(T_\Phi(K))$. However, by proposition 4.2, $U' = END_{\Omega(G)}(K)^{(H)}$, and $END_{\Omega(G)}(K) \cong (\Omega(G)\#G) * G$, where $\Omega(G)\#G$ is the usual smash product [7]. Thus $END_{\Omega(G)}(K)^{(H)} \cong (\Omega(G)\#G)H$, where $(\Omega(G)\#G)H$ is the skew group ring of $H$ over the ring $\Omega(G)\#G$. Hence we have the following corollary:

**Corollary 4.4.** *For a subgroup* $H$ *of a finite group* $G$, *the rings* $(\Omega(G)\#G/H)^{opp}$ *and* $(\Omega(G)\#G) * H$ *are Morita equivalent.*

More information on the ring $(\Omega(G)\#G) * H$ is given by the next corollary:

**Corollary 4.5.** *Let* $G$ *be a finite group and* $H$ *a subgroup of* $G$. *Then* $(\Omega(G)\#G) * H \cong M_{(H)}(\Omega(G)\#G/H)$.

**Proof.** Keeping the notations above, we at once see that $T_\Phi(\Omega(G)(g)) \cong T_\Phi(\Omega(G)(h))$ whenever $g, h \in H$ is in the same coset $gH = hH$ of $H$. This implies that $T_\Phi(K)$ is isomorphic to a direct sum of $|H|$ copies of $V = T_\Phi\left(\bigoplus_{i=1}^{n} \Omega(G)(g_i^{-1})\right)$. Therefore $(\Omega(G)\#G) * H$ is anti-isomorphic to $END_{\Omega(G)}(K)$, which in turn is isomorphic to

$$M_{|H|}\left(END_{\Omega(G)}\left(\bigoplus_{i=1}^{n} \Omega(G)(g_i^{-1})^{(H)}\right)\right).$$

However, $M_{|H|}\left(END_{\Omega(G)}\left(\bigoplus_{i=1}^{n}\Omega(G)(g_i^{-1})^{(H)}\right)\right) \cong M_{|H|}(\Omega(G)\#G/H)$ by theorem 4.1 and proposition 4.2.

**Corollary 4.6.** *Let $\Omega(G) = \bigoplus_{g\in G}\Omega(G)_g$ be strongly $G$-graded and let $H$ be a subgroup of finite index in $G$. Then $\Omega(G)\#G/H$ and $\Omega(G)$ are Morita equivalent.*

**Proof.** Because we have that $\Omega(G)$ is strongly graded, it follows from theorem 1.3.4c of [5] that ${}_{\Omega(G)}\Omega(G)$ is a generator in $\Omega(G)$-gr. So $T_\Phi({}_{\Omega(G)}\Omega(G))$ is a projective generator in $(G/H, \Omega(G))$-gr. Consequently, $(G/H, \Omega(G))$-gr is equivalent to mod-$U$, where

$$U = End_{(G/H,\Omega(G))-gr}(T_\Phi({}_{\Omega(G)}\Omega(G))) = END_{\Omega(G)}({}_{\Omega(G)}\Omega(G))^{(H)}.$$

Since $END_{\Omega(G)}({}_{\Omega(G)}\Omega(G))$ is anti-isomorphic to $\Omega(G)$ as graded rings, it follows that $U$ is anti-isomorphic to $(\Omega(G))^{(H)}$, and thus $\Omega(G)\#G/H$ is Morita equivalent to $(\Omega(G))^{(H)}$.

**Corollary 4.7.** *Let $\Omega(G)$ be a crossed product and let $H$ has finite index $n = [G : H] < \infty$, then $\Omega(G)\#G/H \cong M_n(\Omega(G))^{(H)}$.*

**Proof.** Suppose $\{g_1, \ldots, g_n\}$ is a left transversal for $H$ in $G$. By theorem 4.1 we have that $(\Omega(G)\#G/H)^{(opp)} \cong T,\ T \cong End_{(G/H,\Omega(G))-gr}(V),\ V = T_\Phi\left(\bigoplus_{i=1}^{n}\Omega(G)(g_i^{-1})\right)$. Since $\Omega(G)$ is a crossed product, we have that $\Omega(G) = \Omega(G)(g)$ in $\Omega(G)$-gr for all $g \in G$. Therefore $\bigoplus_{i=1}^{n}\Omega(G)(g_i^{-1}) \cong \Omega(G)^n$. So $V \cong (T_\Phi({}_{\Omega(G)}\Omega(G)))^n$ and it follows also that
$T \cong End_{(G/H,\Omega(G))-gr}((T_\Phi({}_{\Omega(G)}\Omega(G)))^n) \cong M_n(End_{(G/H,\Omega(G))-gr}(T_\Phi({}_{\Omega(G)}\Omega(G))))$. But

$$End_{(\Omega(G),G/H)-gr}(T_\Phi({}_{\Omega(G)}\Omega(G))) \quad = \quad End_{\Omega(G)}(\Omega(G))^H$$

is anti-isomorphic to $(\Omega(G))^{(H)}$ and so $M_n((\Omega(G))^{(H)})$ is anti isomorphic to $T$. Therefore $\Omega(G)\#G/H \cong M_n(\Omega(G)^{(H)})$.

## 5. Functors

Dade [2] constructed several functors associated with homogenous $G$-sets $S$ (that is the set of the form $G/H$, where $H$ is a subgroup of $G$). Here, we extend the ideas in [2] along with the functor ${}^G(-)$ constructed in ([5], p.4) to arbitrary $G$-sets. So let $\Omega(G) = \underset{g\in G}{\oplus}\Omega(G)_g$ be $G$-graded and let $\Phi : S \to S'$ be a morphism of $G$-sets. As in the previous section, consider the canonical functor $T_\Phi : (G, S, \Omega(G))$-gr $\longrightarrow (G, S', \Omega(G))$-gr. Our first result in this section is the following:

**Theorem 5.1.** *$T_\Phi$ has a right adjoint $U^\Phi$. Moreover $U^\Phi$ is an exact functor. Furthermore, if $\Phi^{-1}(s')$ is a finite set for all $s' \in S'$, then $U^\Phi$ is also a left adjoint for $T_\Phi$.*

**Proof.** Let $N[S]$ denote the additive group which is the direct sum $\bigoplus\limits_{s\in S} {}^sN$ of copies ${}^sN$ of an $\Omega(G)$-module $N$. If $s \in S$ and $n \in N$, then ${}^sn$ will denote the natural image of $n$ in the subgroup ${}^sN$ of $N[S]$. We turn $N[S]$ into an $S$-graded $\Omega(G)$-module by putting $N[S]_s = {}^sN$ for all $s \in S$ and $\lambda_g \cdot {}^sn = {}^{gs}(\lambda_g n)$ for all $s \in S, g \in G,\ \lambda_g \in \Omega(G)_g$ and $n \in N$. Let $N \in (G, S', \Omega(G))$-gr. Then $N$ is also an $\Omega(G)$-module and $N[S] \in (G, \ldots \Omega(G))$-gr. We define an additive subgroup $U^\Phi(N)$ of $N[S]$ by $U^\Phi(N) = \bigoplus\limits_{s\in S} {}^sN_{\Phi(s)}$.

Note that $U^\Phi(N)$ is a subobject of $N[S]$ in $(G, S, \Omega(G))$-gr. More especially $U^\Phi(N) \in (G, S, \Omega(G))$-gr with $U^\Phi(N)_s = {}^s(N_{\Phi(s)})$, for all $s \in S$. We check this by looking at $\lambda_g \in \Omega(G)_g,\ n \in N_{\Phi(s)}$ and calculating $\lambda_g \cdot {}^sn = {}^{gs}(\lambda_g n)$. Doing this is easy because since $\lambda_g s \in \lambda_g N_{\Phi(s)} \subset N_{g\Phi(s)} = N_{\Phi(gs)}$, we have that ${}^{gs}(\lambda_g s) = \lambda_g \cdot^s n \in {}^{gs}(N_{\Phi(gs)})$, hence $U^\Phi(N)$ is a subobject of $N(S)$ in $(G, S, \Omega(G))$-gr, as claimed. If $N \to N'$ is a morphism in $(G, S', \Omega(G))$-gr, we define $f[S] : N[S] \to N'[S], {}^sn \to {}^s(f(n))$, and it is clear that the latter is a morphism in $(G, S, \Omega(G))$-gr. From $f(N_{\Phi(s)}) \subset N'_{\Phi(s)}$ it follows that $f[S](U^\Phi(N)) \subset U^\Phi(N')$ and so we may define $U^\Phi(f)$ to be the restriction of $f[S]$ to $U^\Phi(N)$. Exactness of the functor $U^\Phi$ is clear. We now demonstrate that $U^\Phi$ is a right adjoint for $T_\Phi$. Let $M \in (G, S, \Omega(G))$-gr, $N \in (G, S, \Omega(G))$-gr, and define the canonical morphism $\alpha : Hom_{(G,S',\Omega(G))-gr}(T_\Phi(M), N) \ \longrightarrow\ Hom_{(G,S',\Omega(G))-gr}(M, U^\Phi(N))$ as follows: for $u : T_\Phi(M) \to N$ we have $u(T_\Phi(M)_{s'}) \subset N_{s'}$ for all $s' \in S'$; so $u(\bigoplus\{M_s \mid \Phi(s) = s'\}) \subset N_{\Phi(s)}$ or $u(M_s) \subset N_{\Phi(s)}$ for all $s \in S$. So we may define $\alpha(u)(m) = \sum\limits_{s\in S} {}^su(m_s)$ in $U^\Phi(N)$, where $m = \sum\limits_{s\in S} m_s \in M$.

Conversely, let $\psi : N[S] \to N$ denote the natural $\Omega(G)$-morphism sending ${}^sn$ to $n$ for any $n \in N$. Define the canonical morphism

$$\beta : Hom_{(G,S,\Omega(G))-gr}(M, U^\Phi(N)) \ \longrightarrow\ Hom_{(G,S',\Omega(G))-gr}(T_\Phi(M), N)$$

as follows: for $v \in Hom_{(G,S',\Omega(G))-gr}(M, U^{\Phi}(N))$ we have $v(M_s) \subset (U^{\Phi}(N))_s = {}^sN_{\Phi(s)}$, therefore $\psi v(M_s) = \psi v(\bigoplus\{M_s \mid \Phi(s) = s'\}) \subset N_{\Phi(s)} = N_{s'}$. Thus we set $\beta(v) = \psi \circ v$. One can easily verify that $\alpha$ and $\beta$ are inverse to each other.

Now suppose that $\Phi^{-1}(s')$ is a finite set for all $s' \in S'$, and let us show that in this case $U^{\Phi}$ is also a left adjoint for $T_{\Phi}$. Let $M \in (G, S, \Omega(G))$-gr, and $N \in (G, S', \Omega(G))$-gr , $M = \bigoplus\limits_{s\in S} M_s,\ N = \bigoplus\limits_{s\in S} N_{s'}$. We can define the morphisms:

$$\psi : Hom_{(G,S,\Omega(G))-gr}(U^{\Phi}(N), M) \ \longrightarrow\ Hom_{(G,S',\Omega(G))-gr}(N, T_{\Phi}(M))$$

by setting for each $u \in Hom_{(G,S,\Omega(G))-gr}(U^{\Phi}(N), M)$ and $n_{s'} \in N_{s'}$

$$\psi(u)(n_{s'}) = u\left(\sum_{\{s\in S \ \mid\ \Phi(s)=s'\}} {}^sn_{s'}\right)$$

and $\delta : Hom_{(G,S',\Omega(G))-gr}(N, T_{\Phi}(M)) \ \longrightarrow\ Hom_{(G,S,\Omega(G))-gr}(U^{\Phi}(N), M)$ by setting for each $v \in Hom_{(G,S,\Omega(G))-gr}(N, T_{\Phi}(M))$ and ${}^sn_{\Phi(s)} \in (U^{\Phi}(N))_s,\ \delta(v)({}^sn_{\phi(s)}) = v(n_{\Phi(s)})_s$. Now let $v \in Hom_{(G,S',\Omega(G))-gr}(N, T_{\Phi}(M))$ and $n_{s'} \in N_{s'}$. Then $(\psi\circ\delta)(v)(n_{s'}) = \psi(\delta(v))(n_{s'})$

$$= \delta(v)\left(\sum_{\{s\in S \ \mid\ \Phi(s)=s'\}} {}^sn_{s'}\right) = \sum_{\{s\in S \ \mid\ \Phi(s)=s'\}} \delta(v)({}^sn_{s'}) = \sum_{\{s\in S \ \mid\ \Phi(s)=s'\}} v(n_{s'})_s = v(n_{s'}).$$

Hence $\psi \circ \delta = 1_{Hom_{(G,S',\Omega(G))-gr}}(N, T_{\Phi}(M))$.

Let $u \in Hom_{(G,S,\Omega(G))-gr}(U^{\Phi}(N), M)$ and ${}^sn_{\Phi(s)} \in {}^sN_{\Phi(s)} = (U^{\Phi}(N))_s$. Then

$$(\delta \circ \psi)(u)({}^sn_{\Phi(s)}) = (\psi(u)(n_{\Phi(s)}))_s = \left(\sum_{\{t\in S \ \mid\ \Phi(t)=\Phi(s)\}} u({}^tn_{\Phi(s)})\right)_s = u({}^sn_{\Phi(s)}).$$

Therefore $\delta \circ \psi = 1_{Hom_{(G,S,\Omega(G))-gr}}(U^{\Phi}(N), M)$.

**Corollary 5.2.** *Let $S$ be a finite $G$-set. Then $Q \in (G, S, \Omega(G))$-gr is an injective object in this category if and only if $Q$ is an injective $\Omega(G)$-module.*

**Proof.** The "if"-part is immediate from proposition 2.2. Now for the converse, let $s \in S$ and $\Phi : S \to \{s\}, \Phi(t) = s$ for all $t \in S$. Then $T_{\Phi} : (G, S, \Omega(G))\text{-gr} \to (G, \{s\}, \Omega(G))\text{-gr}$ $\approx \Omega(G)$-mod is the functor which "'forgets" the graded structure. This functor has an exact left adjoint,by theorem 5.1 and so it preserves injectivity.

We also have the following corollaries. For similar proofs, we refer the reader to [4].

**Corollary 5.3.** *Let $G$ be a finite group. Then $Q \in \Omega(G)$-gr is gr-injective if and only if $Q$ is injective in $\Omega(G)$-mod.*

**Corollary 5.4.** *Let $H \subset G$ be a subgroup of finite index. Then $Q \in (G/H, \Omega(G))$-gr is injective in this category if and only if $Q$ is an injective $\Omega(G)$-module.*

**Corollary 5.5.** *Let $K \subset H \subset G$ be subgroups of $G$, and $\Phi : G/K \longrightarrow G/H, gK \longmapsto gH$, be the canonical morphism of $G$-sets, $K$ has finite index in $H$, and $Q \in (G/K, \Omega(G))$-gr be an injective object. Then $T_\Phi(Q)$ is an injective object in the category $(G/K, \Omega(G))$-gr.*

Now let $S$ be a $G$-set, $H$ a subgroup of $G$. Assume that $B$ is a subset of $S$ such that $gB \subset B$ for all $g \in H$. Define the functor $T^B : (G, S, \Omega(G))$-gr $\to (H, B, \Omega(G)^{(H)})$-gr by setting $T^B(M) = M^{(B)} = \bigoplus_{s\in B} M_s$. If $f : M \to N$ is a morphism in $(G, S, \Omega(G))$-gr, we set $T^B(f) = f^{(B)} = f|_{M^{(B)}}$.

**Theorem 5.6.** *$T^B$ possesses a left adjoint $U^B$ and a right adjoint $U_B$. If $B \subset S$ and $H$ is a subgroup of $G$ such that $gs = t$ with $s, t \in B$ implies that $g \in H$, then $T^B \circ U^B = T^B \circ U_B = 1_{(H,B,\Omega(G))-gr}$, where $1_{(H,B,\Omega(G))-gr}$ is the identity of $(H, B, \Omega(G))$-gr.*

**Proof.** First we show the existence of a left adjoint for $T^B$. To $N$ in $(N, B, \Omega(G)^{(H)})$-gr we associate $\Omega(G) \otimes_Z N \in \Omega(G)$-mod where the $\Omega(G)$-module structure is given by: $r(\lambda \bigotimes s) = r\lambda \otimes n$, for $r \in \Omega(G)$, $\lambda \in B, n \in N$ As additive groups: $\Omega(G) \bigotimes_Z N \approx \bigoplus_{g\in G} \bigoplus_{t\in B} \Omega(G)_g \bigotimes N_t$. Put $(\Omega(G) \bigotimes_Z N)_s = \bigoplus\{\Omega(G)_g \bigotimes N_t \mid g \in G, t \in B, gt = s\}$ (where by convention direct sum of an empty family is zero). It is clear that $\Omega(G) \bigotimes_Z N = \bigoplus_{s\in S} (\Omega(G) \bigotimes_Z N)_s$ as additive groups. If $\lambda_h \bigotimes n_t \in (\Omega(G) \otimes_Z N)_s$, then $ht = s$. If $r_g \in \Omega(G)_g$, then $r_g(\lambda_h \bigotimes n_t) = r_g\lambda_h \bigotimes n_t$ and $(gh)t = g(ht) = gs$. Therefore $\Omega(G)_g(\Omega(G) \bigotimes_Z N)_s \subset (\Omega(G) \bigotimes_Z N)_{gs}$ and so $\Omega(G) \bigotimes_Z N$ is an object of $(G, S, \Omega(G))$-gr.

Next consider the natural epimorphism $\phi :\ \Omega(G) \otimes_Z N \to \Omega(G) \bigotimes_{\Omega(G)^{(H)}} N$, where $K = Ker(\phi)$ is the $\Omega(G)$-submodule generated by the elements $\{a\lambda \bigotimes n - a \bigotimes \lambda n \mid a \in$

$\Omega(G), \lambda \in \Omega(G)^{(H)}, n \in N\}$. Each such generator may be decomposed as a sum of elements of the same form, with $a, \lambda, n$ being homogeneous; so $K$ is a graded $\Omega(G)$-submodule of $\Omega(G) \bigotimes_Z N$ and therefore $\Omega(G) \bigotimes_{\Omega(G)^{(H)}} N$ is $S$-graded by setting $(\Omega(G) \bigotimes_Z N)_s = \phi((\Omega(G) \bigotimes_Z N)_s)$. We can define $U^B : (H, B, \Omega(G)^{(H)})$-gr $\to (G, S, \Omega(G))$-gr by $U^B(N) = \Omega(G) \bigotimes_{\Omega(G)^{(H)}} N$. To a morphism $f : N \to N'$ in $(H, B, \Omega(G)^{(H)})$-gr we associate the $\Omega(G)$-morphism $1 \bigotimes f$ and because the latter acts well on the generators of $K$, it induces a unique morphism: $U^B(f) : \Omega(G) \bigotimes_{\Omega(G)^{(H)}} N \to \Omega(G) \bigotimes_{\Omega(G)^{(H)}} N$, such that $U^B(f)(\lambda \bigotimes n) = \lambda \bigotimes f(n)$. To establish that $U^B$ is a left adjoint for $T^B$, consider $M \in (G, S, \Omega(G))$-gr and $N \in (H, B, \Omega(G)^{(H)})$-gr, and define

$$\alpha : Hom_{(G,S,\Omega(G))-gr}(U^B(N), M) \to Hom_{(H,B,\Omega(G)^{(H)})-gr}(N, T^B(M))$$

and

$$\beta : Hom_{(H,B,\Omega(G)^{(H)})-gr}(N, T^B(M)) \to Hom_{(G,S,\Omega(G))-gr}(U^B(N), M)$$

as follows: to $u \in Hom_{(G,S,\Omega(G))-gr}(U^B(N), M)$ we associate $\alpha(u)$ given by $\alpha(u)(n) = u(1 \bigotimes n)$. If $\lambda \in \Omega(G)^{(H)}$, we have $\alpha(u)(\lambda n) = u(1 \bigotimes \lambda n) = u(\lambda \bigotimes n) = u(\lambda(1 \bigotimes n)) = \lambda u(1 \bigotimes n) = \lambda \alpha(u)(n)$.

On the other hand, if $n \in N_t, t \in B$, then $1 \bigotimes n \in (\Omega(G) \otimes_{\Omega(G)^{(H)}} N)_t$ and hence $\alpha(u)(n) \in M_t$, that is, $\alpha(u)(Nt) \subset T^B(M)_t$. Therefore $\alpha$ is well-defined. To $v \in Hom_{(H,B,\Omega(G)^{(H)})-gr}(N, T^B(M))$ we associate $\beta(v) : \Omega(G) \otimes_{\Omega(G)^{(H)}} N \to M$ given by $\beta(v)(\lambda \bigotimes n) = \lambda v(N)$. If $s \in S$ and there are no $t \in B,\ g \in G$ such that $gt = s$, then $(\Omega(G) \bigotimes_{\Omega(G)^{(H)}} N)_s = 0$. Otherwise, let $t \in B,\ g \in G$ be such that $gt = s$ and let $\lambda \in \Omega(G)_g,\ n \in N_t$. Then $v(n) \in T^B(M)_t = M_t$, and $\lambda v(n) \in \Omega(G)_g M_t \subset M_{gt} = M_s$. Therefore $\beta$ is well defined too. Now if $u \in Hom_{(G,S,\Omega(G))-gr}(U^B(N), M)$ then we have: $\beta(\alpha(u))(\lambda \bigotimes n) = \lambda\alpha(u)(n) = \lambda u(1 \bigotimes n) = u(\lambda \bigotimes n)$ hence $\beta \circ \alpha$ is the identity on $Hom_{(G,S,\Omega(G))-gr}(U^B(N), M)$. If $v \in Hom_{(H,B,\Omega(G)^{(H)})-gr}(N, T^B(M))$ then we get

$$\alpha(\beta(v))(n) \quad = \quad \beta(v)(1 \bigotimes n) \quad = \quad v(n),$$

hence $\alpha \circ \beta$ is the identity of $Hom_{(H,B,\Omega(G)^{(H)})-gr}(N, T^B(M))$. Assume now that $H$ is as in the last part of the statement and let $N \in (H, B, \Omega(G)^{(H)})$-gr. We have the functorial morphism $\psi(N) : N \to (T^B \circ U^B)(N),\ \psi(N(n) = \alpha(1_{U^B(N)})(n) = 1 \bigotimes n$. But $(T^B \circ$

$U^B)(N) = (\Omega(G)\bigotimes_{\Omega(G)^{(H)}} N)^{(B)} = \bigoplus_{s\in B}(\Omega(G)\bigotimes_{\Omega(G)^{(H)}} N)_s$. Since $(\Omega(G)\bigotimes_{\Omega(G)^{(H)}} N)_s$ is an image $(\Omega(G)\bigotimes_Z N)_s$, the former is generated by elements of the form $\lambda\bigotimes n$, where $\lambda\in\Omega(G)_g, n\in Nt$, and $gt=s$. By our hypothesis $g\in H$ follows, and therefore $(\Omega(G)\bigotimes_{\Omega(G)^{(H)}} N)_s\subset\Omega(G)^{(H)}\bigotimes_{\Omega(G)^{(H)}} N$. This clearly yields that $(T^B\circ U^B)(N)$ is isomorphic to $N$, and $\psi(N)$ is an isomorphism.

We now show that the right adjoint for $T^B$ exits. To do this, let $N\in(H,B,\Omega(G)^{(H)})$-gr, $N=\bigoplus_{t\in B} N_t$ and define for each $s\in S$,
$U_B(N)_s=\{f\in Hom_{\Omega(G)^{(H)}}(\Omega(G),N)\mid f(\Omega(G)_\theta)=0$ if there are no $h\in H, g\in G$ and $t\in B$ such that $gt=s$ and $\theta=hg^{-1}$, and $f(\Omega(G)_{hg^{-1}})\subseteq N_{ht}$ for all $h\in H$ if there exist $g\in G$ and $t\in B$ such that $gt=s\}$. We set $U_B(N)=\sum_{s\in S}U_B(N)_s$. Let us show that the sum is direct. Let $f\in U_B(N)_s\cap\sum_{s'\neq s}U_B(N)_{s'}$. Then $f=\sum_{i=1}^k f_{s_i},\ s_i\neq s,\ i=1,\dots k$. Let $r_\theta\in\Omega(G)_\theta$. If there are no $h\in H, t\in B, g\in G$ such that $gt=s$ and $\theta=hg^{-1}$, then $f(r_\theta)=0$. Suppose there are $h\in H, g\in G, t\in B$ such that $gt=s$ and $\theta=hg^{-1}$, and $h_i\in H, g_i\in G, t_i\in B$ such that $g_it_i=s_i, \theta=h_ig_i^{-1}, i=1,\dots,l$. We have that $f(r_\theta)=f(r_{hg^{-1}})\in N_{ht}$. On the other hand, $f(r_\theta)=\sum_{i=1}^l f_{s_i}(r_\theta)=\sum_{i=1}^l f_{s_i}(r_{h_ig_i^{-1}})\in\sum_{i=1}^l N_{h_it_i}$. Suppose there exists $i$ such that $ht=h_it_i$. Then $s=gt$, so $g^{-1}s=t$, that is $hg^{-1}s=ht$, that is $h_ig_i^{-1}s=h_it_i$ and so $g_i^{-1}s=t_i$, hence $s=g_it_i=s_i$, a contradiction. Therefore $f(r_\theta)\in N_{h_it_i}\cap\sum_{i=1}^l N_{h_it_i}=0$, that is $f=0$. Now we check that $U_B(N)=\bigoplus_{s\in S}U_B(N)_s\in(G,S,\Omega(G))$-gr. Let $r_\lambda\in\Omega(G)_\lambda, f\in U_B(N)_s$. We set $(rf)(a)=f(ar)$ for all $a,r\in\Omega(G)$, and show that $r_\lambda f\in U_B(N)_{\lambda s}$. Let $r_\theta\in\Omega(G)_\theta$. Then $(r_\lambda f)(r_\theta)=f(r_\theta r_\lambda)$ If there are $h\in H,\ g\in G,\ t\in B$ such that $\lambda gt=\lambda s$, and $\theta=hg^{-1}\lambda^{-1}$, then $r_\theta r_\lambda\in(\Omega(G))_{hg^{-1}}$ and $gt=s$, so $(r_\lambda f)(r_\theta)\in N_{\lambda t}$. If not, then $(r_\lambda f)(r_\theta)=0$. Thus $U_B(N)\in(G,S,\Omega(G))$-gr. If $N,N'\in(H,B,\Omega(G)^{(H)})$-gr and $\phi: Hom_{(H,B,\Omega(G)^{(H)})-gr}(N,N')$, then we put $U_B(\varphi): U_B(N)\to U_B(N'),\ U_B(\varphi)(f)=\varphi\circ f$. Then $U_B(\varphi)\in Hom_{(G,S,\Omega(G))-gr}(U_B(N)_0,U_B(N'))$, as one can easily check. So we have defined a functor $U_B(-):(H,B,\Omega(G)^{(H)})$-gr $\to(G,S,\Omega(G))$-gr. We now show that $U_B(-)$ is a right adjoint for $T^B(-)$. To show this, define for all $M\in(G,S,\Omega(G))$-gr, $N\in(H,B,\Omega(G)^{(H)})$-gr, $M=\bigoplus_{s\in S}M_s,\ N=\bigoplus_{t\in B}N_t$, the morphisms $\gamma: Hom_{(H,B,\Omega(G))-gr}(T^B(M),N)\to Hom_{(G,S,\Omega(G))-gr}(M,U_B(N))$ and $\delta: Hom_{(G,S,\Omega(G))-gr}(M,U_B(N))\to Hom_{(H,B,\Omega(G))-gr}(T^B(M),N)$ as follows:

to $u \in Hom_{(H,B,\Omega(G))-gr}(T^B(M), N)$ we associate $\gamma(u)$ given by (using where necessary, the convention that a sum indexed by an empty family is zero)

$$\gamma(u)(m_s)(a) = u\left(\sum_{h\in H, g\in G; t\in B, gt=s} a_{hg^{-1}} m_s\right),\ m_s \in M_s, a \in \Omega(G).$$

It is easy to check that $\gamma(u)(m_s) \in U_B(N)_s$, and so $\gamma$ is well defined. Now to $v \in Hom_{(G,S,\Omega(G))-gr}(M, U_B(N))$ we associate $\delta(v)$ given by $\delta(v)(m_t) = v(m_t)(1),\ t \in B,\ m_t \in M_t$. Because $v(m_t) \in U_B(N)_t$, we take $h \in H,\ g = h,\ h^{-1}t \in B$ and we have $h(h^{-1}t) = t$, thus $v(mt)(\Omega(G)_1) = v(m_t)(\Omega(G)_{hh^{-1}}) \subset N_{h(h^{-1}t)} = N_t$, and so $\delta$ is also well defined. Now let $v \in Hom_{(G,S,\Omega(G))-gr}(M, U_B(N)),\ m_s \in M_s,\ a \in \Omega(G)$. Then

$$\gamma(\delta(v))(m_s)(a) = \delta(v)\left(\sum_{h\in H, g\in G; t\in B, gt=s} a_{hg^{-1}} m_s\right) = v\left(\sum_{h\in H, g\in G; t\in B, gt=s} a_{hg^{-1}} m_s\right)(1)$$

$$= \sum_{h\in H, g\in G; t\in B, gt=s} a_{hg^{-1}} v(m_s)(1) = v(m_s)\left(\sum_{h\in H, g\in G; t\in B, gt=s} a_{hg^{-1}}\right) = v(m_s)(a);$$

since $v(m_s) \in U_B(N)_s$, and hence $v(m_s)(a_\lambda) = 0$ if there are no $h \in H, g \in G, t \in B$ such that $gt = s$ and $\lambda = hg^{-1}$. So $\gamma \circ \delta = id$.

Conversely, let $u \in Hom_{(H,B,\Omega(G))-gr}(T^B(M), N),\ m_t \in M_t$ and $t \in B$. We have $\delta(\gamma(u))(m_t) = \gamma(u)(m_t)(1) = u(m_t)$, so $\delta \circ \gamma = Id$. Now suppose that $H$ is again as in the last part of the statement, and let $N \in (H, B, \Omega(G)^{(H)})$-gr. Then $T^B(U_B(N)) = \bigoplus_{t\in B} U_B(N)_t$. We have the canonical morphism $\phi(N) : \bigoplus_{t\in B} U_B(N)_t \longrightarrow N = \bigoplus_{t\in B}(N)_t$ defined by $\phi(N)(f) = f(1) \in N_t$ for each $f \in U_B(N)_t$. From the condition $gt = t',\ ,\ t' \in B$ implies $g \in H$, it follows that for all $t \in B$ and each $f \in U_B(N)_t$, we have that $f$ is zero outside $\Omega(G)^{(H)}$. It is clear that $\phi(N)$ is an isomorphim in $(H, B, \Omega(G)^{(H)})$-gr.

## References


[1] M. Cohen, S. Montgomery, Groups graded rings, smash products and group actions, Trans. Am. Math. Soc., 284 (1984), 237 - 258.

[2] E. Dade, Clifford theory for group graded rings, J. Reine Angew., Math. 369 (1986), 40-86.

[3] B. Mitchell, Theory of categories, New York: Academic, (1965).

[4] C. Nastaseescu, Group rings of graded rings, applications, J. Algebra, A33 (1985), 313-167.

[5] C. Nastaseescu, F. Van Oystaeyen, Graded ring theory, North Holland: Mathematical Library, 28 (1982).

[6] C. Nastaseescu, S. Raianu, F. Van Oystaeyen, Modules graded by $G$-sets, Math. Z., 203 (1990), 605-627.

[7] C. Nastaseescu, N. Rodino, Group graded rings and smash products, Rend. Semin. Mat. Univ. Padova, 74 (1985).

[8] K. Nwabueze, The category of graded Burnside modules and associated functors, Inter. J. Algebra, 6 (2012), 17, 825-835 .

[9] T. Tom-Dieck, Transformation groups and representation theory, Lecture Notes in Maths, Springer Verlag 766 (1979).

[10] D. Quinn, Graded rings and duality, Trans. Am. Math. Soc., 292 (1985), 155-167.