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SYMMETRIC DERIVATIONS ON KÄHLER MODULES

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Abstract. In this paper, we define generalized symmetric derivations on Kähler modules. We give the relationships between the projective dimensions of $\Omega^{(1)}(R/k)$ and $\Omega^{(2)}(R/k)$ by using the symmetric derivation. We then give some interesting results related to Kähler modules and symmetric derivations.

Keywords: Kähler module; symmetric derivation; regular ring.

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1. Introduction

The concept of a Kähler module of q th order was introduced by H.Osborn in 1965 [10]. Same notion has appeared in R.G. Heyneman and M.E. Sweedler [12]. They introduced differential operators on a commutative algebra and extended the notion of derivations. J. Johnson [3] introduced differential module structures on certain modules of Kähler differentials. J. Lipman [16] studied the Jacobian ideal of the module of differentials. Y. Nakai [7] developed the fundamental theories for the calculus of high order derivations and some functorial properties of the module of high order differentials in his paper. M.E. Sweedler [13] introduced right differential operators on a noncommutative algebra, and extended the notion of right derivations. Komatsu [4] introduced right differential operators on a noncommutative ring extension. In [8] author

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characterized the homological dimension of n-th order Kähler differentials of R over k, and examined functorial properties of the module of n-th order Kähler differentials of $R\bigotimes S$ over k in [9]. Then many authors studied the properties of Kähler modules [2,5,14,15]. The purpose of this paper is to introduce generalized (high order) symmetric derivation on high order Kähler modules which have not been considered before. We will construct the generalized symmetric derivation on high order Kähler modules of commutative ring extension R/k and show their fundamental properties. We give the relationships between the projective dimensions of $\Omega^{(1)}(R/k)$ and $\Omega^{(2)}(R/k)$ by using the symmetric derivation. In particular the relation between high order Kähler modules and the regularity of commutative rings was not known.

Throughout this paper we will let R be a commutative algebra over an algebraically closed field k with characteristic zero. When R is a k-algebra,$J_n(R/k)$ or $J_n(R)$ denotes the universal module of n-th order differentials of R over k and $\Omega^{(q)}(R/k)$ denotes the module of q-th order Kähler differentials of R over k and $\delta^{(q)}_{R/k}$ or $\delta^{(q)}$ denotes the canonical q-th order k-derivation $R\to\Omega^{(q)}(R/k)$ of R. The pair $\{\Omega^{(q)}(R/k),\delta^{(q)}_{R/k}\}$ has the universal mapping property with respect to the q-th order k-derivations of R. $I_{R/k}$ or I_R denotes the kernel of the canonical mapping $R\otimes_k R\to R$ $(a\otimes b\to ab)$. $\Omega^{(q)}(R/k)$ is identified with I_R/I_R^{q+1}.

It is well known that $J_n(R)\cong\Omega^{(n)}(R)\oplus R$.

$\Omega^{(q)}(R/k)$ is generated by the set $\{\delta^{(q)}(r):\quad r\in R\}$. Hence if R is a finitely generated k-algebra,then $\Omega^{(q)}(R/k)$ will be a finitely generated R-module.

2. Preliminaries

In this section, we give some basic definitions and results about the symmetric derivation and Kähler modules of differentials.

Definition 2.1. *([10]) Let R be any k-algebra (commutative with unit), $R\to\Omega^{(q)}(R/k)$ be q-th order Kähler derivation of R and let $S(\Omega^{(1)}(R/k))$ be the symmetric algebra $\bigoplus_{p\geq 0}S^p(\Omega^{(1)}(R/k))$ generated over R by $\Omega^{(1)}(R/k)$.*

A symmetric derivation is any linear map D of $S(\Omega^{(1)}(R/k))$ into itself such that

i) $D(S^p(\Omega^{(1)}(R/k)))\subset S^{p+1}(\Omega^{(1)}(R/k))$

ii) D is a first order derivation over k and *iii) the restriction of D to R ($R \simeq S^0(\Omega^{(1)}(R/k))$) is the Kähler derivation $\delta^{(1)} : R \to \Omega^{(1)}(R/k)$*

Theorem 2.2. *([10]) Let R be an affine k-algebra. Then there exists a short exact sequence of R modules*

$$0 \to Ker\theta \to \Omega^{(2)}(R/k) \xrightarrow{\theta} \Omega^{(1)}(R/k) \to 0$$

such that $\theta(\delta^{(2)}(f)) = \delta^{(1)}(f)$ for all $f \in R$ and $Ker\theta$ is generated by the set $\{\delta^{(2)}(ab) - a\delta^{(2)}(b) - b\delta^{(2)}(a)\}$ for all $a, b \in R$.

Proposition 2.3. *([10]) $S^2(\Omega^{(1)}(R/k)) \simeq Ker\theta$*

Proof. :$\delta^{(1)}(a).\delta^{(1)}(b) = (1 \otimes a - a \otimes 1).(1 \otimes b - b \otimes 1)$

$= 1 \otimes ab - b \otimes a - a \otimes b + ab \otimes 1$

$= (1 \otimes ab - ab \otimes 1) - a(1 \otimes b - b \otimes 1) - b(1 \otimes a - a \otimes 1)$

$= \delta^{(2)}(ab) - a\delta^{(2)}(b) - b\delta^{(2)}(a)$ Then we have $S^2(\Omega^{(1)}(R/k)) \simeq Ker\theta$ as required.

□

Theorem 2.4. *([11]) Let R be an affine k-algebra. If R is a regular ring, then $\Omega^{(q)}(R/k)$ is a projective R-module.*

Theorem 2.5. *([6]) Let R be an affine k-algebra. R is regular ring if and only if $\Omega^{(1)}(R/k)$ is a projective R-module.*

3. Main Results

Definition 3.1. *Let R be any k-algebra (commutative with unit), $R \to \Omega^{(q)}(R/k)$ be q-th order Kähler derivation of R and let $S(\Omega^{(q)}(R/k))$ be the symmetric algebra $\bigoplus_{p\geq 0} S^p(\Omega^{(q)}(R/k))$ generated over R by $\Omega^{(q)}(R/k)$.*

A generalized symmetric derivation is any k-linear map D of $S(\Omega^{(q)}(R/k))$ into itself such that

i) $D(S^p(\Omega^{(q)}(R/k))) \subset S^{p+1}(\Omega^{(q)}(R/k))$

ii) D is a q-th order derivation over k and

iii) the restriction of D to R ($R \simeq S^0(\Omega^{(q)}(R/k))$) is the Kähler derivation $\delta^{(q)} : R \to \Omega^{(q)}(R/k)$

Example 3.2. *Let $R = k[x_1, ..., x_s]$ be a polynomial algebra of dimension s. Then $\Omega^{(q)}(R/k)$ is a free R-module of rank* $\binom{q+s}{s} - 1$ *with basis* $\{\delta^{(q)}(x_1^{i_1} .....x_s^{i_s}) : i_1 + ... + i_s \leq q\}$

$S^2(\Omega^{(q)}(R/k))$ *is a free R-module of rank* $\binom{t+1}{t-1}$

where $t = \binom{q+s}{s} - 1$ *with basis* $\{\delta^{(q)}(x_1^{i_1} .....x_s^{i_s}) \otimes \delta^{(q)}(x_1^{i_1} .....x_s^{i_s}) : i_1 + ... + i_s \leq q\}$

Theorem 3.3. *Let R be an affine k-algebra. Then there exists a long exact sequence of R modules*

$$0 \to Ker\theta \to \Omega^{(2q)}(R/k) \xrightarrow{\theta} J_q(\Omega^{(q)}(R/k)) \to Coker\theta \to 0$$

for all $q \geq 0$

Proof. Let R be any k-algebra, $\Omega^{(q)}(R/k)$ be q-th order Kähler derivation of R. Let $J_q(\Omega^{(q)}(R/k))$ be q-th order universal module of differential operators of order less than or equal to q on $\Omega^{(q)}(R/k)$ with the universal differential operator $\Delta_q : \Omega^{(q)}(R/k) \to J_q(\Omega^{(q)}(R/k))$.

By the universal mapping property of $\Omega^{(2q)}(R/k)$ there exists a unique R-module homomorphism $\theta : \Omega^{(2q)}(R/k) \to J_q(\Omega^{(q)}(R/k))$ such that $\theta\delta^{2q} = \Delta_q\delta^q$ and the following diagram commutes.

$$\begin{array}{ccc} R & \to^{\delta^q} & \Omega^{(q)}(R/k) \\ \downarrow \delta^{2q} & & \downarrow \Delta_q \\ \Omega^{(2q)}(R/k) & \to^{\theta} & J_q(\Omega^{(q)}(R/k)) \end{array}$$

By the homological properties we obtain the sequence as required. □

Corollary 3.4. *If R is any regular k-algebra with dimension s, then θ is injective.*

Proof. We define $\theta(\delta^{2q}(x^\alpha)) = \Delta_q(x^\beta\delta^q(x^\gamma))$

where

$x^\alpha = x_1^{i_1} .....x_s^{i_s} : i_1 + ... + i_s \leq 2q$

$x^\beta = x_1^{i_1} .....x_s^{i_s} : i_1 + ... + i_s \leq q$

$x^\gamma = x_1^{i_1} .....x_s^{i_s} : i_1 + ... + i_s \leq q$ □

Corollary 3.5 (Erdogan)**.** *If R is any regular k-algebra with dimension 1 and $q=1$, then $\Omega^{(2)}(R/k))$ is isomorphic to $J_1(\Omega^{(1)}(R/k))$.*

Proof. Rank of the module $\Omega^{(2)}(R/k))$ is $t = \binom{2+s}{s} - 1$ and it is equal to rank of $J_1(\Omega^{(1)}(R/k))$.

But this is not true in other cases ($q>1$).[see 2]

□

Example 3.6. *Let $R=k[x,y]$ be a polynomial algebra of dimension 2. Then $\Omega^{(1)}(R/k)$ is a free R-module of rank 2 with basis $\{\delta^{(1)}(x), \delta^{(1)}(y)\}$*

$\Omega^{(2)}(R/k)$ is a free R-module of rank 5 with basis

$\{\delta^{(2)}(x), \delta^{(2)}(y), \delta^{(2)}(x^2), \delta^{(2)}(xy), \delta^{(2)}(y^2)\}$

$J_1(\Omega^{(1)}(R/k))$ is a free R-module of rank 6 with basis

$\{\Delta_1(\delta^{(1)}(x)), \Delta_1(\delta^{(1)}(y)), \Delta_1(x\delta^{(1)}(x)), \Delta_1(x\delta^{(1)}(y)), \Delta_1(y\delta^{(1)}(x)), \Delta_1(y\delta^{(1)}(y))\}$

$J_2(\Omega^{(2)}(R/k))$ is a free R-module of rank 30 with basis

$\{\Delta_2(\delta^{(2)}(x)), \Delta_2(\delta^{(2)}(y)), \Delta_2(\delta^{(2)}(x^2)), \Delta_2(\delta^{(2)}(xy)), \Delta_2(\delta^{(2)}(y^2)),$

$\Delta_2(x\delta^{(2)}(x)), \Delta_2(x\delta^{(2)}(y)), \Delta_2(x\delta^{(2)}(x^2)), \Delta_2(x\delta^{(2)}(xy)), \Delta_2(x\delta^{(2)}(y^2)),$

$\Delta_2(y\delta^{(2)}(x)), \Delta_2(y\delta^{(2)}(y)), \Delta_2(y\delta^{(2)}(x^2)), \Delta_2(y\delta^{(2)}(xy)), \Delta_2(y\delta^{(2)}(y^2)),$

$\Delta_2(x^2\delta^{(2)}(x)), \Delta_2(x^2\delta^{(2)}(y)), \Delta_2(x^2\delta^{(2)}(x^2)), \Delta_2(x^2\delta^{(2)}(xy)), \Delta_2(x^2\delta^{(2)}(y^2)),$

$\Delta_2(xy\delta^{(2)}(x)), \Delta_2(xy\delta^{(2)}(y)), \Delta_2(xy\delta^{(2)}(x^2)), \Delta_2(xy\delta^{(2)}(xy)), \Delta_2(xy\delta^{(2)}(y^2)),$

$\Delta_2(y^2\delta^{(2)}(x)), \Delta_2(y^2\delta^{(2)}(y)), \Delta_2(y^2\delta^{(2)}(x^2)), \Delta_2(y^2\delta^{(2)}(xy)), \Delta_2(y^2\delta^{(2)}(y^2)),\}$

$S^2(\Omega^{(1)}(R/k))$ is a free R-module of rank 3 with basis $\{\delta^{(1)}(x)\otimes\delta^{1}(x), \delta^{(1)}(x)\otimes\delta^{(1)}(y), \delta^{(1)}(y)\otimes\delta^{(1)}(y)\}$.

In this example $\Omega^{(2)}(R/k))$ is not isomorphic to $J_1(\Omega^{(1)}(R/k))$ and we obtain the exact sequence

$$0 \to S^2(\Omega^{(1)}(R/k)) \to \Omega^{(2)}(R/k) \to \Omega^{(1)}(R/k) \to 0$$

of R modules.

Theorem 3.7. *Let R be an affine k-algebra. Then there exists a long exact sequence of R modules*

$$0 \to Ker\beta \to J_q(\Omega^{(q)}(R/k)) \xrightarrow{\beta} S^2(\Omega^{(q)}(R/k)) \to Coker\beta \to 0$$

for all $q \geq 0$

Proof: Let $D_q : \Omega^{(q)}(R/k)) \to S^2(\Omega^{(q)}(R/k))$ be any generalized symmetric derivation and let $J_q(\Omega^{(q)}(R/k))$ be q-th order universal module of differential operators of order less than or equal to q on $\Omega^{(q)}(R/k)$ with the universal differential operator $\Delta_q : \Omega^{(q)}(R/k) \to J_q(\Omega^{(q)}(R/k))$.

By the universal mapping property of $J_q(\Omega^{(q)}(R/k))$ there exists a unique R-module homomorphism $\beta : J_q(\Omega^{(q)}(R/k)) \to S^2(\Omega^{(q)}(R/k))$ such that the following diagram commutes.

$$\begin{array}{ccc} \Omega^{(q)}(R/k) & \to^{D_q} & S^2(\Omega^{(q)}(R/k)) \\ \downarrow \Delta_q & & \downarrow \\ J_q(\Omega^{(q)}(R/k)) & \to^{\beta} & S^2(\Omega^{(q)}(R/k)) \end{array}$$

By the homological properties we have the sequence as required.

Lemma 3.8. *Let R be an affine domain with dimension s. Then $\Omega^{(q)}(R/k)$ is a free R-module if and only if $S^2(\Omega^{(q)}(R/k))$ is a free R-module.*

Proof. Without loss of generality we may assume that R is local domain of dimension s. Suppose that $\Omega^{(q)}(R/k)$ is free R-module. From the property of symmetric algebra $S^2(\Omega^{(q)}(R/k))$ is a free R-module.

Conversly, suppose that $S^2(\Omega^{(q)}(R/k))$ is a free R-module. If $dimR = s$ then the rank of $\Omega^{(q)}(R/k)$ is $\left(\begin{array}{c} q+s \\ s \end{array} \right) - 1$. Let $\left(\begin{array}{c} q+s \\ s \end{array} \right) - 1 = t$. Then the rank of $S^2(\Omega^{(q)}(R/k))$ is $\left(\begin{array}{c} t+1 \\ t-1 \end{array} \right)$.

Let m be the maximal ideal of R. Then $S^2(\Omega^{(q)}(R/k)) \otimes_R R/m$ is an R/m vector space of dimension $\left(\begin{array}{c} t+1 \\ t-1 \end{array} \right)$. $S^2(\Omega^{(q)}(R/k)) \otimes_R R/m$ is isomorphic to $S^2(\frac{\Omega^{(q)}(R/k)}{m\Omega^{(q)}(R/k)})$. Then $S^2(\frac{\Omega^{(q)}(R/k)}{m\Omega^{(q)}(R/k)})$is an R/m vector space of dimension $\left(\begin{array}{c} t+1 \\ t-1 \end{array} \right)$ if and only if $\frac{\Omega^{(q)}(R/k)}{m\Omega^{(q)}(R/k)}$ is an R/m vector space of dimension t. Hence $\frac{\Omega^{(q)}(R/k)}{m\Omega^{(q)}(R/k)}$ is an R/m vector space of dimension t if and only if the number of minimal generators of $\Omega^{(q)}(R/k)$ is t. As the rank of $\Omega^{(q)}(R/k)$ is t, it is obtained that $\Omega^{(q)}(R/k)$ is a free R-module as required. $\square$

Theorem 3.9. *Let R be an affine k-algebra and let $S(\Omega^{(1)}(R/k))$ have at least one symmetric derivation. $\Omega^{(1)}(R/k)$ is a projective R-module if and only if $\Omega^{(2)}(R/k)$ is a projective R-module.*

Proof. Suppose that $\Omega^{(1)}(R/k)$ is a projective R-module. By Theorem 2.5 R is a regular ring and $\Omega^{(2)}(R/k)$ is a projective R-module by Theorem 2.4.

Conversely, From by Proposition 2.3. and Theorem 2.2 we have the exact sequence

$$0 \to S^2(\Omega^{(1)}(R/k)) \to \Omega^{(2)}(R/k) \to \Omega^{(1)}(R/k) \to 0$$

of R-modules. This sequence is a split exact sequence. It is sufficient to prove the splitting of the sequence . It follows from the definition of the symmetric derivation any symmetric derivation D is uniquely determined by its restriction $D_1 : \Omega^{(1)}(R/k) \to S^2(\Omega^{(1)}(R/k))$, and it is observed that the composition $D_1\delta^{(1)}$

$$R \to \Omega^{(1)}(R/k) \to S^2(\Omega^{(1)}(R/k))$$

is a second order derivation of R. By the universal mapping property of $\{\Omega^{(2)}(R/k), \delta^{(2)}\}$ there is a unique R module homomorphism $t : \Omega^{(2)}(R/k) \to S^2(\Omega^{(1)}(R/k))$ such that $t\delta^{(2)} = D_1\delta^{(1)}$

$$t(\delta^{(2)}(ab) - a\delta^{(2)}(b) - b\delta^{(2)}(a)) = t\delta^{(2)}(ab) - at\delta^{(2)}(b) - bt\delta^{(2)}(a)$$

$$= D_1\delta^{(1)}(ab) - aD_1\delta^{(1)}(b) - bD_1\delta^{(1)}(a)$$

$$= D_1(a\delta^{(1)}(b) + b\delta^{(1)}(a)) - aD_1\delta^{(1)}(b) - bD_1\delta^{(1)}(a)$$

$$= aD_1\delta^{(1)}(b) + \delta^1(a)\delta^{(1)}(b) + bD_1\delta^{(1)}(a)) + \delta^1(b)\delta^{(1)}(a) - aD_1\delta^{(1)}(b) - bD_1\delta^{(1)}(a)$$

$$= 2\delta^{(1)}(a).\delta^{(1)}(b)$$

It follows that $(1/2)t$ splits as required.

Thus $\Omega^{(2)}(R/k)$ is isomorphic to $\Omega^{(1)}(R/k) \bigoplus S^2(\Omega^{(1)}(R/k))$. Therefore $\Omega^{(1)}(R/k)$ is a projective R-module. □

For an affine local k algebra we give the following result.

Corollary 3.10. *Let R be an affine local k-algebra and let $S(\Omega^{(1)}(R/k))$ have at least one symmetric derivation.$\Omega^{(1)}(R/k)$ is a free R module if and only if $\Omega^{(2)}(R/k)$ is a free R module.*

Theorem 3.11. *Let R be an affine k-algebra and let $S(\Omega^{(1)}(R/k))$ have at least one symmetric derivation. R is a regular ring if and only if $\Omega^{(2)}(R/k)$ is a projective R module.*

Proof. This follows from by Theorem 2.4., Theorem 2.5. and Theorem 3.9. □

Corollary 3.12. *Let R be an affine k-algebra and let $S(\Omega^{(1)}(R/k))$ have at least one symmetric derivation. R is a regular local ring if and only if $\Omega^{(2)}(R/k)$ is a free R module.*

Now, we will use the split exact sequence

$$0 \to S^2(\Omega^{(1)}(R/k)) \to \Omega^{(2)}(R/k) \to \Omega^{(1)}(R/k) \to 0$$

of R-modules in the proof of Theorem 3.9. and homological properties to obtain the following important results related to projective dimension of Kähler Modules.

Corollary 3.13. *Let R be an affine k-algebra and let $S(\Omega^{(1)}(R/k))$ have at least one symmetric derivation. If the projective dimension of $\Omega^{(2)}(R/k)$ is finite then the projective dimension of $\Omega^{(1)}(R/k)$ is finite.*

Proof. $\Omega^{(2)}(R/k)$ is isomorphic to $\Omega^{(1)}(R/k) \bigoplus S^2(\Omega^{(1)}(R/k))$ from the split exact sequence

$$0 \to S^2(\Omega^{(1)}(R/k)) \to \Omega^{(2)}(R/k) \to \Omega^{(1)}(R/k) \to 0$$

of R-modules. □

Corollary 3.14. *Let R be an affine k-algebra and let $S(\Omega^{(1)}(R/k))$ have at least one symmetric derivation. If the projective dimension of $\Omega^{(1)}(R/k)$ is infinite then the projective dimension of $\Omega^{(2)}(R/k)$ is infinite.*

Example 3.15. *Let S be the coordinate ring of the cups $y^2 = x^3$. Then $S = k[x,y]/(f)$ where $f = y^2 - x^3$. We can find the projective dimension of $\Omega^{(1)}(S/k)$, $\Omega^{(2)}(S/k)$ and $J_{(1)}(\Omega^{(1)}(S/k))$*

$\Omega^{(1)}(S/k) \simeq F/N$ where F is a free S module on $\{\delta^1(x), \delta^1(y)\}$ and N is a submodule of F generated by $\delta^1(f) = 2y\delta^1(y) - 3x^2\delta^1(x)$. Certainly N is a free on $\delta^1(f)$. Therefore we have

$$0 \longrightarrow N \xrightarrow{\phi} F \xrightarrow{\pi} \Omega^{(1)}(S/k) \simeq F/N \longrightarrow 0$$

a free resolution of $\Omega^{(1)}(S/k)$. *In this sequence the homomorphism* ϕ *is a matrix* $\begin{pmatrix} -3x^2 \\ 2y \end{pmatrix}$ *and projective dimension of* $\Omega^{(1)}(S/k)$ *is less than or equal to one.*

By the same argument $\Omega^{(2)}(S/k) \simeq F'/N'$ *where* F' *is a free* S *module on*

$$\{\delta^2(x), \delta^2(y), \delta^2(xy), \delta^2(x^2), \delta^2(y^2)\}$$

and N' *is a submodule of* F' *generated by* $\{\delta^2(f), \delta^2(xf), \delta^2(yf)\}$ *where*

$\delta^2(f) = \delta^2(y^2) - 3x\delta^2(x^2) + 3x^2\delta^2(x)$

$\delta^2(xf) = x\delta^2(y^2) - 6x^2\delta^2(x^2) + 2y\delta^2(xy) + 7x^3\delta^2(x) - 2xy\delta^2(y)$

$\delta^2(yf) = 3y\delta^2(y^2) - 3xy\delta^2(x^2) - 3x^2\delta^2(xy) + 6x^2y\delta^2(x) - y^2\delta^2(y)$

Since $rank\Omega^{(2)}(S/k) = 2$ *we have* $rankN' = rankF' - rank\Omega^{(2)}(S/k) = 5 - 2 = 3$. *So* N' *is free S-module. Therefore we have*

$$0 \longrightarrow N' \xrightarrow{\phi} F' \xrightarrow{\pi} \Omega^{(2)}(S/k) \simeq F'/N' \longrightarrow 0$$

a free resolution of $\Omega^{(2)}(S/k)$. *Here* π *is the natural surjection and* ϕ *is given by the following matrix*

$$\begin{pmatrix} -3x & 1 & 0 & 3x^2 & 0 \\ -6x^2 & x & 2y & 7x^3 & -2xy \\ -3xy & 3y & -3x^2 & 6x^2y & -y^2 \end{pmatrix}$$

and projective dimension of $\Omega^{(2)}(S/k)$ *is less than or equal to one.*

Similarly $J_1(\Omega^{(1)}(S/k)) \simeq F''/N''$ *where* F'' *is a free S-module with basis* $\{\Delta_1(\delta^{(1)}(x)), \Delta_1(\delta^{(1)}(y)), \Delta_1(x\delta^{(1)}(x)), \Delta_1(x\delta^{(1)}(y)), \Delta_1(y\delta^{(1)}(x))\}$ *and* N'' *is a submodule of* F'' *generated by*

$a = 3x^2\Delta_1(x\delta^{(1)}(x)) - 2y\Delta_1(x\delta^{(1)}(y)) - 3x^3\Delta_1(\delta^{(1)}(x)) + 2xy\Delta_1(\delta^{(1)}(y))$

$b = 3x^2\Delta_1(x\delta^{(1)}(x)) - 2y\Delta_1(y\delta^{(1)}(x)) - x^3\Delta_1(\delta^{(1)}(x))$

$c = -6xy\Delta_1(x\delta^{(1)}(x)) + 3x^2\Delta_1(x\delta^{(1)}(y)) + 3x^2y\Delta_1(\delta^{(1)}(x)) + x^3\Delta_1(\delta^{(1)}(y))$

Since $rankJ_1(\Omega^{(1)}(S/k)) = 2$ *we have* $rankN'' = rankF'' - rankJ_1(\Omega^{(1)}(S/k)) = 5-2 = 3$. *So* N'' *is a free S-module of rank 3. Therefore*

$$0 \longrightarrow N'' \longrightarrow F'' \xrightarrow{\pi} J_1(\Omega^{(1)}(S/k)) \simeq F''/N'' \longrightarrow 0$$

is a free resolution of $J_1(\Omega^{(1)}(S/k))$ *and projective dimension of* $J_1(\Omega^{(1)}(S/k))$ *is less than or equal to one.*

$\Omega^{(1)}(S/k) \simeq F/N$ *where* F *is a free* S *module on* $\{\delta^1(x), \delta^1(y)\}$ *and* N *is a submodule of* F *generated by* $\delta^1(f) = 2y\delta^1(y) - 3x^2\delta^1(x)$. *Using these modules,* $S^2(\Omega^{(1)}(S/k)) \simeq S^2(F)/l_N$ *where* $S^2(F)$ *is a free module with basis*

$\{\delta^{(1)}(x) \bigvee \delta^{(1)}(x), \delta^{(1)}(x) \bigvee \delta^{(1)}(y), \delta^{(1)}(y) \bigvee \delta^{(1)}(y)\}$ *and* l_N *is a submodule of* $S^2(F)$ *generated by*

$\delta^1(f) \bigvee \delta^1(x) = (2y\delta^1(y) - 3x^2\delta^1(x)) \bigvee \delta^1(x) = 2y\delta^1(y) \bigvee \delta^1(x) - 3x^2\delta^1(x) \bigvee \delta^1(x)$

$\delta^1(f) \bigvee \delta^1(y) = (2y\delta^1(y) - 3x^2\delta^1(x)) \bigvee \delta^1(y) = 2y\delta^1(y) \bigvee \delta^1(y) - 3x^2\delta^1(x) \bigvee \delta^1(y)$

Since $rankS^2(\Omega^{(1)}(S/k)) = 1$ *we have* $rankl_N = rankS^2(F) - rankS^2(\Omega^{(1)}(S/k)) = 3-1 = 2$. *So* l_N *is a free S-module of rank 2. Therefore*

$$0 \longrightarrow l_N \longrightarrow S^2(F) \xrightarrow{\pi} S^2(\Omega^{(1)}(S/k)) \simeq S^2(F)/l_N \longrightarrow 0$$

is a free resolution of $S^2(\Omega^{(1)}(S/k))$ *and projective dimension of* $S^2(\Omega^{(1)}(S/k))$ *is less than or equal to one..*

Example 3.16. *Let* $R = k[x,y,z]$ *with* $y^2 = xz$ *and* $z^2 = x^3$. *It can be found that the projective dimension of* $\Omega^{(1)}(S/k) = 1$ *and the projective dimension of* $\Omega^{(2)}(S/k)$ *is infinite.*

4. Conclusion

In this work we have introduced the concept of higher symmetric derivation on Kähler modules and studied some of its properties. An application of this theory is given in solving a projective dimension problem. Consequently, it is not defined a symmetric derivation for an arbitrary ring (see Corollary 3.13 and example 3.16). Now we present the following two problems for further works,

First problem: On which rings can we define a symmetric derivation?

Second problem: Can we find a relationship between the projective dimensions of $\Omega^{(1)}(R/k)$ and $\Omega^{(n)}(R/k)$?

CONFLICT OF INTERESTS

The author declares that there is no conflict of interests.

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