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ON A NON-SOLVABLE GROUP SATISFYING $x^G = (x^{-1})^G$

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Abstract. A group G satisfies Syskin's condition if elements of same order are conjugates. If a group G satisfies Syskin's condition, then each element and its inverse are conjugate to each other, i.e., for all $x \in G$, $x^G = (x^{-1})^G$, but not conversely. Thus, the class of those groups satisfying Syskin's condition forms a proper subclass of groups satisfying $x^G = (x^{-1})^G$. In this note, it is proved that if a group G meets the condition $x^G = (x^{-1})^G$, then G cannot be of odd order. As the main result, it is shown that if $x^G = (x^{-1})^G$ holds for a centreless and non-solvable group G of order 120 such that $G \neq G'$, then $G \cong S_5$.

Keywords: conjugate elements; meta cyclic group; symmetric group.

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1. Introduction

A group G satisfies Syskin's condition if elements of same order lie in same conjugacy class. Equivalently, these elements are conjugate to each other. Fiet, Sietz [2] and Zhang[4] proved that if a finite group G meet this condition, then $G \cong S_i, i = 1,2,3$. Furthermore, You, Qian and Shi [3] generalised Syskin's problem and proved that if all non-cental elements of a finite group G are conjugate to each other, then G is either abelian or isomorphic to S_3. It is straightforward to see that if a group G meets Syskin's condition, then for all $x \in G$, x^G and $(x^{-1})^G$ coincide,

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but not conversely. Thus, the class of those groups satisfying Syskin's condition forms a proper subclass of groups satisfying $x^G = (x^{-1})^G$. Berggren, J.L.[1] proved that the class $\Im$ of all finite groups whose irreducible characters over $\mathbf{C}$ are real, is equivalent to those finite groups in which x^G and $(x^{-1})^G$ coincides. He proved that alternating group $A_n \in \Im$ if $n \in \{1,2,5,6,10,14\}$ and conversely. In this paper, it is shown that if $x^G = (x^{-1})^G$ for each $x \in G$, then G must be of even order. As a main result it is shown that if G is a centreless, non-perfect, non-solvable group of order 120 and x^G concides with $(x^{-1})^G$ for each $x \in G$, then $G \cong S_5$.

2. MAIN RESULTS

Lemma 2.1. *If G is a finite group with each element conjugate to its inverse, then G can not be of odd order.*

Proof. Let G be a finite group with $x^G = (x^{-1})^G$ for every $x \in G$. If G is abelian, then the given condition implies that $x = x^{-1}$ for each $x \in G$ and hence G turns out to be an elementary abelian 2-group. Suppose $G$is non-abelian. Because $x^G = (x^{-1})^G$, for some $u \in G$, $x^{-1} = u^{-1}xu$. Now

$$\begin{aligned} u^{-2}xu^2 &= u^{-1}(u^{-1}xu)u \\ &= u^{-1}x^{-1}u \\ &= (x^{-1})^{-1} = x. \end{aligned}$$

An easy induction shows that $u^{-k}xu^k = x$, if k is even, and $u^{-k}xu^k = x^{-1}$ if k is odd. If $|G| = m$, where m is an odd positive integer, then $u^{-m}xu^m = x^{-1}$. Since $u^m = 1$, $x = x^{-1}$ for each $x \in G$. This results in G being an elementary abelian 2-group, which is a contradiction. Thus, unless $G = 1$, G cannot be of odd order.

Theorem 2.2. *Let G be a meta-cyclic group with a finite centre such that for any $x \in G$, $x^G = (x^{-1})^G$. In that case, G is a finite group.*

Proof. Assume that G is a meta-cyclic group.Then there is a normal subgroup H of G for which both G/H and H are cyclic. Assume $G/H = < xH >$ and $H = < y >$, for some $x \in G$ and $y \in H$. Each element of G can now be expressed as $x^i y^j$, for some integers i, j. Because, for all $xH \in$

G/H, $(xH)^{G/H} = (x^{-1}H)^{G/H}$, there exists some $bH \in G/H$ such that $xH = b^{-1}Hx^{-1}HbH$. But then $xH = x^{-1}H$. This shows that $x^2 \in H =< y >$. Hence, $x^2 = y^t$, for some $t \in \mathbf{Z}$. Since $x \in (x^{-1})^G$, $x = g^{-1}x^{-1}g$, for some $g \in G$. Let $g = x^r y^s$. Then $x = y^{-s}x^{-1}y^s$ and $x^2 = y^{-s}x^{-2}y^s$. Since $x^{-2} \in H =< y >$, $x^4 = 1$. If $t \neq 0$, then $x^2 = y^t \Rightarrow y^{2t} = x^4 = 1$, i.e., $|y| < \infty$. Hence, in this case G is finite. But, $x^2 = 1$ if $t = 0$, and in this case, $x = x^{-1}$. Now

$$\begin{aligned} x &= y^{-s}x^{-1}y^s, \\ x &= y^{-s}xy^s. \end{aligned}$$

Thus, $y^s \in Z(G)$. Because $Z(G)$ is finite, order of y must be finite. But then G is a finite group.

Theorem 2.3. *Let G be a non-solvable group of order 120 with the property that for all $x \in G$, x^G coincides with $(x^{-1})^G$. If $G \neq G'$ and center of G is trivial, then $G \cong S_5$.*

Proof. To prove the theorem, we use the fact that every group of order < 60 is solvable and if a subgroup H with index n exists within a group G, then then there exists a homomorphism $f : G \to S_n$, whose kernel is contained in H. Since $(xG')^{G/G'} = (x^{-1}G')^{G/G'}$ holds for each $xG' \in G/G'$, G/G' turns out to be an elementary abelian 2-group. As a result, $|G/G'|$ has order 2^m, for some $m \geq 1$. Because $Z(G) = \{e\}$, G can no longer be a 2-group. Consequently, $m \geq 2$, and hence $|G/G'| < 60$ and $2 < |G'| < 60$. However, since both G' and G/G' are solvable, G is solvable as well, which is a contradiction. Thus, $|G/G'| = 2$ and $|G'| = 60$. Now $|G| = 2^3.3.5$. Let $n_p(G)$ represents the number of Sylow p-subgroups of G. Since $|G| = 2^3.3.5$, $n_2(G)$= 1,3,5 or 15. If $n_2(G) = 1$, then G posseses a unique Sylow 2-subgroup, say H of order 8, which is normal in G. Now $|G/H| = 15$ indicates that G is a solvable group, which is a contradiction. Assume $n_2(G) = 3$ and $H \in Syl_2(G)$. As $n_2(G)$ is the number of conjugates of H in G, G has a subgroup of order 40, which is precisely $N_G(H)$. Let $A = N_G(H)$. A homomorphism $f : G \to S_3$ now exists such that $ker(f) \subseteq A$. Since $|A| = 40$ and $|S_3| = 6$, $20 \leq |ker(f)| \leq 40$. But then G is solvable, a contradiction.

Similarly, if $n_2(G) = 5$, there exists a subgroup A of order 24 in G and hence we can find a homomorphism $f : G \to S_5$. Let $K = Ker(f)$. Then $1 \leq |K| \leq 24$. If $|K| > 2$, then both K and G/K are solvable, and thus G is solvable; this is a contradiction. If $|K| = 2$, then $K \subseteq G'$, otherwise $G = K \oplus G'$, and finally $K \subseteq Z(G) = 1$, resulting in another contradiction. Hence,

$K = \{e\}$. But then, $G \cong S_5$. Now suppose that $n_2(G) = 15$. As $|G^{'}| = 2^2.3.5$ and $G^{'}$ is a normal subgroup of G, G and $G^{'}$ have identical Sylow 3- and Sylow 5-subgroups. Clearly $n_3(G)$ =1,4 or 10. As before, if $n_3(G) = 1$ or 4, then G is solvable. Hence $n_3(G) = 10$. Let $P \in Syl_3(G)$. Then $|N_{G'}(P)| = 6$. A similar argument shows that $n_5(G) = 6$. Since $n_3(G) = n_3(G^{'}) = 10$ and $n_5(G) = n_5(G^{'}) = 6$, G and $G^{'}$ have 44 elements of order 3 or 5.

Let H_i $(1 \leq i \leq 15)$ be Sylow 2-subgroups of G. Let $A = H_i \cap H_j$. If for some $i \neq j$, $|H_i \cap H_j| = 4$, then $A \Delta H_i$, $A \Delta H_j$, as $[H_i : A] = [H_j : A] = 2$. Thus $H_iH_j \subseteq N_G(A)$. From this it follows that, $|N_G(A)| \geq 16$.Hence either $|N_G(A)| = 24$ or 40. If $|N_G(A)| = 40$, then G is solvable; again a contradiction. So, $|N_G(A)| = 24$, but then $G \cong S_5$.

Now $n_2(G^{'}) = 1$ or 3 or 5 or 15. Suppose $n_2(G^{'}) = 15$ and let K_i, $1 \leq i \leq 15$ be Sylow 2-subgroups of $G^{'}$. Suppose there exist two Sylow 2-subgroups, say K_1 and K_2, of $G^{'}$ such that $|K_1 \cap K_2| = 2$. (note that $|K_1 \cap K_2| \neq 4$, as $|K_i| = 4$, for all i). Let $K = K_1 \cap K_2$ and A be the normalizer of K in $G^{'}$, i.e., $A = N_{G'}(K)$. Then K is a normal subgroup of K_1 and K_2. Thus, $K_1K_2 \subseteq N_{G'}(K) = A$. Since $|K_1K_2| = 8$, $|A| \geq 12$. If $|A| > 12$, then $|A| = 20$ or 60 and therefore $G^{'}$ is solvable; a contradiction. Hence $|A| = 12 = 2^2.3$. Now $K_1, K_2 \in Syl_2(A)$, and thus $n_2(A) = 3$. But then A has a unique Sylow 3-subgroup, say $\hat{P}$ of order 3. Now $\hat{P} \Delta A$ and hence $A \subseteq N_{G'}(\hat{P})$. But then $|N_{G'}(\hat{P})| \geq 12$; a contradiction.Thus $|N_{G'}(\hat{P})| = 6$. Hence $K_i \cap K_j = \{e\}$, for all $i \neq j$. But then $G^{'}$ has at least 20+24+45+1=90 elements; a contradiction. Hence, $n_2(G^{'}) = 5$, i.e., K_i, $1 \leq i \leq 5$ are Sylow 2-subgroup of $G^{'}$. Now $H_i \in Syl_2(G)$ and $G^{'} \Delta G$, therefore $H_i \cap G^{'} \in Syl_2(G^{'})$. Since $1 \leq i \leq 15$ and $n_2(G^{'}) = 5$, there are two Sylow 2-subgroups of G, say H_1 and H_2, respectively such that $H_1 \cap G^{'} = H_2 \cap G^{'} = K_1$ (say). But then $H_1 \cap H_2 = K_1$, i.e., $|H_1 \cap H_2| = 4$. Finaly, using the same argument as above, $G \cong S_5$.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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