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NOTE ON THE HOMOLOGY EQUIVALENCE OF TOPOLOGICAL OPERADS

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Abstract. In this paper, we prove that the equivalence homotopy of topological operads gives rise to an associated equivalence of monads in the category of $\mathbb{K}$-modules when $\mathbb{K}$ is of characteristic zero. Some consequences hold in the case of topological monoids associated to group operads.

Keywords: operad; monad; monoid; homology.

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1. Introduction

The Homology of operad and the homology of algebras over operad has been hugely studied by many authors [4], [6], [8]... . Indeed, in [4], Fresse gave a complete description of (co) homology of algebras over operads by using the module of Kähler differentials and the coefficients consist of right modules over the enveloping algebra and gave some classical examples. In [6], Markl has studied the properties of differential graded (dg) operads modulo weak equivalences, that is, modulo the relation given by the existence of a chain of dg operad maps inducing a homology isomorphism. And in [8], Sinha proved the well-known result that the homology of the little disks operad is the graded Poisson operad.

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Our result is based on two classical May's theorems that gives an equivalence of monads associated to topological operad, and we finish the paper by using the standard result that describes the homology in term of derived functor Tor.

We recall P. May's definition of operads and symmetric operads, and discuss some basic properties.

Definition 1.1. *An operad $\mathscr{C}$ consists of $k-$modules $\mathscr{C}(j), j \geqslant 0$, together with a unit map $\eta : k \to \mathscr{C}(1)$, a right action by the symmetric group Σ_j on $\mathscr{C}(j)$ for each j, and maps*

$\gamma : \mathscr{C}(k) \otimes \mathscr{C}(j_1) \otimes \ldots \otimes \mathscr{C}(j_k) \to \mathscr{C}(j)$ for $k \geqslant 1$ and $j_s \geqslant 0$, where $\Sigma j_s = j$. The γ are required to be associative, unital, and equivariant in the following senses.

(i) The following associativity diagrams commute, where $\Sigma j_s = j$ and $\Sigma i_t = i$; we set $g_s = j_1 + \ldots + j_s$, and $h_s = i_{g_{s-1}} + 1 + \ldots + i_{g_s}$ for $1 \leqslant s \leqslant k$:

$$\begin{array}{ccc}
\mathscr{C}(k) \otimes (\bigotimes_{s=1}^{k} \mathscr{C}(j_s)) \otimes (\bigotimes_{r=1}^{j} \mathscr{C}(i_r)) & \xrightarrow{\gamma \otimes Id} & \mathscr{C}(j) \otimes (\bigotimes_{r=1}^{j} \mathscr{C}(i_r)) \\
\downarrow{\scriptstyle shuffle} & & \searrow{\scriptstyle \gamma} \\
 & & \mathscr{C}(i) \\
 & & \nearrow{\scriptstyle \gamma} \\
\mathscr{C}(k) \otimes (\bigotimes_{s=1}^{k} \mathscr{C}(j_s) \otimes (\bigotimes_{q=1}^{j_s} \mathscr{C}(i_{g_{s-1}+q}))) & \xrightarrow{Id \otimes (\gamma \otimes \ldots \otimes \gamma)} & \mathscr{C}(k) \otimes (\bigotimes_{s=1}^{k} \mathscr{C}(h_s))
\end{array}$$

(ii) The following unit diagrams commute:

$$\begin{array}{ccc}
\mathscr{C}(k) \otimes \mathbb{K}^{\otimes k} & \xrightarrow{\cong} & \mathscr{C}(k) \\
\downarrow{\scriptstyle Id \otimes \eta^{\otimes k}} & \nearrow{\scriptstyle \gamma} & \\
\mathscr{C}(k) \otimes \mathscr{C}(1)^{k} & &
\end{array}
\qquad
\begin{array}{ccc}
\mathbb{K} \otimes \mathscr{C}(j) & \xrightarrow{\cong} & \mathscr{C}(k) \\
\downarrow{\scriptstyle Id \otimes \eta} & \nearrow{\scriptstyle \gamma} & \\
\mathscr{C}(1) \otimes \mathscr{C}(j) & &
\end{array}$$

(iii) Equivariance axiom: The following equivariance diagrams commute, where $\sigma \in \Sigma_k$, $\tau_s \in \Sigma_{j_s}$; $\sigma(j_1, \ldots, j_k) \in \Sigma_k$ permutes k blocks of letter as σ permutes k letters, and $\tau_1 \oplus \ldots \oplus \tau_k \in \Sigma_k$ is the block sum:

$$\begin{array}{ccc}
\mathscr{C}(k)\otimes\mathscr{C}(j_1)\otimes\ldots\otimes\mathscr{C}(j_k) & \xrightarrow{\sigma\otimes\sigma^{-1}} & \mathscr{C}(k)\otimes\mathscr{C}(j_{\sigma(1)})\otimes\ldots\otimes\mathscr{C}(j_{\sigma(k)}) \\
\downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\gamma} \\
\mathscr{C}(j) & \xrightarrow{\sigma} & \mathscr{C}(j)
\end{array}$$

$$\begin{array}{ccc}
\mathscr{C}(k)\otimes\mathscr{C}(j_1)\otimes\ldots\otimes\mathscr{C}(j_k) & \xrightarrow{Id\otimes\tau_1\otimes\ldots\otimes\tau_k} & \mathscr{C}(k)\otimes\mathscr{C}(j_1)\otimes\ldots\otimes\mathscr{C}(j_k) \\
\downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\gamma} \\
\mathscr{C}(j) & \xrightarrow{\tau_1\otimes\ldots\otimes\tau_k} & \mathscr{C}(j)
\end{array}$$

The $\mathscr{C}(j)$ are to be thought of as modules of parameters for "j-ary operations" that accept j inputs and produce one output. Thinking of elements as operations, we think of $\gamma(c\otimes d_1\otimes\ldots\otimes d_k)$ as the composite of the operation c with the tensor product of the operations d_s.

2. Algebras Over Operads

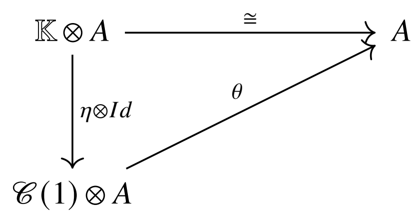
Let X^j denote the j-fold tensor power of a $\mathbb{K}$-module X, with Σ_j acting on the left. Again, $X^0=\mathbb{K}$.

Definition 2.1. *Let $\mathscr{C}$ be an operad. A $\mathscr{C}$-algebra is a $\mathbb{K}$-module A together with maps $\theta:\mathscr{C}(j)\otimes A^j\to A$, $j\geqslant 0$, that are associative, unital, and equivariant in the following senses.*

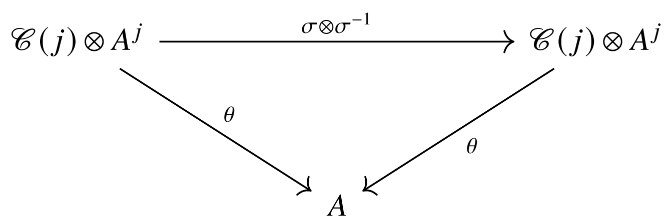
(i) The following associativity diagrams commute, where $j=\Sigma j_s$:

$$\begin{array}{ccccc}
\mathscr{C}(k)\otimes\mathscr{C}(j_1))\otimes\ldots\otimes\mathscr{C}(j_k)) & \xrightarrow{\theta\otimes Id} & \mathscr{C}(j)\otimes A^j & & \\
 & & & \searrow{\scriptstyle\theta} & \\
\downarrow{\scriptstyle shuffle} & & & & A \\
 & & & \nearrow{\scriptstyle\theta} & \\
\mathscr{C}(k)\otimes\mathscr{C}(j_1)\otimes A^{j_1}\bigotimes\ldots\otimes\mathscr{C}(j_k)\otimes A^{j_k} & \xrightarrow{Id\otimes\theta^k} & \mathscr{C}(k)\otimes A^k & &
\end{array}$$

(ii) The following unit diagram commutes:



(iii) The following equivariance diagrams commute, where $\sigma \in \Sigma_j$;



To define the homology of $\mathscr{C}$-algebras, we use the module of Kähler differentials $\Omega^1_{\mathscr{C}}(A)$ and the coefficients consist of right modules over the enveloping algebra $U_{\mathscr{C}}(A)$. These objects are defined in §4 [4] for operads and algebras in k-modules . The principle of generalized point tensors can be used to extend the construction to the dg-context. Then the definition returns a dg-algebra $U_{\mathscr{C}}(A)$ and a dg-module $\Omega^1_{\mathscr{C}}(A)$ so that we have an isomorphism

$$\mathrm{Der}_{\mathscr{C}}(A,E) \cong Hom_{U_{\mathscr{C}}(A)}\mathrm{Mod}(\Omega^1_{\mathscr{C}}(A),E)$$

in the category of dg-modules, E is a representation of A, and Q_A is any cofibrant replacement of A in the category of $\mathscr{C}$-algebras.

The homology of A with coefficients in a right $U_{\mathscr{C}}(A)$-module E is defined by:

$$H^{\mathscr{C}}_*(A,E) = H_*(E \otimes_{U_{\mathscr{C}}(Q_A)} \Omega^1_{\mathscr{C}}(Q_A)),$$

3. Topological Operad

Throughout this paper, all topological spaces are assumed to be compactly gen- erated Hausdorff spaces [8]. So, we denote the category of compactly gen- erated Hausdorff spaces by $\mathcal{T}$. For two points x, x' in a space X, let $x \sim x'$ denote that x, x' are in the same path-connected component of X.

We recall the topological operad notion as defined in [7], def.1.1.

Definition 3.1. *A topological operad $\mathscr{C}$ consists of*

1) a sequence of topological spaces $\{\mathscr{C}(n)\}_{n\geq 0}$ with $\mathscr{C}(0)=*$,

2) a family of maps,

$$\gamma : \mathscr{C}(k)\times\mathscr{C}(m_1)\times\ldots\times\mathscr{C}(m_k)\to\mathscr{C}(m),\ k\geqslant 1,\ m_i\geqslant 0,\ m=m_1+\ldots+m_k,$$

3) an element $1\in\mathscr{C}(1)$ called the unit, satisfying the following two coherence properties: for $a\in\mathscr{C}(k)$, $b_i\in\mathscr{C}(m_i)$, and $c_j\in\mathscr{C}(n_j)$, $n_j\geqslant 0$,

i) Associativity:

$$\gamma(\gamma(a;b_1,\ldots,b_k);c_1,\ldots,c_m)=\gamma(a;\gamma(b_1;c_1,\ldots,c_{m_1}),\ldots,\gamma(b_k;c_{m_1+\ldots+m_{k-1}+1},\ldots,c_m));$$

ii) Unitality: $\gamma(1;a)=a$ and $\gamma(a;1^{(k)})=a$.

A symmetric topological operad is a topological operad with a left action of Σ_n on $\mathscr{C}(n)$ for each n, satisfying the following equivariance property: for $\sigma\in\Sigma_k$, and $\tau_i\in\Sigma_{m_i}$,

$$\gamma(\sigma a;\tau_1 b_1,\ldots,\tau_k b_k)=\gamma(\sigma;\tau_1,\ldots,\tau_k)\gamma(a;b_{\sigma^{-1}(1)},\ldots,b_{\sigma^{-1}(k)}).$$

The most important examples of symmetric topological operads are the little n-cubes operads and n-little disks operads, refer to [May] for details. A very important symmetric discrete operad is the symmetric groups operad $\mathscr{S}=\{\Sigma_n\}_{n\geqslant 0}$ introduced in Definition 3.1 (i) of [7].

Definition 3.2. *An operad $\mathscr{C}$ is said to be Σ-free if Σ_j acts freely on $\mathscr{C}(j)$ for all j. A morphism $\psi:\mathscr{C}\to\mathscr{C}'$ of operads is a sequence of Σ_j-equivariant maps $\psi_j:\mathscr{C}(j)\to\mathscr{C}'(j)$ such that $\psi_1(1)=1$ the following diagram commutes*

$$\begin{array}{ccc}
\mathscr{C}(k)\otimes\mathscr{C}(j_1)\otimes\ldots\otimes\mathscr{C}(j_k) & \xrightarrow{\quad\gamma\quad} & \mathscr{C}(j)\\
\downarrow{\scriptstyle \psi_k\times\psi_{j_1}\times\ldots\times\psi_{j_k}} & & \downarrow{\scriptstyle \psi_j}\\
\mathscr{C}'(k)\otimes\mathscr{C}'(j_1)\otimes\ldots\otimes\mathscr{C}'(j_k) & \xrightarrow{\quad\gamma'\quad} & \mathscr{C}'(j)
\end{array}$$

The May's recognition theorem (theorem 1.3. [7]) plays a primordial role in the study of topological operads

Theorem 3.3. *There exist Σ-free operads $\mathscr{C}_n$, $1<n<\infty$, such that every n-fold loop space is a $\mathscr{C}_n$ -space and every connected $\mathscr{C}_n$-space has the weak homotopy type of an n-fold loop space.*

Proposition 3.4. *For a topological operad $\mathscr{C}$, $\mathscr{C}(1)$ is an associative $H-$space.*

4. The Associated Monad of an Operad.

Definition 4.1. *Any topological Σ-free-operad $\mathscr{C}$ de- termines a monad from the category of pointed compactly generated Hausdorff spaces to itself. Let X be a pointed space. Let $\mathscr{C}X = \coprod_{k\geq 0} \mathscr{C}(k) \times_{\Sigma_k} X^k$ with the weak topology, where the equivalence relation is generated by $[d_i c, x] \sim [c, dix]$, for $c \in \mathscr{C}(k)$, $1 \leqslant i \leqslant k$, $x = (x1, ..., x_{k-1}) \in X^{k-1}$.*

This construction CX is based on the Δ-set structure on C and is similar to the geometric realization of a Δ-set. The two natural maps $\mathscr{C}(\mathscr{C}X) \to \mathscr{C}X$ and $X \to \mathscr{C}X$ are defined as Construction 2.4 of [7].

In the category of $\mathbb{K}$-modules, we have the following simple construction of the monad of free algebras over an operad $\mathscr{C}$.

Definition 4.2. *Define the monad C associated to an operad $\mathscr{C}$ by letting*

$$CX = \bigoplus_{k\geq 0} \mathscr{C}(k) \otimes_{\mathbb{K}[\Sigma_k]} X^k.$$

The unit $\eta : X \to CX$ is $\eta \otimes Id : X = \mathbb{K} \otimes X \to \mathscr{C} \otimes X$ and the map $\mu : CCX \to CX$ is induced by the maps ($j = \sum j_s$)

$$\begin{array}{c}
\mathscr{C}(k) \otimes \mathscr{C}(j_1) \otimes X^{j_1} \otimes \ldots \otimes \mathscr{C}(j_k) \otimes X^{j_k} \\
\Big\downarrow {\scriptstyle shuffle} \\
\mathscr{C}(k) \otimes \mathscr{C}(j_1) \otimes \ldots \otimes \mathscr{C}(j_k) \otimes X^j \\
\Big\downarrow {\scriptstyle \gamma \otimes Id} \\
\mathscr{C}(j) \otimes X^j
\end{array}$$

Proposition 4.3. [5] *A $\mathscr{C}$-algebra structure on a $\mathbb{K}$-module A determines and is deter mined by a C-algebra structure on A. Formally, the identity functor on the category of $\mathbb{K}$-modules restricts to give an isomorphism between the categories of $\mathscr{C}$-algebras and of C-algebras.*

Definition 4.4. *A morphism $\psi : \mathscr{C} \to \mathscr{C}'$ of topological operads is called an equivalence, if each $\psi : \mathscr{C}(k) \to \mathscr{C}'(k)$ is (1) a Σ_k-equivariant homotopy equivalence, or (2) a homotopy equivalence and the actions of Σ_k on $\mathscr{C}(k)$, $\mathscr{C}'(k)$ are covering actions for all k. Two topological equivariant-operads are equivalent if there is a chain of equivalences connecting them.*

This definition is a slight modification of P. May's Definition 3.3 in [6]. It should be mentioned that an equivalence $\psi : \mathscr{C} \to \mathscr{C}'$ need not have an inverse morphism. We recall the following key result due to May.

Proposition 4.5. [5] *Let $\mathscr{C}$ be any operad of spaces. Let C denote both the monad in the category of spaces associated to $\mathscr{C}$ and the monad in the category of $\mathbb{K}$-modules associated to $H_*(\mathscr{C})$. If $\mathbb{K}$ is a field of characteristic zero, then*

$$H_*(CX) \cong CH_*(X)$$

And now, we announce our main result in this paper, that gives equivalence of homology of monads associated to connected topological operad.

Theorem 4.6 (Main Theorem)**.** *Let $\mathscr{C}$ and $\mathscr{C}'$ be two path-connected topological operads, and $\psi : \mathscr{C} \to \mathscr{C}'$ is an equivalence. Let C and C' denote both the monads in the category of spaces associated respectively to $\mathscr{C}$ and $\mathscr{C}'$ and the monad in the category of $\mathbb{K}$-modules associated to $H_*(\mathscr{C})$ and $H_*(\mathscr{C}')$. Then, $CH_*(X) \cong C'H_*(X)$, for all connected CW-complex X.*

Proof. Since $\psi : \mathscr{C} \to \mathscr{C}'$ is an equivalence of path connected topological operads, then $\psi : \mathscr{C}X \to \mathscr{C}'X$ is a homotopy equivalence of connected CW-complex, by Hurewicz theorem, we deduce that $H_*(CX) \cong H_*(C'X)$. The isomorphism $H_*(CX) \cong CH_*(X)$ leads to $CH_*(X) \cong C'H_*(X)$. □

In [10], W. Zhang proved the following results, which can be useful to study the topological monoids.

Proposition 4.7. *If $\mathscr{G}$ is a group operad, then $\mathscr{G}X$ is a topological monoid.*

Proposition 4.8. *For a group operad $\mathscr{G}$, $\mathscr{G}X$ is a free monoid.*

For a monoid M, recall that its universal group UM, cf. Proposition 4.2 of [1], is constructed as follows. Let FM be the free group on the pointed (by the unit) set M and denote the image of $a \in M$ in FM by $[a]$. Let N be the normal subgroup of FM generated by all elements of the form $[a].[b].[ab]^{-1}$, $a, b \in M$. Then define $UM = FM/N$. UM has the property that UM is a free group if M is a free monoid. Consequently $U\mathcal{G}X$ is a free group since $\mathcal{G}X$ is a free monoid.

We can immediatly deduce the following

Proposition 4.9. *For each left $\mathcal{G}X$-module A, $H_p(\mathcal{G}X, A) = Tor_p^{\mathcal{G}X}(\mathbb{Z}, A)$.*

Conflict of Interests

The author declares that there is no conflict of interests.

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