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GENERALIZED ORDER DIVISOR GRAPHS ASSOCIATED WITH FINITE GROUPS

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Abstract. For each finite group G we associate a simple graph $\Omega(G)$. We investigate the coaction between the group-theoretic properties of G and the graph-theoretic properties of $\Omega(G)$.

Keywords: p-groups; cyclic groups; dihedral groups; complete graphs; connected graphs; diameter; girth; clique number.

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1. Introduction

Algebra and graph theory are incredibly useful in many contexts. Group theory studies symmetry, found in crystals, art, architecture and music. Many other algebraic structures are used in theoretical computer science. Lattice theory is used in the development of semantics. Universal algebra is used for defining algebraic specifications of data types. Category theory is the foundation for type theory. Graph theory is applied in different fields of computer science like image segmentation, data mining, image capturing, clustering, and networking etc. The concept

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coloring of graph is used in scheduling and resource allocation. Also, paths, walks and circuits are used in database design concepts, traveling salesman problem, and resource networking.

Cayley's Theorem from group theory motivates the study of algebraic combinatorics. Group action is a dynamical process that partition the elements of an object into orbits. The study of the structure and quantity of these orbits implies important combinatorial results. Algebraic combinatorics and graph theory are applicable in different areas of computer science such as complexity theory, automata theory, and polya enumeration theory.

The automorphism group of a graph and Cayley graph of a group motivates to study the interplay between groups and graphs. The Cayley graph $Cay(X:G)$ of a finite group G with generating set X is a directed graph in which each element of G is a vertex and there exists an arc from vertex a to b iff $ax = b$ for some $x \in X$. The set acquiring all the automorphisms of a graph is a group under the operation of composition of functions, called the automorphism group of a graph. Some important readings related to these topics are [13, 15, 16, 19, 20].

There are many ways to associate a graph with a given group. There exists a large amount of literature devoted to study the graphs associated to finite groups, for instance commuting graphs [2, 4, 6, 10, 21, 27], non-commuting graphs [3, 23], intersection graphs[1], prime graphs [5, 17, 18], conjugacy class graphs [7], power graphs [11, 12, 22] , inverse graphs [30], and order divisor graphs [28].

In this paper, we associate a simple graph $\Omega(G)$ to a finite group G, called by us the *generalized order divisor graph associated with finite group* G, in which the elements of G are vertices and two distinct vertices a and b are adjacent if $|a|$ divides $|b|$ or $|b|$ divides $|a|$. Our main objective here is to study the interplay of group-theoretic properties of G with graph-theoretic properties of $\Omega(G)$. This newly associated graph $\Omega(G)$ is a generalization of the order divisor graph $OD(G)$, introduced and studied in [28]. The definition of $OD(G)$ is slightly different than $\Omega(G)$. In $OD(G)$ the adjacent vertices must have different orders, however this condition is not imposed in $\Omega(G)$. We find that $\Omega(G)$ better illustrates the coaction between groups and graphs as compared to $OD(G)$.

For the convenience of readers we provide a brief explanation here for the notions used. A group G is known as a *p-group* if $\forall a \in G$, $|a| = p^{\alpha}$, for some prime p. The group of symmetries of a regular n-gon ($n \geq 3$) is known as *dihedral group* that have $2n$ elements. The dihedral group can be presented by $D_n =< r,s \mid r^n = s^2 = (rs)^2 = e >$. We symbolize the group of units of $\mathbb{Z}_n$ by $U(n)$, i.e., $U(n) = \{\bar{a} \in \mathbb{Z}_n \mid (a,n) = 1\}$. An undirected graph that have no multiple edges and loops is known as simple graph. A complete bipartite graph is a bipartite graph (i.e., a graph in which vertices are decomposed into two disjoint sets such that no two vertices are adjacent if they belong to the same set) such that every vertex in one set is connected with every vertex in other set. The star graph S_n with order n is a tree such that degree of one vertex is $n-1$ and all remaining vertices have degree 1, i.e., $S_n \cong K_{1,n}$. The chromatic number $\chi(G)$ of a graph G is the least number of colors required for coloring its vertices. Girth of a graph G, denoted by $g(G)$, is the length of smallest cycle within G. The graph G has girth infinity if there does not exist any cycle in G. The clique number $\omega(G)$ of a graph G is the order of maximal clique in G.

In Section 2, we prove that $\Omega(G)$ is a connected graph with $diam(\Omega(G)) \leq 2$ and $\Omega(G)$ contains a cycle if $|G| > 2$ and $g(\Omega(G)) = 3$. We conclude that for $|G| > 2$, $\Omega(G)$ cannot be a bipartite graph and hence it cannot be a tree, star or path. In Section 3, we obtain some characterizations about $\Omega(G)$ to be complete and regular. As a consequence, we investigate the completeness of $\Omega\big(U(n)\big)$, $\Omega(\mathbb{Z}_n)$ and $\Omega(D_n)$. Moreover, we obtain that $\Omega(D_n)$ is one-vertex union of K_n and K_{n+1} if and only if $n = p^m$ for some odd prime p and positive integer m. In Section 4, we determine the clique number of $\Omega(D_n)$ and $\Omega(\mathbb{Z}_n)$.

Throughout this paper, all groups and graphs will be finite. See [8, 14, 29] for basic references.

2. Main Results

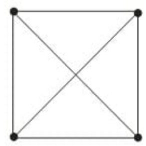
2.1. Properties of $\Omega(G)$.

In this section, we initiate with few examples and later show that $\Omega(G)$ is connected always and has small girth and diameter. We also determine that a bipartite graph with more than two vertices cannot be realized as $\Omega(G)$.

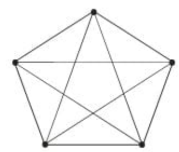
Example 2.1. The generalized order divisor graphs for several groups are given below.

Z_2

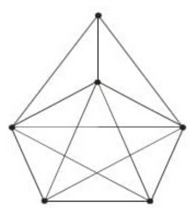
Z_3



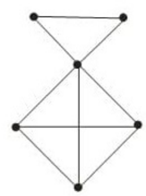
$Z_4, Z_2 \times Z_2$



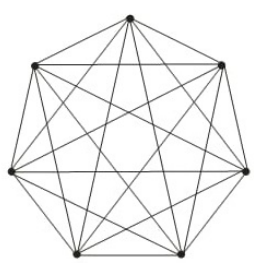
Z_5



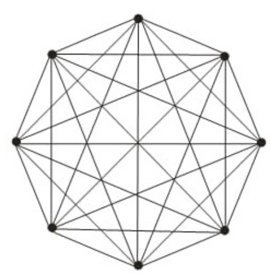
Z_6



D_3 or S_3



Z_7



$Z_8, Z_2 \times Z_4, Z_2 \times Z_2 \times Z_2$, D_3

Remark 2.2. Above examples show that generalized order divisor graph may be same for non-isomorphic groups.

Theorem 2.3. *Let G be a group with $|G| < \infty$. Then $\Omega(G)$ is always connected and $diam(\Omega(G)) \leq 2$. Moreover, if $|G| > 2$ then $\Omega(G)$ contains a cycle and $g(\Omega(G)) = 3$.*

Proof. The vertex which is associated with identity is adjacent to all other vertices, so $\Omega(G)$ is always connected. If two distinct vertices a, b of G are adjacent then $d(a,b) = 1$. If they are not adjacent, then $a - e - b$ is a path of length 2. Hence $diam(\Omega(G)) \leq 2$. Now suppose $|G| > 2$. If every element other than identity has order 2, then $\Omega(G)$ is complete and hence contain a cycle of length 3. If G contains and element a of order more than 2, then $a \neq a^{-1}$ and so $a - e - a^{-1} - a$ is a cycle of length 3. □

Corollary 2.4. *Following assertions are true for a finite group G.*

(a) *$\Omega(G)$ is a cycle iff $|G| = 3$.*
(b) *$\Omega(G)$ is bipartite iff $|G| = 2$.*

Proof. The proof of (a) is straightforward. For the proof of (b), note that a graph is bipartite iff it does not have a cycle of odd length, cf. [29, Theorem 1.2.18]. □

Remark 2.5. Note that the family of bipartite graphs include trees, star graphs, and path graphs, so if $|G| > 2$, then $\Omega(G)$ cannot be a tree, star or path.

2.2. Certain Characterizations of $\Omega(G)$. In this section we retrieve some characterizations about $\Omega(G)$. We find out when $\Omega(G)$ is complete and when $\Omega(G)$ is regular. As a consequence, we determine those values of n for which $\Omega\big(U(n)\big)$, $\Omega(\mathbb{Z}_n)$ and $\Omega(D_n)$ are complete. We also determine that for specific values of n, the $\Omega(D_n)$ is one-vertex union of two complete graphs.

Theorem 2.6. *The following assertions are equivalent for a group G with $|G| < \infty$.*

(a) *G is a p-group for some prime p.*
(b) *$\Omega(G)$ is complete.*
(c) *$\Omega(G)$ is regular.*

Proof. (a)⇒ (b): Let $|G| = p^n$ for some positive integer n. Then order of every element of G is a power of p. Moreover, for each pair p^i and p^j with $i \leq j$, we have $p^i | p^j$. Hence $\Omega(G)$ is complete.

(b)$\Rightarrow$ (a): Deny. Then $|G|$ has at least two distinct prime divisors, say, p and q. According to Cauchy's theorem, G must have an element with order p and an element with order q. Therefore, the vertex associated to p and q are not adjacent, which is a contradiction

(b)$\Rightarrow$(c): Every complete graph is regular.

(c)$\Rightarrow$(b): The graph $\Omega(G)$ is regular and the vertex associated to the identity element has degree $|G|-1$. $\square$

Recall that the integers $F_n = 2^{2^n}+1$ are called Fermat numbers. Note that not all Fermat numbers are prime. It is known only from $n=0$ to $n=4$. The first non-prime Fermat number is when $n=5$. A *Fermat prime* is a Fermat number that is also a prime number, cf. [26, Section 3.6].

Corollary 2.7. *The following assertions are equivalent.*

(a) $\Omega\big(U(n)\big) \cong K_{\phi(n)}$.

(b) $U(n)$ *is a* 2*-group.*

(c) $n = 2^k p_1 p_2 \cdots p_s$*, where* p_i *are distinct Fermat primes and* $k \in Z^+ \cup \{0\}$.

(d) *A regular polygon of n sides can be constructed using a ruler and compass.*

Proof. (a) $\Leftrightarrow$ (b): As $\phi(n)$ is even for $n>2$, so by Theorem 2.6 $\Omega(U(n))$ is complete iff $|U(n)| = \phi(n) = 2^m$ for some positive integer m.

(b) $\Leftrightarrow$ (c): Suppose $U(n)$ is a 2-group. This implies that, if $n = p_1^{a_1} p_2^{a_2} \cdots p_t^{a_t}$ is the prime power decomposition of n, then each factor $p_i^{a_i-1}(p_i-1)$ should be some power of 2. This implies that either $p_i = 2$ or $a_i = 1$ and $p_i - 1 = 2^{k_i}$ and thus p_i will be a Fermat prime. Hence $U(n)$ is a 2-group iff $n = 2^k p_1 p_2 \cdots p_s$, where p_i are distinct Fermat primes and $k \in Z^+ \cup \{0\}$.

(c) $\Leftrightarrow$ (d): See [26, Theorem 3.22] or [24]. $\square$

Corollary 2.8. $\Omega(\mathbb{Z}_n) \cong K_n$ *iff* $n = p^m$*, where p is a prime and m is an integer greater than or equal to* 1.

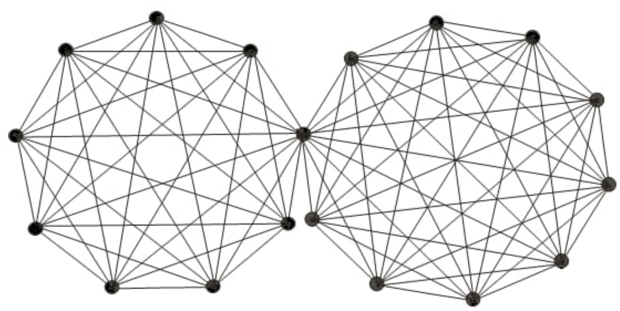
Corollary 2.9. $\Omega(D_n) \cong K_{2n}$ *iff* $n = 2^m$ *for some integer* $m \geq 1$.

Recall that one-vertex union of finite number of connected graphs can be grabbed by identifying one vertex from each graph.

Theorem 2.10. *$\Omega(D_n)$ is one-vertex union of K_n and K_{n+1} iff $n = p^m$ for some odd prime p and $m \in Z^+$.*

Proof. We have $D_n =< a,b \mid a^n = b^2 = (ab)^2 = e >$. Since, $|a^i b| = 2$ for each i; $0 \le i \le n-1$, therefore the vertices associated to $e,b,ab,a^2b,...,a^{n-1}b$ forms a complete subgraph K_{n+1} in $\Omega(D_n)$. Moreover, the vertices associated to elements in $< a >= \{e,a,a^2,...,a^{n-1}\}$ forms a complete subgraph K_n in $\Omega(D_n)$ iff $n = | < a > | = p^m$, where p is prime and m is some positive integer. Hence $\Omega(D_n)$ is one-vertex union of K_n and K_{n+1} iff $n = p^m$ for some odd prime p and positive integer m. □

Example 2.11.



$\Omega(D_9)$

2.3. Clique number of $\Omega(G)$.

Recall that the clique number $\omega(\mathscr{G})$ of a graph $\mathscr{G}$ is the order of a maximal clique of $\mathscr{G}$. In this section we determine the clique number of $\Omega(G)$, where G is a dihedral group or a cyclic group.

Theorem 2.12. *$\omega(\Omega(D_n)) = 1+n$ iff n is odd positive integer.*

Proof. ($\Rightarrow$:) Let n be even. Then the subgroup $< a >= \{e,a,a^2,...,a^{n-1}\}$ in $D_n =< a,b \mid a^n = b^2 = (ab)^2 = e >$ has an element of order 2, say, x. This implies that the elements e, x, $b,ab,a^2b,...,a^{n-1}b$ forms a clique of order $n+2$, a contradiction.

($\Leftarrow$:) The subgroup $< a >$ in $D_n =< a,b \mid a^n = b^2 = (ab)^2 = e >$ has no element of order 2. So, e, $b,ab,a^2b,...,a^{n-1}b$ forms a largest clique in $\Omega(D_n)$. □

Theorem 2.13. *Let $n = 2^k p^\alpha$, where p is odd prime and k, α be positive integers. Then the clique number of $\Omega(D_n)$ is given by*

$$\omega(\Omega(D_n)) = 1 + n + p^{\alpha-1} + (2^k - 1)\phi(p^\alpha),$$

where ϕ denotes the Euler's Phi function.

Proof. We have $D_n =< a,b \mid a^n = b^2 = (ab)^2 = e >$. The largest clique in $\Omega(D_n)$ consists of $\{e,b,ab,a^2b,...,a^{n-1}b\} \cup \{x \in < a > \mid |x| \in A\}$, where $A = \{2,2p,2p^2,...,2p^{\alpha-1},2p^\alpha,2^2p^\alpha,...,2^kp^\alpha\}$. Since for every positive divisor d of n, the subgroup $< a >= \{e,a,a^2,...,a^{n-1}\}$ has $\phi(d)$ elements with order d, cf. [14, Theorem 4.4], the clique number of $\Omega(D_n)$ is given by

$$\begin{aligned}\omega(\Omega(D_n)) =& n + 1 + \phi(2) + \phi(2p) + \phi(2p^2) + \cdots + \phi(2p^{\alpha-1}) \\ & + \phi(2p^\alpha) + \phi(2^2p^\alpha) + \cdots + \phi(2^kp^\alpha) \\ =& n + 2 + \phi(p) + \phi(p^2) + \cdots + \phi(p^{\alpha-1}) \\ & + \phi(p^\alpha) + 2\phi(p^\alpha) + \cdots + 2^{k-1}\phi(p^\alpha) \\ =& n + 1 + p^{\alpha-1} + (2^k - 1)\phi(p^\alpha)\end{aligned}$$

□

Example 2.14. $\omega(\Omega(D_{36})) = \omega(\Omega(D_{2^23^2})) = 1 + 36 + 3^{2-1} + (2^2 - 1)\phi(3^2) = 58.$

Theorem 2.15. *Let $n = 2^k p_1p_2\cdots p_m$, where $p_1 < p_2 < \cdots < p_m$ are distinct odd primes and $k \geq 1$. Then*

$$\omega\big(\Omega(D_n)\big) = n + 2 + \phi(p_m) + \phi(p_{m-1}p_m) + \cdots + \phi(p_2p_3\cdots p_m) + (2^k - 1)\phi(p_1p_2\cdots p_m).$$

Proof. We have $D_n =< a,b \mid a^n = b^2 = (ab)^2 = e >$. The largest clique in $\Omega(D_n)$ consists of $\{e,b,ab,a^2b,...,a^{n-1}b\} \cup \{x \in < a > \mid |x| \in A\}$, where $A = \{2,2p_m,2p_{m-1}p_m,...,2p_2p_3\cdots p_{m-1}p_m,$ $2p_1p_2\cdots p_m,2^2p_1p_2\cdots p_m,...,2^kp_1p_2\cdots p_m\}$. Since for every positive divisor d of n, there are $\phi(d)$ elements with order d in the subgroup $< a >= \{e,a,a^2,...,a^{n-1}\}$, cf. [14, Theorem 4.4], the clique number of $\Omega(D_n)$ is given by

$$\begin{aligned}\omega(\Omega(D_n)) =& n+1+\phi(2)+\phi(2p_m)+\phi(2p_{m-1}p_m)+\cdots+\phi(2p_2p_3\cdots p_m)\\ &+\phi(2p_1p_2\cdots p_m)+\phi(2^2p_1p_2\cdots p_m)+\cdots+\phi(2^kp_1p_2\cdots p_m)\\ =& n+2+\phi(p_m)+\phi(p_{m-1}p_m)+\cdots+\phi(p_2p_3\cdots p_m)\\ &+\phi(p_1p_2\cdots p_m)(1+2+2^2+\cdots+2^{k-1})\\ =& n+2+\phi(p_m)+\phi(p_{m-1}p_m)+\cdots+\phi(p_2p_3\cdots p_m)\\ &+\phi(p_1p_2\cdots p_m)(2^k-1)\end{aligned}$$

□

Example 2.16.

$$\omega(\Omega(D_{840}))=\omega(\Omega(D_{2^3\cdot 3\cdot 5\cdot 7}))=840+2+\phi(7)+\phi(5\cdot 7)+\phi(3\cdot 5\cdot 7)(2^3-1)=1208.$$

Theorem 2.17. *If $n=2^k$ for some integer $k\geq 2$, then $\omega(\Omega(D_n))=2^{k+1}$.*

Proof. If $n=2^k$, then D_n is a 2-group. Now apply Theorem 2.6. □

Remark 2.18. Note that $\Omega(D_3)$ has clique number 2^2, which shows that converse of above result is false.

Theorem 2.19. *Let $n=p_1p_2\cdots p_m$, where $p_1<p_2<\cdots<p_m$ are distinct primes. Then $\omega\big(\Omega(\mathbb{Z}_n)\big)=\phi(n)+\phi(p_2p_3\cdots p_m)+\phi(p_3p_4\cdots p_m)+\cdots+\phi(p_m)+1$.*

Proof. The largest clique in $\Omega(\mathbb{Z}_n)$ consists of $\{x\in\mathbb{Z}_n \mid |x|\in A\}$, where $A=\{n,p_2p_3\cdots p_m, p_3p_4\cdots p_m,\dots,p_{m-1}p_m,p_m,1\}$. Hence, by using the fact that for every positive divisor d of n, there are $\phi(d)$ elements with order d in $\mathbb{Z}_n$, we get $\omega\big(\Omega(\mathbb{Z}_n)\big)=\phi(n)+\phi(p_2p_3\cdots p_m)+\phi(p_3p_4\cdots p_m)+\cdots+\phi(p_{m-1}p_m)+\phi(p_m)+1$. □

Example 2.20.

$$\omega\big(\Omega(\mathbb{Z}_{210})\big)=\omega\big(\Omega(\mathbb{Z}_{2\cdot 3\cdot 5\cdot 7})\big)=\phi(2\cdot 3\cdot 5\cdot 7)+\phi(3\cdot 5\cdot 7)+\phi(5\cdot 7)+\phi(7)+1=127.$$

Theorem 2.21. *Let $n=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_m^{\alpha_m}$, where $p_1<p_2<\cdots<p_m$ are distinct primes and $\alpha_1\leq\alpha_2\leq\cdots\leq\alpha_m$. Then*

$$\omega\big(\Omega(\mathbb{Z}_n)\big)=\sum_{i=0}^{\alpha_1}\phi(p_1^{\alpha_1-i}p_2^{\alpha_2}p_3^{\alpha_3}\cdots p_m^{\alpha_m})+\sum_{i=0}^{\alpha_2}\phi(p_2^{\alpha_2-i}p_3^{\alpha_3}\cdots p_m^{\alpha_m})+\cdots+\sum_{i=0}^{\alpha_m}\phi(p_m^{\alpha_m-i})$$

Proof. The largest clique in $\Omega(\mathbb{Z}_n)$ consists of $\{x \in \mathbb{Z}_n \mid |x| \in A\}$, where $A = \{p_1^{\alpha_1 - i} p_2^{\alpha_2} \cdots p_m^{\alpha_m}, p_2^{\alpha_2 - i} p_3^{\alpha_3} \cdots p_m^{\alpha_m}, \ldots, p_m^{\alpha_m - i} \mid 0 \leq i \leq m\}$. Since, for every positive divisor d of n the group $\mathbb{Z}_n$ has $\phi(d)$ elements with order d, therefore

$$\omega\big(\Omega(\mathbb{Z}_n)\big) = \sum_{i=0}^{\alpha_1} \phi(p_1^{\alpha_1 - i} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_m^{\alpha_m}) + \sum_{i=0}^{\alpha_2} \phi(p_2^{\alpha_2 - i} p_3^{\alpha_3} \cdots p_m^{\alpha_m}) + \cdots + \sum_{i=0}^{\alpha_m} \phi(p_m^{\alpha_m - i}). \qquad \square$$

Corollary 2.22. *If $p_1 < p_2$ are distinct primes, then $\omega(\Omega(\mathbb{Z}_{p_1 p_2})) = p_1 p_2 - p_1 + 1$.*

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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