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THE INTERSECTION OF FUZZY SUBGROUPS AND RELATION

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Abstract. In this paper we have defined fuzzy set, fuzzy relation, fuzzy subgroup, complex fuzzy subgroup, and fuzzy normal subgroup. By using our new perspective, we have introduced the idea of the intersection of fuzzy sets and provided several theorems on fuzzy subgroup. Additionally, we have used our novel notion for fuzzy relations and provided several instances.

Keywords: fuzzy subgroup; fuzzy relation; fuzzy sets; fuzzy normal subgroup; complex fuzzy subgroup.
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1. Introduction

In 1965, L.A. Zadeh [13] proposed fuzzy sets and a number of set operations, including intersection, union, and complement of fuzzy sets. The fuzzy subgroup was then defined by Rosenfeld [1], utilizing the fuzzy set and its operations. The year 1992 was the pinnacle of fuzzy sets. In the same year, Saxena [14] researched the non-trivial union and decomposition of fuzzy subgroups, and Kumar [9] established conclusions for normal fuzzy subgroups. In 1989, Buckley [11] proposed fuzzy complex numbers. This two-dimensional technique covers every branch of the fuzzy universe after fuzzy complex numbers. Buckley's two-dimensional method

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was employed as a complex fuzzy set [3]. Work on the idea of fuzzy relation equations with applications in transportation issues and a focus on the medical elements of difficulties was done by Sanchez in 1976 [6]. Dubois [2] worked on the fuzzy set's application component in 2000. He used fuzzy sets in modeling and system control, approximation reasoning and information systems, pattern recognition and image processing, decision analysis, operation research, and statistics. Using complex vague lattices, Singh [15] examined ambiguity and uncertainty in a data set in 2017. In 2019, Lee and Hur [10] proposed the idea of a bipolar fuzzy connection and explored how it related to their (a,b)-level sets. Emam [5] worked on the suitability of finite intuitionistic fuzzy relations with the group Z_n. In 2021, Kumar and Gangwal [17] investigated the use of fuzzy relations in conjunction with Sanchez's method for medical diagnosis.

The intersection of two or more complex fuzzy subgroups was defined by Alsarahead [12] and Ramot [4].In this paper, we have applied this concept for fuzzy subgroups, given a new result by defining the intersection of fuzzy sets in different domains, and illustrated our main theorem by example 4.1

2. PRELIMINARIES

Definition 2.1 (Fuzzy set)**.** [13] Let $\mathbf{X}$ be any set, then a fuzzy set $\tilde{A}$ in $\mathbf{X}$ is a set of ordered pairs:

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in \mathbf{X}\}$$

here $\mu_{\tilde{A}} : \mathbf{X} \longrightarrow [0,1]$ is called membership function.

Definition 2.2. [13] $\tilde{A}$ and $\tilde{B}$ are fuzzy sets in $\mathbf{X}$ then $\tilde{A} \cap \tilde{B}$ is a fuzzy set in $\mathbf{X}$

$$\tilde{A} \cap \tilde{B} = \{(x, \mu_{\tilde{A}\cap\tilde{B}}(x)) : x \in \mathbf{X}\}$$

here $\mu_{\tilde{A}\cap\tilde{B}}(x) = \mu_{\tilde{A}}(x) \wedge \mu_{\tilde{B}}(x) = min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}$

Definition 2.3 (Crisp Relation)**.** [7] A relation among crisp sets $X_1, X_2, \ldots, X_n$ is a subset of the cartesian product $\prod_{i=1}^{n} X_i$. It is denoted by $\mathbf{R}(X_1, X_2, \ldots, X_n)$. For every object $x \in \prod_{i=1}^{n} X_i$, if $x \in \mathbf{R}$ means relation $\mathbf{R}$ hold for x, and if $x \notin \mathbf{R}$ then relation $\mathbf{R}$ does not hold for x.

In light of these two facts, the following characteristic function for relation **R** is defined:

$$\mathbf{R}(x) = \begin{cases} 1 & if\ x \in \mathbf{R} \\ 0 & \text{otherwise} \end{cases}$$

A crisp relation is constructed only for a specific argument in which the truth value can be true or false. In reality, there are a lot of statements that might be true or false, e.g., *"The price of product **X** is higher than product **Y**,"* but it does not provide any information about the price difference between **X** and **Y**. By establishing fuzzy relation on this type of argument or relationship, we can account all possible outcomes.

Definition 2.4 (Fuzzy Relation)**.** [8] Let $X_1, X_2, \dots, X_n$ be any set, then

$$\tilde{R} = \{(x, \mu_{\tilde{R}}(x)) : x \in \prod_{i=1}^{n} X_i\}$$

is called a fuzzy relation on $\prod_{i=1}^{n} X_i$.

Example 2.5. If **X** is a set of some district of Chhattisgarh and Y is set of the capital of Chhattisgarh. For $x \in \mathbf{X}$ and $y \in \mathbf{Y}$ relation **R** stated as follows

R: *x is far from y*

Here $\mathbf{Y} = \{Raipur\}$

X	Kanker	Bilaspur	Korba	Rajnandgaon	Durg	Raipur	Balod	Kondagaon	Sukma
Distance (in km) $\{d(x,y)\}$	130	106	169	64.1	36.3	0	74	184	318

TABLE 1. **X** and distance from **Y**

Table-1 shows the distance of Raipur from elements of **X**. Now define $\mu_{\tilde{R}} : \mathbf{X} \times \mathbf{Y} \to [0,1]$ such that

$$\mu_{\tilde{R}}(x,y) = \frac{d(x,y)}{350}\ \forall\ \mathbf{X} \times \mathbf{Y} \tag{1}$$

Note: For this function, we randomly select a denominator that is greater than or equal to the most significant distance.

Here $\mathbf{X} \times \mathbf{Y} \sim \mathbf{X}$ therefore, we write x instead of (x, y), so relation $\tilde{R}$ can be written as

$$\tilde{R} = \frac{0.3714}{Kanker} + \frac{0.3029}{Bilaspur} + \frac{0.4829}{Korba} + \frac{0.1831}{Rajnandgaon} + \frac{0.1037}{Durg} + \frac{0}{Raipur} + \frac{0.2114}{Balod} + \frac{0.5257}{Kondagaon} + \frac{0.9086}{Sukma}$$

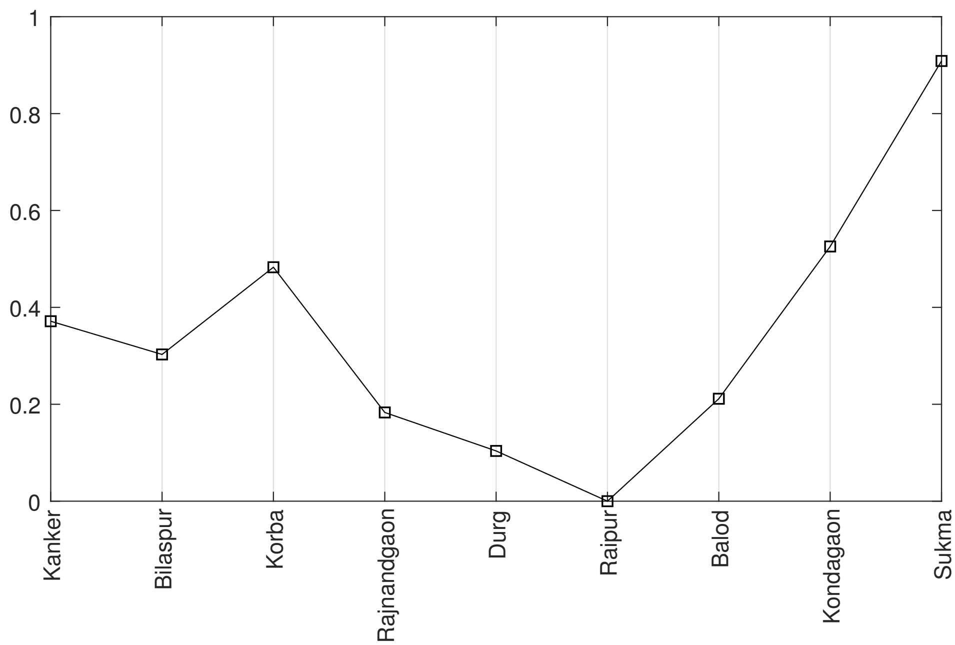

FIGURE 1. Fuzzy Relation on $\mathbf{X} \times \mathbf{Y}$

In (fig.1), Sukma has the highest value, indicating that it is more distant from Raipur compared to other locations.

Definition 2.6 (Fuzzy Subgroup [1])**.** Let $\mathbf{G}$ be any group and $\mathscr{F}\mathscr{P}(\mathbf{G})$ be the fuzzy power set of $\mathbf{G}$. If $\mu \in \mathscr{F}\mathscr{P}(\mathbf{G}) then \mu$ is called a fuzzy subgroup of $\mathbf{G}$ if,

(1) $\mu(xy) \geq \mu(x) \wedge \mu(y) \; \forall \, x, y \in \mathbf{G}$
(2) $\mu(x^{-1}) \geq \mu(x) \; \forall \, x \in \mathbf{G}$

here $\mathscr{F}(\mathbf{G})$ denotes the set of all fuzzy subgroups of $\mathbf{G}$.

Theorem 2.7. *If $\mathbf{G}$ be any group. $\{\tilde{A}_{\mathbf{i}} : \mathbf{i} \in \wedge\}$ be the collection of fuzzy subgroups of $\mathbf{G}$. Then $\bigcap_{\mathbf{i} \in \wedge} \tilde{A}_{\mathbf{i}}$ is a fuzzy subgroup of $\mathbf{G}$.*

Proof. Given $\mathbf{G}$ be a group and $\tilde{A}_{\mathbf{i}} \in \mathscr{F}(\mathbf{G}) \forall \mathbf{i} \in \wedge$

Clearly

$$\bigcap_{\mathbf{i}\in\wedge} \tilde{A}_{\mathbf{i}} \in \mathscr{F}\mathscr{P}(\mathbf{G}) \qquad \text{(from [13])}$$

Let $x, y \in \mathbf{G}$ be arbitrary then

$$\begin{aligned}
(\bigcap_{\mathbf{i}\in\wedge} \tilde{A}_{\mathbf{i}})(xy) &= \min_{\mathbf{i}\in\wedge} \{\tilde{A}_{\mathbf{i}}(xy)\} \\
&\geq \min_{\mathbf{i}\in\wedge} [min\{\tilde{A}_{\mathbf{i}}(x), \tilde{A}_{\mathbf{i}}(y)\}] \\
&= min[\min_{\mathbf{i}\in\wedge}\{\tilde{A}_{\mathbf{i}}(x)\}, \min_{\mathbf{i}\in\wedge}\{\tilde{A}_{\mathbf{i}}(y)\}] \\
&= min\{\bigcap_{\mathbf{i}\in\wedge} \tilde{A}_{\mathbf{i}}(x), \bigcap_{\mathbf{i}\in\wedge} \tilde{A}_{\mathbf{i}}(y)\} \\
&= (\bigcap_{\mathbf{i}\in\wedge} \tilde{A}_{\mathbf{i}})(x) \wedge (\bigcap_{\mathbf{i}\in\wedge} \tilde{A}_{\mathbf{i}})(y)
\end{aligned}$$

Thus for all $x, y \in \mathbf{G}$

$$(\bigcap_{\mathbf{i}\in\wedge} \tilde{A}_{\mathbf{i}})(xy) \geq (\bigcap_{\mathbf{i}\in\wedge} \tilde{A}_{\mathbf{i}})(x) \wedge (\bigcap_{\mathbf{i}\in\wedge} \tilde{A}_{\mathbf{i}})(y) \tag{2}$$

Again let $x \in \mathbf{G}$ be arbitrary

$$\begin{aligned}
\bigcap_{\mathbf{i}\in\wedge} \tilde{A}_{\mathbf{i}})(x^{-1}) &= \min_{\mathbf{i}\in\wedge}\{\tilde{A}_{\mathbf{i}}(x^{-1})\} \\
&\geq \min_{\mathbf{i}\in\wedge}\{\tilde{A}_{\mathbf{i}}(x)\} \\
&= (\bigcap_{\mathbf{i}\in\wedge} \tilde{A}_{\mathbf{i}})(x)
\end{aligned}$$

Thus for all $x \in \mathbf{G}$

$$(\bigcap_{\mathbf{i}\in\wedge} \tilde{A}_{\mathbf{i}})(x^{-1}) \geq (\bigcap_{\mathbf{i}\in\wedge} \tilde{A}_{\mathbf{i}})(x) \tag{3}$$

Thus by inequality (2) and inequality (3)

$$\bigcap_{\mathbf{i}\in\wedge} \tilde{A}_{\mathbf{i}} \in \mathscr{F}(\mathbf{G})$$

□

Corollary 2.8. [12] The intersection of any two fuzzy subgroups of a group is a fuzzy subgroup.

Proof. Let **G** be any group and $\tilde{A},\tilde{B} \in \mathscr{F}(\mathbf{G})$.

Clearly

$$\tilde{A}\cap\tilde{B} \in \mathscr{F}\mathscr{P}(\mathbf{G})[13]$$

Let $x,y \in \mathbf{G}$ be arbitrary then, from Def. 2.2

$$\begin{aligned}
(\tilde{A}\cap\tilde{B})(xy) &= \tilde{A}(xy)\wedge\tilde{B}(xy) \\
&= min\{\tilde{A}(xy),\tilde{B}(xy)\} \\
&\geq min[min\{\tilde{A}(x),\tilde{A}(y)\},min\{\tilde{B}(x),\tilde{B}(y)\}] \\
&= min[min\{\tilde{A}(x),\tilde{B}(x)\},min\{\tilde{A}(y),\tilde{B}(y)\}] \\
&= min\{(\tilde{A}\cap\tilde{B})(x),(\tilde{A}\cap\tilde{B})(y)\} \\
&= (\tilde{A}\cap\tilde{B})(x)\wedge(\tilde{A}\cap\tilde{B})(y)
\end{aligned}$$

Thus for all $x,y \in \mathbf{G}$

$$(\tilde{A}\cap\tilde{B})(xy) \geq (\tilde{A}\cap\tilde{B})(x)\wedge(\tilde{A}\cap\tilde{B})(y) \tag{4}$$

Again let $x \in \mathbf{G}$ be arbitrary

$$(\tilde{A}\cap\tilde{B})(x^{-1}) = \tilde{A}(x^{-1})\wedge\tilde{B}(x^{-1}) \geq min\{\tilde{A}(x),\tilde{B}(x)\} = (\tilde{A}\cap\tilde{B})(x)$$

Thus for all $x \in \mathbf{G}$

$$(\tilde{A}\cap\tilde{B})(x^{-1}) \geq (\tilde{A}\cap\tilde{B})(x) \tag{5}$$

Thus by inequality (4) and inequality (5)

$$\tilde{A}\cap\tilde{B} \in \mathscr{F}(\mathbf{G})$$

□

Following example is an illustration of the Cor. 2.8.

Example 2.9. $(\mathbb{Z}_6, +_6)$ be a group and $\tilde{A}, \tilde{B} \in \mathscr{F}(\mathbf{G})$ such that

$$\tilde{A}(x) = \begin{cases} 1 & \text{if x = e} \\ .25 & \text{else} \end{cases}$$

and

$$\tilde{B}(x) = \begin{cases} .85 & \text{if x = e} \\ .5 & \text{else} \end{cases}$$

we can easily show that

$$(\tilde{A} \cap \tilde{B})(x) = \begin{cases} .85 & \text{if x= e} \\ .25 & \text{else} \end{cases}$$

which gives $\tilde{A} \cap \tilde{B} \in \mathscr{F}(\mathbf{G})$.

Definition 2.10 (Fuzzy Normal Subgroup [9])**.** Let $\mathbf{G}$ be any group and $\mu \in \mathscr{F}(\mathbf{G})$ then μ is said to be a fuzzy normal subgroup if

$$\mu(xyx^{-1}) \geq \mu(y) \; \forall \; x, y \in \mathbf{G}$$

Theorem 2.11. *If $\mathbf{G}$ be any group. $\{\tilde{N}_{\mathbf{i}} : \mathbf{i} \in \wedge\}$ be the collection of fuzzy normal subgroups of $\mathbf{G}$. Then $\bigcap_{\mathbf{i} \in \wedge} \tilde{N}_{\mathbf{i}}$ is a fuzzy normal subgroup of $\mathbf{G}$*

Proof. Here $\mathbf{G}$ be any group and $\tilde{N}_{\mathbf{i}}$ is a fuzzy normal subgroup of $\mathbf{G}$ for all $\mathbf{i} \in \wedge$.
Now for all $\mathbf{i} \in \wedge$, $\tilde{N}_{\mathbf{i}} \in \mathscr{F}(\mathbf{G}) \implies \bigcap_{\mathbf{i} \in \wedge} \tilde{N}_{\mathbf{i}} \in \mathscr{F}(\mathbf{G})$ [*by* 2.7]
Let $x, y \in \mathbf{G}$ be arbitrary.

$$\begin{aligned} \bigcap_{\mathbf{i} \in \wedge} \tilde{N}_{\mathbf{i}}(xyx^{-1}) &= \min_{\mathbf{i} \in \wedge}\{\tilde{N}_{\mathbf{i}}(xyx^{-1})\} \\ &\geq \min_{\mathbf{i} \in \wedge}\{\tilde{N}_{\mathbf{i}}(y)\} \\ &= \bigcap_{\mathbf{i} \in \wedge} \tilde{N}_{\mathbf{i}}(y) \end{aligned}$$

Thus for all $x, y \in \mathbf{G}$

$$\bigcap_{\mathbf{i} \in \wedge} \tilde{N}_{\mathbf{i}}(xyx^{-1}) \geq \bigcap_{\mathbf{i} \in \wedge} \tilde{N}_{\mathbf{i}}(y) \tag{6}$$

Thus $\bigcap\limits_{\mathbf{i}\in\wedge} \tilde{N}_{\mathbf{i}}$ is a fuzzy normal subgroup of $\mathbf{G}$. □

Corollary 2.12. The intersection of two fuzzy normal subgroups of a group is a fuzzy normal subgroup.

Proof. Here $\mathbf{G}$ be any group and $\tilde{A}$ and $\tilde{B}$ are a fuzzy normal subgroup of $\mathbf{G}$.
Now $\tilde{A},\tilde{B} \in \mathscr{F}(\mathbf{G}) \implies \tilde{A}\cap\tilde{B} \in \mathscr{F}(\mathbf{G})$ [*by* 2.8]
Let $x,y \in \mathbf{G}$ be arbitrary.

$$\begin{aligned}(\tilde{A}\cap\tilde{B})(xyx^{-1}) &= min\{\tilde{A}(xyx^{-1}),\tilde{B}(xyx^{-1})\}\\ &\geq min\{\tilde{A}(y),\tilde{B}(y)\}\\ &= (\tilde{A}\cap\tilde{B})(y)\end{aligned}$$

Thus for all $x,y \in \mathbf{G}$

$$(\tilde{A}\cap\tilde{B})(xyx^{-1}) \geq (\tilde{A}\cap\tilde{B})(y) \tag{7}$$

Thus $\tilde{A}\cap\tilde{B}$ is a fuzzy normal subgroup of $\mathbf{G}$. □

Definition 2.13 (Complex Fuzzy Set [4])**.** Let $\mathbf{X}$ be any set, then a complex fuzzy set $\tilde{C}$ in $\mathbf{X}$ is a set of ordered pairs:

$$\tilde{C} = \{(x,\mu_{\tilde{C}}(x) : x \in \mathbf{X}\}$$

here $\mu_{\tilde{C}}\colon \mathbf{X} \longrightarrow \{z \in \mathbb{C}\colon |z| \leq 1\}$ complex valued membership function such that

$$\mu_{\tilde{C}} = r_{\tilde{C}}(x)e^{\omega_{\tilde{C}}(x)}$$

where $r_{\tilde{C}}(x)\colon \mathbf{X} \longrightarrow [0,1]$ and $\omega_{\tilde{C}}(x)\colon \mathbf{X} \longrightarrow [0,2\pi]$
$\{(x,r_{\tilde{C}}(x))\colon x \in \mathbf{X}\}$ is a fuzzy set on $\mathbf{X}$ and $\{(x,\omega_{\tilde{C}}(x))\colon x \in \mathbf{X}\}$ is π- fuzzy set on $\mathbf{X}$[12].

Definition 2.14 (Homogeneous Complex Fuzzy Sets [12])**.** A complex fuzzy set $\tilde{C}$ of $\mathbf{X}$ is said to be homogeneous if for all $x,y \in \mathbf{X}$

$$r_{\tilde{C}}(x) \leq r_{\tilde{C}}(y) \iff \omega_{\tilde{C}}(x) \leq \omega_{\tilde{C}}(y)$$

Definition 2.15 (Complex Fuzzy Subgroup [12])**.** A complex fuzzy set which is a fuzzy subgroup is known as a complex fuzzy subgroup.

Theorem 2.16. [12] *The intersection of two complex fuzzy subgroups is a complex fuzzy subgroup.*

Proof. By Cor. 2.8 and Def. 2.15 we can say, The intersection of two complex fuzzy subgroups is a complex fuzzy subgroup. □

3. Main Results

If two fuzzy sets are defined in a set, their intersection and union can be easily defined, but when the fuzzy sets are defined in different sets then it goes meaningless. In this section we have extended the notion of intersection and union of fuzzy sets.

Definition 3.1. Let $\tilde{A}$ and $\tilde{B}$ are fuzzy sets in set $\mathbf{X}$ and $\mathbf{Y}$ respectively, then $\tilde{A}\cap\tilde{B}$ is a fuzzy set in $\mathbf{X}\cap\mathbf{Y}$, and its membership grade is defined by

$$(\tilde{A}\cap\tilde{B})(x)=\tilde{A}(x)\wedge\tilde{B}(x)\ \forall\ x\in\mathbf{X}\cap\mathbf{Y}$$

3.1. Fuzzy Subgroup.

Theorem 3.1.1. If $\mathbf{G}$ be any group. $\{\mathbf{H_i}:\mathbf{i}\in\wedge\}$ be an arbitrary collection of subgroups of $\mathbf{G}$. If for any $\mathbf{i}\in\wedge$, $\tilde{A}_\mathbf{i}$ is a fuzzy subgroup of $\mathbf{H_i}$, then $\bigcap_{\mathbf{i}\in\wedge}\tilde{A}_\mathbf{i}$ is a fuzzy subgroup of $\bigcap_{\mathbf{i}\in\wedge}\mathbf{H_i}$.

Proof. For all $\mathbf{i}\in\wedge$, $\mathbf{H_i}$ is a subgroup of $\mathbf{G}$, therefore $\bigcap_{\mathbf{i}\in\wedge}\mathbf{H_i}$ is a subgroup of $\mathbf{G}$. Now

$$\tilde{A}_\mathbf{i}\in\mathscr{F}(\mathbf{H_i})\ \forall\ \mathbf{i}\in\wedge$$

Let $x,y\in\bigcap_{\mathbf{i}\in\wedge}\mathbf{H_i}$ be arbitrary then

$$\begin{aligned}(\bigcap_{\mathbf{i}\in\wedge}\tilde{A}_\mathbf{i})(xy) &= \min_{\mathbf{i}\in\wedge}\{\tilde{A}_\mathbf{i}(xy)\}\\ &\geq \min_{\mathbf{i}\in\wedge}[\min\{\tilde{A}_\mathbf{i}(x),\tilde{A}_\mathbf{i}(y)\}]\\ &= \min[\min_{\mathbf{i}\in\wedge}\{\tilde{A}_\mathbf{i}(x)\},\min_{\mathbf{i}\in\wedge}\{\tilde{A}_\mathbf{i}(y)\}]\end{aligned}$$

$$= min\{(\bigcap_{\mathbf{i}\in\wedge}\tilde{A}_{\mathbf{i}})(x),(\bigcap_{\mathbf{i}\in\wedge}\tilde{A}_{\mathbf{i}})(y)\}$$
$$= (\bigcap_{\mathbf{i}\in\wedge}\tilde{A}_{\mathbf{i}})(x)\wedge(\bigcap_{\mathbf{i}\in\wedge}\tilde{A}_{\mathbf{i}})(y)$$

Thus for all $x,y\in\bigcap_{i\in\wedge}\mathbf{H_i}$

$$(\bigcap_{\mathbf{i}\in\wedge}\tilde{A}_{\mathbf{i}})(xy)\geq(\bigcap_{\mathbf{i}\in\wedge}\tilde{A}_{\mathbf{i}})(x)\wedge(\bigcap_{\mathbf{i}\in\wedge}\tilde{A}_{\mathbf{i}})(y) \tag{8}$$

Again let $x\in\bigcap_{i\in\wedge}\mathbf{H_i}$ be arbitrary

$$\begin{aligned}(\bigcap_{\mathbf{i}\in\wedge}\tilde{A}_{\mathbf{i}})(x^{-1}) &= \min_{\mathbf{i}\in\wedge}\{\tilde{A}_{\mathbf{i}}(x^{-1})\}\\ &\geq \min_{\mathbf{i}\in\wedge}\{\tilde{A}_{\mathbf{i}}(x)\}\\ &= (\bigcap_{\mathbf{i}\in\wedge}\tilde{A}_{\mathbf{i}})(x)\end{aligned}$$

Thus for all $x\in\bigcap_{i\in\wedge}\mathbf{H_i}$

$$(\bigcap_{\mathbf{i}\in\wedge}\tilde{A}_{\mathbf{i}})(x^{-1})\geq(\bigcap_{\mathbf{i}\in\wedge}\tilde{A}_{\mathbf{i}})(x) \tag{9}$$

By inequality (8) and inequality (9)

$$\tilde{A}\cap\tilde{B}\in\mathscr{F}(\mathbf{H}\cap\mathbf{K})$$

□

Corollary 3.1.2. Given $\mathbf{G}$ be any group. $\mathbf{H}$ and $\mathbf{K}$ are subgroups of $\mathbf{G}$. If $\tilde{A}$ and $\tilde{B}$ are fuzzy subgroups of $\mathbf{H}$ and $\mathbf{K}$ respectively, then $(\tilde{A}\cap\tilde{B})$ is a fuzzy subgroup of $\mathbf{H}\cap\mathbf{K}$.

Proof. Since $\mathbf{H}$ and $\mathbf{K}$ are subgroups of $\mathbf{G}$, therefore $\mathbf{H}\cap\mathbf{K}$ is a subgroup of $\mathbf{G}$. Now

$$\tilde{A}\in\mathscr{F}(\mathbf{H})\ and\ \tilde{B}\in\mathscr{F}(\mathbf{K})$$

Let $x,y\in\mathbf{H}\cap\mathbf{K}$ be arbitrary then

$$\begin{aligned}(\tilde{A}\cap\tilde{B})(xy) &= \tilde{A}(xy)\wedge\tilde{B}(xy)\\ &\geq min[min\{\tilde{A}(x),\tilde{A}(y)\},min\{\tilde{B}(x),\tilde{B}(y)\}]\\ &= min[min\{\tilde{A}(x),\tilde{B}(x)\},min\{\tilde{A}(y),\tilde{B}(y)\}]\end{aligned}$$

$$
\begin{aligned}
&= min\{(\tilde{A}\cap\tilde{B})(x),(\tilde{A}\cap\tilde{B})(y)\}\\
&= (\tilde{A}\cap\tilde{B})(x)\wedge(\tilde{A}\cap\tilde{B})(y)
\end{aligned}
$$

Thus for all $x,y\in\mathbf{H}\cap\mathbf{K}$

$$(\tilde{A}\cap\tilde{B})(xy)\geq(\tilde{A}\cap\tilde{B})(x)\wedge(\tilde{A}\cap\tilde{B})(y) \tag{10}$$

Again let $x\in\mathbf{H}\cap\mathbf{K}$ be arbitrary

$$
\begin{aligned}
(\tilde{A}\cap\tilde{B})(x^{-1}) &= \tilde{A}(x^{-1})\wedge\tilde{B}(x^{-1})\\
&\geq min\{\tilde{A}(x),\tilde{B}(x)\}\\
&= (\tilde{A}\cap\tilde{B})(x)
\end{aligned}
$$

Thus for all $x\in\mathbf{H}\cap\mathbf{K}$

$$(\tilde{A}\cap\tilde{B})(x^{-1})\geq(\tilde{A}\cap\tilde{B})(x) \tag{11}$$

By inequality (10) and inequality (11)

$$\tilde{A}\cap\tilde{B}\in\mathscr{F}(\mathbf{H}\cap\mathbf{K})$$

□

Following example is an illustration of the Cor. 3.1.2

Example 3.1.3. $\mathbf{Q}_8$ represents a group of quaternions. $\mathbf{H}=\{1,-1,i,-i\}$ and $\mathbf{K}=\{1,-1,j,-j\}$ are the subgroups of $\mathbf{Q}_8$. $\tilde{A}\in\mathscr{F}(\mathbf{H})$ and $\tilde{B}\in\mathscr{F}(\mathbf{K})$ such that

$$\tilde{A}(x)=\begin{cases} .9 & \text{if x = 1}\\ .65 & \text{else}\end{cases}$$

and

$$\tilde{B}(x)=\begin{cases} .76 & \text{if x = 1}\\ .68 & \text{else}\end{cases}$$

Now $(\tilde{A}\cap\tilde{B})$ defined on $(H\cap K)$ such that

$$(\tilde{A}\cap\tilde{B})(x)=\begin{cases} .76 & \text{if x= 1} \\ .65 & \text{if x= -1} \end{cases}$$

which gives $\tilde{A}\cap\tilde{B}\in\mathscr{F}(\mathbf{H}\cap\mathbf{K})$.

3.2. Fuzzy Normal Subgroup.

Theorem 3.2.1. Let $\mathbf{G}$ be any group. $\{\mathbf{H_i} : \mathbf{i}\in\wedge\}$ be an arbitrary collection of subgroups of $\mathbf{G}$. If $\tilde{N}_{\mathbf{i}}$ is a fuzzy normal subgroup of $\mathbf{H_i}$ for every $\mathbf{i}\in\wedge$, then $\bigcap_{\mathbf{i}\in\wedge}\tilde{N}_{\mathbf{i}}$ is a fuzzy normal subgroup of $\bigcap_{\mathbf{i}\in\wedge}\mathbf{H_i}$.

Proof. For any $\mathbf{i}\in\wedge$, $\tilde{N}_{\mathbf{i}}$ is a fuzzy normal subgroup of $\mathbf{H_i}$.
So $\tilde{N}_{\mathbf{i}}$ is a fuzzy subgroup of $\mathbf{H_i}$ for all $\mathbf{i}\in\wedge$, then by Theorem 3.1.1 $\bigcap_{\mathbf{i}\in\wedge}\tilde{N}_{\mathbf{i}}$ is a fuzzy subgroup of $\bigcap_{\mathbf{i}\in\wedge}\mathbf{H_i}$
Now we can easily verify that

$$(\bigcap_{\mathbf{i}\in\wedge}\tilde{N}_{\mathbf{i}})(xyx^{-1})\geq(\bigcap_{\mathbf{i}\in\wedge}\tilde{N}_{\mathbf{i}})(y)\;\forall\, x,y\in\bigcap_{\mathbf{i}\in\wedge}\mathbf{H_i}$$

Hence $\bigcap_{\mathbf{i}\in\wedge}\tilde{N}_{\mathbf{i}}$ is a fuzzy normal subgroup of $\bigcap_{\mathbf{i}\in\wedge}\mathbf{H_i}$. □

Corollary 3.2.2. Let $\mathbf{G}$ be any group. $\mathbf{H}$ and $\mathbf{K}$ are a subgroup of $\mathbf{G}$. If $\tilde{A}$ and $\tilde{B}$ are the fuzzy normal subgroup of $\mathbf{H}$ and $\mathbf{K}$ respectively, then $\tilde{A}\cap\tilde{B}$ is a fuzzy normal subgroup of $\mathbf{H}\cap\mathbf{K}$.

Proof. Since $\tilde{A}$ and $\tilde{B}$ are fuzzy normal subgroups of $\mathbf{H}$ and $\mathbf{K}$, respectively, so $\tilde{A}$ and $\tilde{B}$ are fuzzy subgroups of $\mathbf{H}$ and $\mathbf{K}$ respectively, by Theorem 3.1.2 $\tilde{A}\cap\tilde{B}$ is a fuzzy subgroup of $\mathbf{H}\cap\mathbf{K}$
Now we can easily verify that

$$(\tilde{A}\cap\tilde{B})(xyx^{-1})\geq(\tilde{A}\cap\tilde{B})(y)\;\forall\, x,y\in\mathbf{H}\cap\mathbf{K}$$

Hence $\tilde{A}\cap\tilde{B}$ is a fuzzy normal subgroup of $\mathbf{H}\cap\mathbf{K}$. □

3.3. Complex Fuzzy Subgroup.

Theorem 3.3.1. If $\mathbf{G}$ be any group. $\{\mathbf{H_i} : \mathbf{i} \in \wedge\}$ be an arbitrary collection of subgroups of $\mathbf{G}$. If $\tilde{C}_\mathbf{i}$ is a complex fuzzy subgroup of $\mathbf{H_i}$ for every $\mathbf{i} \in \wedge$, then $\bigcap_{\mathbf{i}\in\wedge} \tilde{C}_\mathbf{i}$ is a complex fuzzy subgroup of $\bigcap_{\mathbf{i}\in\wedge} \mathbf{H_i}$.

Proof. By Theorem 3.1.1 $\bigcap_{\mathbf{i}\in\wedge} \tilde{C}_\mathbf{i}$ is a fuzzy subgroup of $\bigcap_{\mathbf{i}\in\wedge} \mathbf{H_i}$ and by Def. 2.13 $\bigcap_{\mathbf{i}\in\wedge} \tilde{C}_\mathbf{i}$ is a complex fuzzy set. Thus $\bigcap_{\mathbf{i}\in\wedge} \tilde{C}_\mathbf{i}$ is a complex fuzzy subgroup of $\bigcap_{\mathbf{i}\in\wedge} \mathbf{H_i}$. □

Corollary 3.3.2. Let $\mathbf{G}$ be any group. $\mathbf{H}$ and $\mathbf{K}$ are subgroup of $\mathbf{G}$. If $\tilde{A}$ and $\tilde{B}$ are complex fuzzy subgroups of $\mathbf{H}$ and $\mathbf{K}$ respectively, then $\tilde{A} \cap \tilde{B}$ is a complex fuzzy subgroup of $\mathbf{H} \cap \mathbf{K}$.

Proof. Since $\tilde{A}$ and $\tilde{B}$ are fuzzy subgroups of $\mathbf{H}$ and $\mathbf{K}$ respectively. By Cor. 3.1.2 $\tilde{A} \cap \tilde{B}$ is a fuzzy subgroup of $\mathbf{H} \cap \tilde{K}$.
Again $\tilde{A}$ and $\tilde{B}$ are complex fuzzy sets of $\mathbf{H}$ and $\mathbf{K}$ respectively. By Def. 2.13 $\tilde{A} \cap \tilde{B}$ is a complex fuzzy set of $\mathbf{H} \cap \tilde{K}$.
$\tilde{A} \cap \tilde{B}$ is a complex fuzzy set which is a fuzzy subgroup as well.
Hence $\tilde{A} \cap \tilde{B}$ is a complex fuzzy subgroup of $\mathbf{H} \cap \mathbf{K}$.

□

4. Application In Fuzzy Relation

It is clear from Def.2.4, a fuzzy relation is a fuzzy set defined on the cartesian product of sets. The intersection of two different relations defining on the same set can be determined by using Def.2.2, but our Def.3.1 gives us the freedom to find the intersection of fuzzy relation defining on different sets.

Example 4.1. Define $\mathbf{X}$ and $\mathbf{Y}$ as in example 2.5 and $\tilde{R}$ is fuzzy relation(1) on $\mathbf{X} \times \mathbf{Y}$. Let $\mathbf{Z} = \{Rajnandgaon,\ Raigarh,\ Bastar,\ Durg,\ Bijapur,\ Raipur\}$ and relation $\mathbf{Q}$ on $\mathbf{Z} \times \mathbf{Y}$ is stated as follows

$\mathbf{Q}$: area of z is smaller than y.

In Table-2 $d_1(z,y) = 12383 - z$ and negative sign shows the area of z is lesser than y and positive sign implies the area of z is greater than y, here $z \in \mathbf{Z} and y \in \mathbf{Y}$. define $\mu_{\tilde{Q}} : \mathbf{Z} \times \mathbf{Y} \to [0,1]$ such that

Z	Rajnandgaon	Raigarh	Bastar	Durg	Bijapur	Raipur	Surguja
Area of z (in km^2)	8070	7086	10470	8535	8530	12383	15732
$d_1(z,y)$	-4313	-5297	-1913	-3848	-3853	0	3349

TABLE 2. Area of z and $d_1(z,y)$

(12)
$$\mu_{\tilde{Q}}(z,y) = \begin{cases} -\frac{d_1(z,y)}{5300} & \text{if } d_1(z,y) \leq 0 \\ 0 & \text{else} \end{cases}$$

Here $\mathbf{Z} \times \mathbf{Y} \sim \mathbf{Z}$ therefore we write z instead of (z,y), so relation $\tilde{Q}$ can be written as

$$\tilde{Q} = \frac{0.8138}{Rajnandgaon} + \frac{.09994}{Raigarh} + \frac{0.3609}{Bastar} + \frac{.7260}{Durg} + \frac{0.7270}{Bijapur} + \frac{0}{Raipur} + \frac{0}{Surguja}$$

In fig.2, the peak value holds for Raigarh, which means the area of Raigarh is significantly smaller than the area of Raipur. moreover, the area of Raigarh is the smallest in our collection.

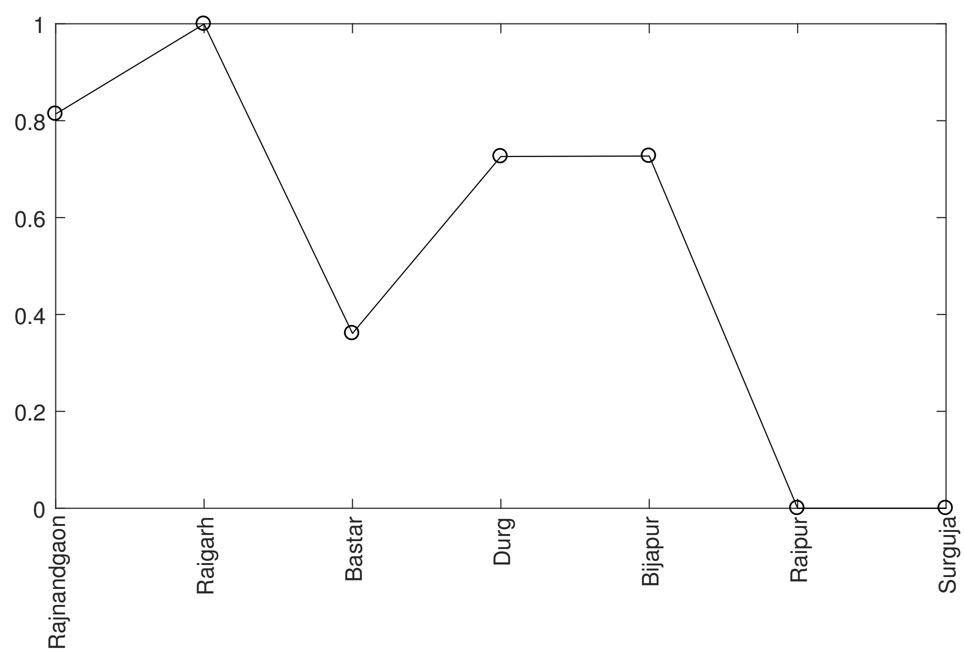

FIGURE 2. Fuzzy Relation $\tilde{Q}$ on $\mathbf{Z} \times \mathbf{Y}$

Now recall collection $\mathbf{X}$ from example2.5. We get

$\mathbf{X}\cap\mathbf{Z} = \{Rajnandgaon, Durg, Raipur\}$ and relation $\mathbf{R}\cap\mathbf{Q}$ defined on $(\mathbf{X}\times\mathbf{Y})\cap(\mathbf{Z}\cap\mathbf{Y})$ and stated as follows

$\mathbf{R}\cap\mathbf{Q}$: $x\ is\ far\ from\ y\ and$

$area\ of\ x\ is\ smaller\ than\ area\ of\ y$

membership fuction $\mu_{\tilde{R}\cap\tilde{Q}}$: $(\mathbf{X}\times\mathbf{Y})\cap(\mathbf{Z}\times\mathbf{Y})\rightarrow[0,1]$ and for all $(x,y)\in(\mathbf{X}\times\mathbf{Y})\cap(\mathbf{Z}\cap\mathbf{Y})$, defined as

$$\mu_{\tilde{R}\cap\tilde{Q}}(x,y) = \min\{\mu_{\tilde{R}}(x,y), \mu_{\tilde{Q}}(x,y)\} \tag{13}$$

Here $(\mathbf{X}\times\mathbf{Y})\cap(\mathbf{Z}\cap\mathbf{Y})\sim\mathbf{X}\cap\mathbf{Z}$ thereffore we write x instead of (x,y), So relation $\tilde{R}\cap\tilde{Q}$ can be written as

$$\tilde{R}\cap\tilde{Q} = \frac{0.131}{Rajnandgaon} + \frac{0.1037}{Durg} + \frac{0}{Raipur}$$

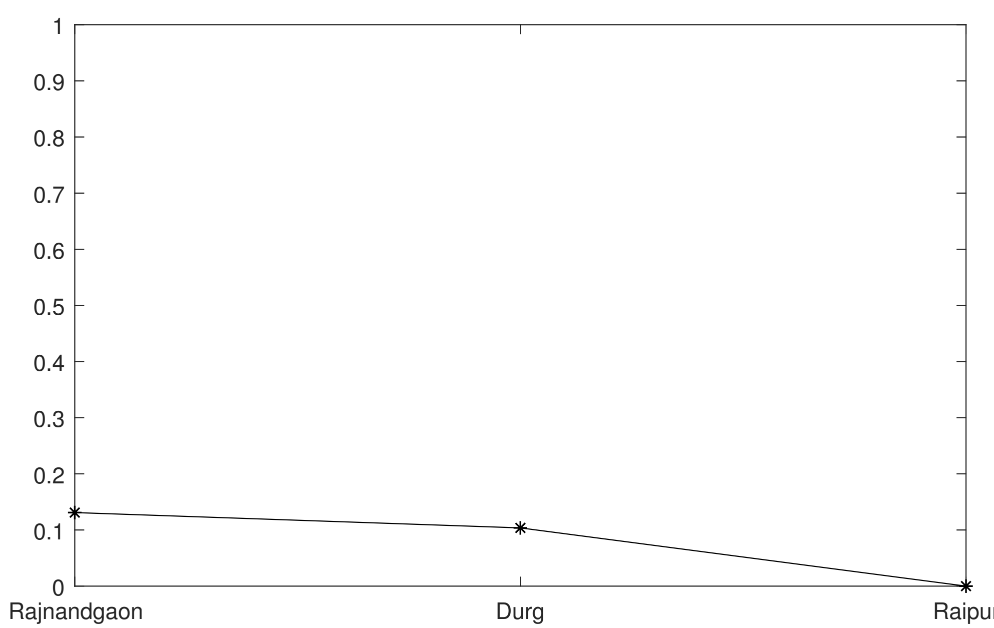

FIGURE 3. Fuzzy Relation $\tilde{R}\cap\tilde{Q}$ on $(\mathbf{X}\times\mathbf{Y})\cap(\mathbf{Z}\cap\mathbf{Y})$

membership value for relation $\tilde{R} \cap \tilde{Q}$ can give an idea about at least one relation, In fig.3 membership value of Rajnadgaon is 0.131, which means relation **R** and relation **Q** are not false for Rajnandgaon but in the case of Raipur membership value is 0, which means at least one of the relation must be false, it can be **R**, **Q** or both, in this case, both the relation is false for Raipur.

5. Conclusion

We can say that our Def. 3.1 is valid for fuzzy subgroup and fuzzy relation. It can be used for sentences in which the decision depends on more than one component. We can make a decision by creating a fuzzy set for each component and finding the intersection of all.

We have applied Def. 3.1 for $\alpha-$ fuzzy subgroups (see [16]).

Conflict of Interests

The authors declare that there is no conflict of interests.

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