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CATEGORICAL ANALYSIS OF FAITHFUL REPRESENTATION OF TRANSLATIONAL HULL OF AMPLE SEMIGROUPS

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Abstract. Representation of a semigroup has to do with obtaining a homomorphism that maps the semigroup into the full transformation semigroup. The representation is said to be faithful when it is an embedding. It is an existing result that ample monoid is embeddable into an inverse semigroup. This result has been extended to translational hulls and, in effect, a faithful representation of translational hull of ample semigroup is also an existing result. This faithful representation will be called categorical if we can establish that it is a class consisting of systems of the same type, referred to as objects and between any pair of objects A and B in the class, there are arrows $f: A \to B$ and each arrow is a structure preserving map referred to as morphism. In this paper, therefore, we want to carry out categorical analysis of faithful representation of translational hull of ample semigroup. The commutative diagrams of the faithful representation of translational hull of ample semigroups shall be very useful tools in the analysis.

Keywords: faithful representation; translational hull; category, subcategory; functor.

2020 AMS Subject Classification: 20M10.

1. INTRODUCTION AND PRELIMINARIES

Let X be a set, and denote by T_X the set of all functions $\alpha : X \to X$. T_X is called the full transformation semigroup on X with the operation of composition of functions. A

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homomorphism $\phi: S \to T_X$ is a representation of the semigroup S. We say that ϕ is a faithful representation, if it is an embedding.

It is well known from [16] that the set of all partial one-one maps of any non-empty set X is an inverse semigroup and it is called symmetric inverse semigroup usually denoted by $\mathfrak{I}_X$.

Let a, b be elements of a semigroup S, we define $a\,\mathcal{R}^* b$ if and only if for all $x, y \in S^1$, $xa = ya \Leftrightarrow xb = yb$. Dually we define the relation $\mathcal{L}^*$. Let S be a semigroup and $a \in S$. The elements $a^{\dagger}$(resp. a^*) will denote an idempotent element in $\mathcal{R}^*$(resp. $\mathcal{L}^*$)-class R_a^* (resp. L_a^*).

A semigroup S with a semilattice of idempotents $E(S)$ is said to be an adequate semigroup if each $\mathcal{R}^*$-class and $\mathcal{L}^*$-class contain an idempotent.

With $E(S)$ being a semilattice such an idempotent is unique. A left adequate semigroup is said to be a left ample (formerly left type A) if for all $e \in E(S)$ and $a \in S$, $ae = (ae)^{\dagger} a$ (see [9]) and dually for right ample (formerly right type A) semigroups. A semigroup S is said to be an ample (formerly a type A) semigroup if it is both left and right type ample. For more results on ample semigroups, see [6], [7], [10], [14], [17] and [18].

It is important to note that from Fountain [9] that every ample semigroup is essentially a special subsemigroup of an inverse semigroup through an embedding, thus several results in ample semigroups are analogous to those of an inverse semigroup. In fact, Offor et al [22] extended this embedding to translational hulls. Other results for translational hulls of a semigroup exist in the literature, for example, see [5], [12], [13], [21], [23], [24], [25] and [26].

A map λ from a semigroup S to itself is a left translation of S if for all elements $a, b \in S$, $\lambda(ab) = (\lambda a)b$. A map ρ from a semigroup S to itself is a right translation of S if $(ab)\rho = a(b\rho)$ for all elements $a, b \in S$. A left translation λ and a right translation ρ are linked if $a(\lambda b) = (a\rho)b$ for all $a, b \in S$. The set of all linked pairs (λ, ρ) of left and right translations is called the translational hull of S and it is denoted by $\Omega(S)$. We denote the set of all the idempotents of $\Omega(S)$ by $E_{\Omega(S)}$. The set of the left translations of S is denoted by $\Lambda(S)$ and the set of the right translations of S is denoted by $P(S)$. $\Omega(S)$ is a subsemigroup of the direct product $\Lambda(S) \times P(S)$. For $(\lambda, \rho)(\lambda', \rho') \in \Omega(S)$, the multiplication is given by $(\lambda, \rho)(\lambda', \rho') = (\lambda\lambda', \rho\rho')$ where $\lambda\lambda'$ denotes the composition of the left maps λ and λ' (that is, first λ' and then λ) and $\rho\rho'$ denotes the composition of the right maps ρ and ρ' (that is, first ρ and then ρ'). For each a in S, there is a linked pair (λ_a, ρ_a) within $\Omega(S)$ defined by $\lambda_a x = ax$ and $x\rho_a = xa$, and called the inner part of $\Omega(S)$ and for all $a, b \in S$, the following is obvious $(\lambda_a, \rho_a)(\lambda_b, \rho_b) =$

$(\lambda_{ab}, \rho_{ab})$. $a \mapsto (\lambda_a, \rho_a)$ is a map of S into $\Omega(S)$ is denoted by Π_S. $\Pi_S(S) = \{(\lambda_a, \rho_a) \,|\, a \in S, \lambda_a x = ax, x\rho_a = xa, \forall x \in S\}$

Theorem 1.2 [1]: Translational hull of an inverse semigroup is an inverse semigroup.

For $(\lambda, \rho) \in \Omega(S)$, the inverse $(\lambda, \rho)^{-1}$ is denoted by $(\lambda^{-1}, \rho^{-1})$ and satisfies the property that

$$\lambda^{-1}x = (x^{-1}\rho)^{-1}, \quad \text{and} \quad x\rho^{-1} = (\lambda x^{-1})^{-1} \forall x \in S$$

Lemma 1.3: Let S be a type Asemigroup. λ, λ' (ρ, ρ') are left (right) translations of S whose restrictions to the set of idempotents of S are equal, then $\lambda = \lambda'$ $(\rho = \rho')$.

If $\Omega(S)$ is adequate, and (λ, ρ) is an element of $\Omega(S)$, then (λ^*, ρ^*) denotes the unique idempotent in the $\mathcal{L}^*$-class of (λ, ρ),and $(\lambda^\dagger, \rho^\dagger)$ denotes the unique idempotent in the $\mathcal{R}^*$-class of (λ, ρ).

For $e \in E(S)$, $\lambda^\dagger e = (\lambda e)^\dagger$; $\lambda^* e = (\lambda e)^*$; $e\rho^\dagger = (e\rho)^\dagger$; $e\rho^* = (e\rho)^*$

$\lambda^\dagger, \lambda^*, \rho^\dagger, \rho^*$satisfy the following properties ;

For $a \in S$, $\lambda^\dagger a = (a^\dagger \rho)^\dagger a$; $\lambda^* a = (\lambda a^\dagger)^* a$; $a\rho^\dagger = a(a^*\rho)^\dagger$; $a\rho^* = a(\lambda a^*)^*$

We notice from the definition that $\lambda^\dagger e, \lambda^* e, e\rho^\dagger$ and $e\rho^*$are idempotents. We also note the following:

i. $\lambda^\dagger e = (e\rho)^\dagger e$ from the definition

$= e(e\rho)^\dagger$ idempotents commute

$= e(e^*\rho)^\dagger$

$= e\rho^\dagger$ by definition.

ii. $\lambda^* e = (\lambda e)^* e$ by definition

$= e(\lambda e)^*$ commutativity of idempotents

$= e\rho^*$ by definition.

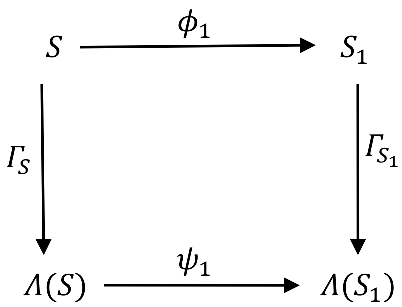
In particular therefore, $\lambda^\dagger b^\dagger$ and $a^*\rho^\dagger$ are idempotent of S.

Theorem 1.4 [11] The translational hull of an ample semigroup is ample.

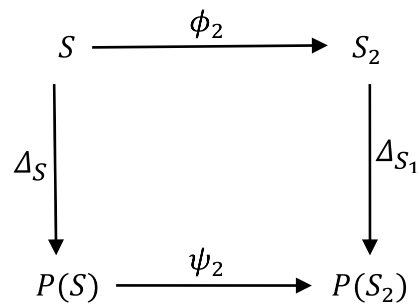
The set of left translations is denoted by $\Lambda(S)$ and right translations by $\Gamma(S)$. The left and the right translations are assumed linked. $\Gamma_S: a \mapsto \lambda_a$, and $\Gamma(S) = \{\lambda_a : a \in S\}$. $\Delta_S: a \mapsto \rho_a$ and $\Delta(S) = \{\rho_a : a \in S\}$.

Theorem 1.5 [2]. Given an ample monoid S, there are inverse semigroups S_1, S_2, and embeddings $\phi_1: S \to S_1$, $\phi_2: S \to S_2$, such that $\phi_1 a^* = (\phi_1 a)^* = (\phi_1 a)^{-1}(\phi_1 a), \phi_2 a^\dagger =$

$(\phi_2 a)^\dagger = (\phi_1 a)(\phi_1 a)^{-1}$, and there are also embeddings $\psi_1 : \Lambda(S) \to \Lambda(S_1)$, $\psi_2 : P(S) \to P(S_2)$ such that each of the diagrams



(i)



(ii)

commutes

and $\psi_1(\lambda^*) = [\psi_1(\lambda)]^* = [\psi_1(\lambda)]^{-1}\psi_1(\lambda)$, $\quad \psi_2(\rho^\dagger) = [\psi_2(\rho)]^\dagger = \psi_2(\rho)[\psi_2(\rho)]^{-1}$.

The theorem is proved through the following propositions and corollaries. Diagram (i) is dual to diagram (ii) and therefore every fact established about diagram (i) applies in dual manner to diagram (ii).

Proposition 1.6. [9]**.** Given an ample monoid S, there are inverse semigroups S_1, S_2, and embeddings $\phi_1 : S \to S_1$, $\phi_2 : S \to S_2$, such that $\phi_1 a^* = (\phi_1 a)^* = (\phi_1 a)^{-1}(\phi_1 a), \phi_2 a^\dagger = (\phi_2 a)^\dagger = (\phi_2 a)(\phi_2 a)^{-1}$.

Corollary 1.7 [22]**.** If $(a, b) \in \mathcal{R}^*(S)$, then $[\phi_1(a), \phi_1(b)] \in \mathcal{R}^*[\phi_1(S)]$

Let H be a subset of a semigroup S. The upper saturation $H\omega$ of H in S is defined by:

$$H\omega = \{s \in S : (\exists h \in H)\ h \leq s\}.$$

Proposition 1.8 [22]**.** ϕ_1 preserves subsemigroups and upper saturations

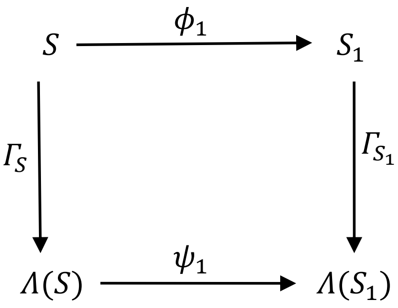
Lemma 1.9 [22]**.** For an inverse semigroup $S_1, \Gamma : a \mapsto \lambda_a$ is an isomorphism from S_1onto $\Gamma(S_1)$.

Lemma 1.10 [22]**.** For an ample semigroup $S, \Gamma : a \mapsto \lambda_a$ is an isomorphism from S onto $\Gamma(S)$.

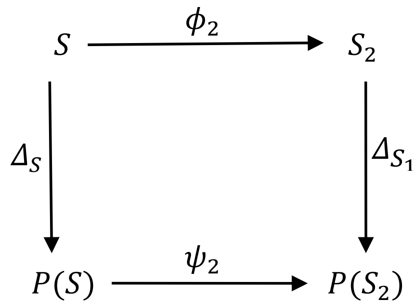
Corollary 1.11 [22]**.** For an inverse semigroup S_1, $\Delta_{S_1} : a \mapsto \rho_a$ is an isomorphism from S_1 onto $\Delta(S_1)$. Similarly, for an ample semigroup $S, \Delta_S : a \mapsto \rho_a$ is an isomorphism from S onto $\Delta(S)$.

Proposition 1.12 [22]**.** Given an ample monoid S, there are inverse semigroups S_1, S_2, and embeddings $\psi_1 : \Lambda(S) \to \Lambda(S_1)$, $\quad \psi_2 : P(S) \to P(S_2)$ such that $\psi_1(\lambda^*) = [\psi_1(\lambda)]^* = [\psi_1(\lambda)]^{-1}\psi_1(\lambda)$, $\psi_2(\rho^\dagger) = [\psi_2(\rho)]^\dagger = \psi_2(\rho)[\psi_2(\rho)]^{-1}$.

Proposition 1.13 [22**].** Each of the diagrams



(i)



(ii)

commutes.

Proposition 1.14. [22] $\psi_1 \circ \psi_1{}^{-1}$ is an idempotent-separating congruence on $\Gamma(S)$

Proposition 1.15 [22**].** ψ_1 is a good homomorphism

Proposition 1.16 [22**].** $\psi_1 : \Gamma(S) \to \Gamma(S_1)$ is a $*$-homomorphism

Proposition 1.17 [22**].** If δ is a congruence on $\Gamma(S)$, then $\psi_1(\delta)$ is a congruence on $\Gamma(S_1)$

Definition 1.18.

- An ideal F of a semilattice E is called a P-ideal if the intersection of F with any other principal ideal of the semigroup is a principal ideal.
- A semigroup homomorphism $\psi : S \to T$ is called a P-homomorphism if $\langle \psi(E_S) \rangle$ is a P-ideal of E_T.

Theorem 1.19 [22]. If ϕ_1 and Γ_{S_1} are P-homomorphisms, then so is the composition $\psi_1 \Gamma_S$.

2. CATEGORY THEORY

According to Hollings [15], category can be viewed in two versions which are indeed implicitly the same. Namely; the object – morphism version of category and the generalized monoid version of category.

2.1. The Object – Morphism Version of Category

According to Asibong-Ibe [3], a category consists of

- a class of objects (usually denoted by $\boldsymbol{C}$- obj)
- a set of morphisms between the objects in $\boldsymbol{C}$ which are denoted by $hom_{\boldsymbol{C}}(A, B)$ or simply $hom\,(A, B)$ for morphisms between A and B, satisfying the following conditions:

i. for any set of objects $A, B, C \in \boldsymbol{C}$, the $\boldsymbol{C}$-morphisms $f \in hom\ (A, B)$, $g \in hom\ (B, C)$ imply $g \circ f \in hom\ (A, C)$

ii. for each object A, an identity morphism $1_A \in hom\ (A, A)$

iii. if $f \in hom\ (A, B)$, $g \in hom\ (B, C)$ and $h \in hom\ (C, D)$, then $h \circ (g \circ f) = (h \circ g) \circ f \in hom\ (A, D)$

iv. for every object A, $1_A \in hom\ (A, A)$ and $f \circ 1_A = f, 1_B \circ g = g$, for every $f, g \in hom\ (A, B)$.

v. every distinct pair of $\boldsymbol{C}$- objects has distinct set of morphisms. That is, if $(A, B) \neq (C, D)$, then $hom\ (A, B) \cap hom\ (C, D) = \emptyset$

So, in a category, there must be a class consisting of systems of the same type, referred to as $objects$ and between any pair of objects A and B in the class, there must arrows $f: A \rightarrow B$ and each arrow is a structure preserving map referred to as morphism.

2.2 Subcategory

According to Asibong-Ibe [3], assuming $\boldsymbol{D}$ is a subclass of a category $\boldsymbol{C}$ such that each object in $\boldsymbol{D}$ is also a $\boldsymbol{C}$- object. Then $\boldsymbol{D}$ is a subcategory if

i. for any pair of objects A, B in $\boldsymbol{D}$, each morphism $f: A \rightarrow B$ in $\boldsymbol{D}$ is also a morphism in $\boldsymbol{C}$

ii. each object in $\boldsymbol{D}$ has an identity morphisms in $\boldsymbol{D}$ and

iii. $\boldsymbol{D}$ contains the product of its morphisms. That is, the products of $\boldsymbol{D}$-morphisms $f: A \rightarrow B$ and $g: B \rightarrow C$ which is $g \circ f: A \rightarrow C$ is also a $\boldsymbol{D}$-morphism.

Reader is referred to [3], [4] and [8] for the numerous examples of category and other details. Right now, we are particularly interested in the following example as it will help to link us to the generalized monoid version of category.

Example 2.2.1 [3]

Define the object set in M as elements of the underlying set A of M and between any objects a, b, is the monoid operation $'' \circ ''$. That is, for a, b in object M, $(a, b) \rightarrow a \circ b$.

Thus, $((a, b), c) \rightarrow ((a \circ b), c) \rightarrow (a \circ b) \circ c = a \circ (b \circ c) = \big(a, (b, c)\big)$.

Also $1_A \in \text{hom}\ (A, A)$ and 1_A is the unit in $M. a \circ 1_A = 1_A \circ a = a$.

Thus, M is a category with object A and morphism $'' \circ ''$.

2.3 The Generalized Monoid Version of Category

Let C be a class and $'' \cdot ''$ be a partial binary operation on C. For $x, y \in C$, we write $\exists x \cdot y$ if $x \cdot y \in C$. An element $e \in C$ is called an idempotent if $\exists e \cdot e$ and $e \cdot e = e$. The idempotents $e \in C$

which satisfy the conditions that for $x \in C, \exists e \cdot x \Rightarrow e \cdot x = x$ and $\exists x \cdot e \Rightarrow x \cdot e = x$, are called the identities of C. We denote the set identities of C by C_o.

According to [15], the pair $(C,\cdot)$ is called a category if the following hold:

i. $\exists x \cdot (y \cdot z) \Leftrightarrow \exists (x \cdot y) \cdot z$ and in which case, $x \cdot (y \cdot z) = (x \cdot y) \cdot z$

ii. $\exists x \cdot (y \cdot z) \Leftrightarrow \exists x \cdot y$ and $\exists y \cdot z$

iii. $\forall x \in C$, there exist unique identities $\boldsymbol{d}(x), \boldsymbol{r}(x) \in C_o$ such that $\exists\ \boldsymbol{d}(x) \cdot x$ and $\exists x \cdot \boldsymbol{r}(x)$

Whenever the partial multiplication in category $(C,\cdot)$ is clear, we simply refer to category C. The identity $\boldsymbol{d}(x)$ is called the domain of x and the identity $\boldsymbol{r}(x)$ is called the range of x. Since $\boldsymbol{d}(x)$, $\boldsymbol{r}(x) \in C_o$, $\boldsymbol{d}(x) \cdot x = x$ and $x \cdot \boldsymbol{r}(x) = x$. Thus, for any identity e, $\boldsymbol{d}(e) = \boldsymbol{r}(e) = e$.

2.4. Functor

Let $\boldsymbol{C}$ and $\boldsymbol{D}$ be categories. A function $\phi: \boldsymbol{C} \to \boldsymbol{D}$ is called a functor if it satisfies the following conditions:

i. If $\exists\ a \cdot b$ in $\boldsymbol{C}$, then $\exists\ a\phi \cdot b\phi$ in $\boldsymbol{D}$ and

ii. $a\phi \cdot b\phi = (a \cdot b)\phi$

A functor $\phi: \boldsymbol{C} \to \boldsymbol{D}$ is called an ordered functor (or order preserving functor) if $a \leq b$ in $\boldsymbol{C}$, then $a\phi \leq b\phi$ in $\boldsymbol{D}$.

An ordered functor $\phi: \boldsymbol{C} \to \boldsymbol{D}$ is called inductive functor if $\forall e, f \in \boldsymbol{C}_o$, then $e\phi \wedge f\phi$ exists in $\boldsymbol{D}_o$.

Lemma 2.5. [15]. Let $(C,\cdot)$ be a category with $x, y \in C$.

i. $\exists x \cdot y \Leftrightarrow \boldsymbol{r}(x) = \boldsymbol{d}(y)$

ii. If $\exists x \cdot y$, then $\boldsymbol{d}(x \cdot y) = \boldsymbol{d}(x)$ and $\boldsymbol{r}(x \cdot y) = \boldsymbol{r}(y)$.

Let $(C,\cdot)$ be a category. For $e, f \in C_o$, Hollings [15] defined the set $mor(e,f)$ by:

$$mor(e,f) = \{x \in C: \boldsymbol{d}(x) = e, \boldsymbol{r}(x) = f\}.$$

When $e = f, mor(e,f)$ is a monoid. To see this, for $x \in mor(e,e), e \cdot x = \boldsymbol{d}(x) \cdot x = x$, $x \cdot e = \boldsymbol{r}(x) = x$. Therefore, e is the identity in $mor(e,e)$. Let $x, y \in mor(e,e)$. Then, $\boldsymbol{d}(x \cdot y) = \boldsymbol{d}(x) = e$ and $\boldsymbol{r}(x \cdot y) = \boldsymbol{r}(y) = e$. Therefore, $x \cdot y \in mor(e,e)$. It then follows that $\exists x \cdot (y \cdot z)$ and $\exists (x \cdot y) \cdot z, \forall\ x, y, z \in mor(e,e)$, and since $mor(e,e) \in C, x \cdot (y \cdot z) = (x \cdot y) \cdot z$.

$mor(e,e)$ is called the local submonoid of C at e. Thus, category is regarded as a generalization of a monoid. According to [15], a unipotent category is a category in which every local submonoid contains only one idempotent.

Definition 2.6. [15] A cancellative category is a category $(C,\cdot)$ inwhich $\forall\ a, b, z \in C$,

i. If $\exists z \cdot a, \exists z \cdot b$ and $z \cdot a = z \cdot b$, then $a = b$ and

ii. If $\exists a \cdot z, \exists b \cdot z$ and $a \cdot z = b \cdot z$, then $a = b$.

Definition 2.7. [15]. Let $(C, \cdot)$ be a category and let C be partially ordered by $\leq$ with $a, b, c, d \in C$. According to [15], the triple $(C, \cdot, \leq)$ is called an ordered category if the following conditions hold:

i. If $a \leq c, b \leq d, \exists a \cdot b$ and $\exists c \cdot d$, then $a \cdot b \leq c \cdot d$

ii. If $a \leq b$, then $\boldsymbol{d}(a) \leq \boldsymbol{d}(b)$ and $\boldsymbol{r}(a) \leq \boldsymbol{r}(b)$

iii. (1) for each $f \in C_o$ and $a \in C$ with $f \leq \boldsymbol{r}(a)$, there exists an element of C, denoted by $a|f$ which is the unique element with the properties $a|f \leq a$ and $\boldsymbol{r}(a|f) = f$.

(2) for each $f \in C_o$ and $a \in C$ with $f \leq \boldsymbol{d}(a)$, there exists an element of C, denoted by $f|a$, which is the unique element with the properties $f|a \leq a$ and $\boldsymbol{d}(f|a) = f$.

The element $a|f$ is called the corestriction of a to f, while $f|a$ is called restriction of f to a.

Assuming $e, f \in C_o$ and $\exists e|f$, then by definition, $e \leq \boldsymbol{d}(f)$. But $\boldsymbol{d}(f) = f$. Therefore, $e|f \Rightarrow e \leq f$. Similarly, if corestriction $f|e$ is defined, $f \leq e$.

Let $(C, \cdot, \leq)$ be an ordered category with $a, b, x, y, z \in C$ and $e, f \in C_o$. Suppose $a \leq b$. Then $\boldsymbol{d}(a) \leq \boldsymbol{d}(b)$ and $\boldsymbol{r}(a) \leq \boldsymbol{r}(b)$. Since $\boldsymbol{d}(a)$ and $\boldsymbol{r}(a)$ are identities by definition, then by (vi) & (vii), the restriction $\boldsymbol{d}(a)|b$ and corestriction $b|\boldsymbol{r}(a)$ are defined in C and $\boldsymbol{d}(a)|b \leq b$; $\boldsymbol{d}[\boldsymbol{d}(a)|b] = \boldsymbol{d}(a)$. Since $\boldsymbol{d}(a)|b$ is unique, $\boldsymbol{d}(a)|b = a$. Similarly, $b|\boldsymbol{r}(a) = a$.

So, if $a \leq b$, then $\boldsymbol{d}(a)|b = a = b|\boldsymbol{r}(a)$ (2.7.1)

Let $f \leq \boldsymbol{r}(a)$. By definition, $\exists a|f, \boldsymbol{r}(a|f) = f$ and since $\boldsymbol{r}(a|f)$ is an identity such that $(a|f) \cdot \boldsymbol{r}(a|f) = (a|f)$, we have $(a|f) \cdot f = a|f$. Similarly, $f \cdot (f|a) = f|a$. Thus, for $f \leq \boldsymbol{r}(a), \exists (a|f) \cdot f$ with $(a|f) \cdot f = a|f$ and for $f \leq \boldsymbol{d}(a), \exists f \cdot (f|a)$ with

$$f \cdot (f|a) = f|a \qquad (2.7.2)$$

Suppose $\exists\, c \in C$ such that $a \leq c$ and $b \leq c$ and that $\boldsymbol{r}(a) = \boldsymbol{r}(b)$. Then by equation (1), $a \leq c$ and $b \leq c$ imply that $a = c|\boldsymbol{r}(a)$ and $b = c|\boldsymbol{r}(b)$. Therefore, $a = c|\boldsymbol{r}(a) = c|\boldsymbol{r}(b) = b$. Similarly, if $a \leq c$ and $b \leq c$ and that $\boldsymbol{d}(a) = \boldsymbol{d}(b)$, then $a = b$. Thus, if $\exists\, c \in C$ such that $a \leq c$ and $b \leq c$ and either $\boldsymbol{r}(a) = \boldsymbol{r}(b)$ or $\boldsymbol{d}(a) = \boldsymbol{d}(b)$, then $a = b$

Thus, $\boldsymbol{r}(a) = \boldsymbol{r}(b)$ or $\boldsymbol{d}(a) = \boldsymbol{d}(b)$, then $a = b$ (2.7.3)

From equation (1), we have that $a \leq b \Rightarrow \boldsymbol{d}(a)|b = a = b|\boldsymbol{r}(a)$. Now, if $a = b$, we have $\boldsymbol{d}(a)|a = a = a|\boldsymbol{r}(a)$.

Thus, $\forall a \in C, \boldsymbol{d}(a)|a = a = a|\boldsymbol{r}(a)$ (2.7.4)

Assuming $e \leq f$, then by equation (1) and since $e = \boldsymbol{d}(e) = \boldsymbol{r}(e)$, we have $e|f = \boldsymbol{d}(e)|f = e = f|\boldsymbol{r}(e) = f|e$.

Thus, $e \leq f \Rightarrow e|f = e = f|e$ (2.7.5)

Lemma 2.8 [20]: Let $(C,\cdot,\leq)$ be an ordered category and suppose that $a \in C$ and $e \in C_o$. If $a \leq e$, then $a \in C_o$.

Consequently, in an ordered category $(C,\cdot,\leq)$, if the greatest lower bound $(the\ meet)$ of two identities – e, f , denoted by $e \wedge f$ (with respect to $\leq$) exists, then it is an identity.

An inductive category is an ordered category $(C,\cdot,\leq)$ in which $\forall e, f \in C_o, e \wedge f$ exists in C_o.

Let $(C, \cdot, \leq)$ be an ordered category. The Pseudoproduct $\otimes$ in $(C, \cdot, \leq)$ is the binary operation given by

$$a \otimes b = [a|\boldsymbol{r}(a) \wedge \boldsymbol{d}(b)] \cdot [\boldsymbol{r}(a) \wedge \boldsymbol{d}(b)|b]$$

If $\exists a \cdot b$, then by lemma 2.8.4, $\boldsymbol{r}(a) = \boldsymbol{d}(b)$. So that $a \otimes b = [a|\boldsymbol{r}(a)] \cdot [\boldsymbol{d}(b)|b]$

By equation (2.7.4), $a|\boldsymbol{r}(a) = a$ and $\boldsymbol{d}(b)|b = b$. Hence, $a \otimes b = [a|\boldsymbol{r}(a)] \cdot [\boldsymbol{d}(b)|b] = a \cdot b$.

Thus, if both $a \cdot b$ and $a \otimes b$ are defined in C, then $a \cdot b = a \otimes b$

(2.7.6)

Proposition 2.9 [19]. Let $(C,\cdot,\leq)$ be an ordered category. If both $a \otimes (b \otimes c)$ and $(a \otimes b) \otimes c$ are defined, then they equal. Hence, in an inductive category, $\otimes$ is an everywhere defined associative binary operation.

Suppose $(C,\cdot,\leq)$ be is an inductive category with $a \in C$ and $e \in C_o$. Then, $e \otimes a = [e|\boldsymbol{r}(e) \wedge \boldsymbol{d}(a)] \cdot [\boldsymbol{r}(e) \wedge \boldsymbol{d}(a)|a] = [e|\boldsymbol{e} \wedge \boldsymbol{d}(a)] \cdot [\boldsymbol{e} \wedge \boldsymbol{d}(a)|a]$. Notice that $\boldsymbol{e} \wedge \boldsymbol{d}(a)$ is an identity and $\boldsymbol{e} \wedge \boldsymbol{d}(a) \leq e$. Therefore, by (xii), $e|\boldsymbol{e} \wedge \boldsymbol{d}(a) = \boldsymbol{e} \wedge \boldsymbol{d}(a)$. So that $e \otimes a = \boldsymbol{e} \wedge \boldsymbol{d}(a) \cdot [\boldsymbol{e} \wedge \boldsymbol{d}(a)|a]$. Furthermore, $\boldsymbol{e} \wedge \boldsymbol{d}(a) \leq \boldsymbol{d}(a)$ and by equation (2.7.2),

$e \otimes a = \boldsymbol{e} \wedge \boldsymbol{d}(a) \cdot [\boldsymbol{e} \wedge \boldsymbol{d}(a)|a] = \boldsymbol{e} \wedge \boldsymbol{d}(a)|a$ (2.7.7)

Similarly, $a \otimes e = a|\boldsymbol{r}(a) \wedge e$ (2.7.8)

It is important to note that $a|\boldsymbol{r}(a) \wedge \boldsymbol{d}(b)$ means $a|[\boldsymbol{r}(a) \wedge \boldsymbol{d}(b)]$ and not $[a|\boldsymbol{r}(a)] \wedge \boldsymbol{d}(b)$. Similarly, $\boldsymbol{r}(a) \wedge \boldsymbol{d}(b)|b$ should be read as $[\boldsymbol{r}(a) \wedge \boldsymbol{d}(b)]|b$ and not $\boldsymbol{r}(a) \wedge [\boldsymbol{d}(b)|b]$.

Theorem 2.10. [3]. Every functor preserves identities, isomorphisms and commutative diagrams.

3. RESULTS: CATEGORICAL ANALYSIS OF THE EMBEDDING

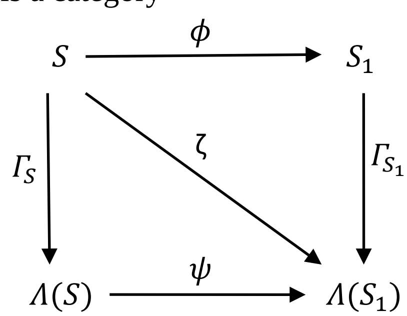
We recall that in a category, there must be a class consisting of systems of the same type, referred to as objects and between any pair of objects A and B in the class, there must arrows $f: A \to B$ and each arrow is a structure preserving map referred to as morphism.

Proposition 3.1. Let S_1, S_2 be inverse semigroups and S an ample monoid. Then the embeddings $\psi_1: \Lambda(S) \to \Lambda(S_1)$, $\psi_2: P(S) \to P(S_2)$ such that $\psi_1(\lambda^*) = [\psi_1(\lambda)]^* = [\psi_1(\lambda)]^{-1}\psi_1(\lambda)$, $\psi_2(\rho^\dagger) = [\psi_2(\rho)]^\dagger = \psi_2(\rho)[\psi_2(\rho)]^{-1}$ is categorical.

Proof. Here is the commutative diagram. We show that it is a category

The objects are:

i. The ample semigroup S;
ii. The inverse semigroup S_1;
iii. The left translational hull $\Lambda(S)$ of S and
iv. The left translational hull $\Lambda(S_1)$ of S_1.



We showed that the diagram commutes. That is, $\psi\Gamma_S = \Gamma_{S_1}\phi = \zeta$ where $\zeta: a \to \theta_{\lambda_a}(a \in S)$.

$\phi \in hom(S, S_1)$, $\Gamma_{S_1} \in hom(S_1, \Lambda(S_1))$ and $\Gamma_{S_1}\phi, \zeta \in hom(S, \Lambda(S_1))$

$\Gamma_S \in hom(S, \Lambda(S))$, $\psi \in hom(\Lambda(S), \Lambda(S_1))$ and $\psi\Gamma_S, \zeta \in hom(S, \Lambda(S_1))$

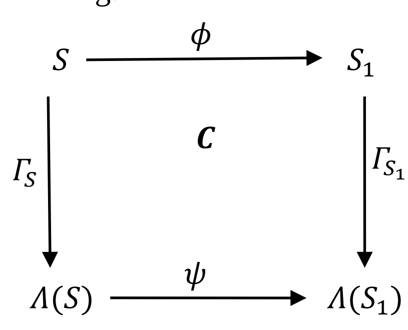
For each object, there is an identity map. For instance, $1_S : a \to 1 \cdot a$(where $a \in S$ and 1 is the identity in S), is an identity map on the object S. $\phi 1_S = \phi$.

Notice that $\phi_1\phi_a = \phi_{1a} = \phi_a$ and $\phi_a\phi_1 = \phi_a$. So that ϕ_1 is the identity in S_1. $1_{S_1}: \phi_a \to \phi_a\phi_1$ is an identity map on the object S_1. $1_{S_1}\phi = \phi$

Every arrow in the commutative diagram is distinct. Each arrow is not just a homomorphism but a $*$-homomorphism. Hence, the arrows are all structure preserving.

Thus, the diagram is a category.

From this point, we call the commutative diagram – category $\boldsymbol{C}$.



3.2 Construction of a Category from an inverse semigroup

We do this construction in analogy with Lawson [19] construction of inductive category from a restriction semigroup.

Given an inverse semigroup S_1, we define a product in S_1 by

$$a \cdot b = \begin{cases} ab & if\ \ a^{-1}a = bb^{-1} \\ \text{undefined}, & \text{otherwise} \end{cases} \quad a, b \in S_1 \tag{3.2.0}$$

Theorem 3.2.1 Let S_1 be an inverse semigroup with the natural partial order $\leq$. Then $(S_1, \cdot, \leq) = \boldsymbol{C}(\boldsymbol{S_1})$ is a category with $\boldsymbol{C}(\boldsymbol{S_1})_o = E(S_1)$, $\boldsymbol{d}(a) = aa^{-1}$, $\boldsymbol{r}(a) = a^{-1}a$, $\forall a \in S_1$, where $'' \cdot ''$ is the product defined in equation (3.2.0) above.

Proof. Assuming e is an identity in $(S_1,\cdot)$ such that $\exists\ e \cdot x$ for $x \in S_1$. Then, by the definition of $'' \cdot ''$, $e = xx^{-1}$. Similarly, if f is an identity in $(S_1,\cdot)$ such that $\exists x \cdot f$ for $x \in S_1$. Then $f = x^{-1}x$. Thus, idempotents in S_1 are the identities in $(S_1,\cdot)$. $xx^{-1} \cdot x$ exists since $(xx^{-1})^{-1}(xx^{-1}) = xx^{-1}$. Of course, $xx^{-1} \cdot x = x$ and by uniqueness of $\boldsymbol{d}(x)$, $xx^{-1} = \boldsymbol{d}(x)$. Similarly, $x^{-1}x = \boldsymbol{r}(x)$.

Next, we show that $(\forall x, y, z \in S_1)\ \exists x \cdot (y \cdot z) \Leftrightarrow \exists (x \cdot y) \cdot z$ and that $x \cdot (y \cdot z) = (x \cdot y) \cdot z$, $\exists x \cdot y\,; \exists y \cdot z$.

Assuming $\exists x \cdot (y \cdot z)$, then $x^{-1}x = (y \cdot z)(y \cdot z)^{-1}$

But $(y \cdot z) = yz$ such that $y^{-1}y = zz^{-1}$

Therefore, $\exists x \cdot (y \cdot z) \Rightarrow x^{-1}x = (yz)(yz)^{-1}$ and $y^{-1}y = zz^{-1}$

So that $x^{-1}x = (yz)(yz)^{-1} = yzz^{-1}y^{-1} = yzz^{-1}zz^{-1}y^{-1} = yy^{-1}yy^{-1}yy^{-1}$ (since $y^{-1}y = zz^{-1}$) $= yy^{-1}$

Therefore, $\exists x \cdot y$. Similarly, $\exists y \cdot z$.

Again, $x \cdot (y \cdot z) = xyz$ such that $x^{-1}x = (yz)(yz)^{-1}$ and $y^{-1}y = zz^{-1}$. But $x^{-1}x = (yz)(yz)^{-1} = yy^{-1}$.

Therefore, $x \cdot (y \cdot z) = xyz$ such that $x^{-1}x = yy^{-1}$; $y^{-1}y = zz^{-1}$.

On the other hand, $(x \cdot y) \cdot z = xyz$ such that $(xy)^{-1}(xy) = zz^{-1}$ and $x^{-1}x = yy^{-1}$.

But $(xy)^{-1}(xy) = y^{-1}x^{-1}xy = y^{-1}x^{-1}xx^{-1}xy = y^{-1}yy^{-1}yy^{-1}y = y^{-1}y$.

So that $(x \cdot y) \cdot z = xyz$ such that $y^{-1}y = zz^{-1}$; $x^{-1}x = yy^{-1}$

Thus, $\exists x \cdot (y \cdot z) \Rightarrow \exists (x \cdot y) \cdot z$ and $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.

Hence, $(S_1,\cdot)$ is a category. We denote by $\boldsymbol{C}(\boldsymbol{S_1})$ this category associated with an inverse semigroup S_1, and the set of identities of $\boldsymbol{C}(\boldsymbol{S_1})$ by $\boldsymbol{C}(\boldsymbol{S_1})_o$

Proposition 3.2.2. $\boldsymbol{C}(\boldsymbol{S_1})$ is inductive

Proof. Suppose $a, b, c, d \in \boldsymbol{C}(\boldsymbol{S_1})$ such that $a \leq c, b \leq d, \exists a \cdot b, \exists c \cdot d$. Then for some $e, f \in \boldsymbol{C}(\boldsymbol{S_1})_o, a = ec$ and $b = fd$. So that $ab = ecfd = ecc^{-1}cfd = ecfc^{-1}cd$

$ecfc^{-1}$ is an idempotent. Therefore, $ab = ecfc^{-1}cd \leq cd$. Thus, $a \cdot b \leq c \cdot d$

Now, suppose $a \leq b$. This implies that $a = eb = bf$ for some $e, f \in \boldsymbol{C}(\boldsymbol{S_1})_o$.

With $a^{-1} = b^{-1}e, aa^{-1} = ebb^{-1}e = ebb^{-1} \leq bb^{-1}$

So that $\boldsymbol{d}(a) \leq \boldsymbol{d}(b)$.

With $a = bf, a^{-1} = fb^{-1}$ and $a^{-1}a = fb^{-1}bf = b^{-1}bf \leq b^{-1}b$. So that $\boldsymbol{r}(a) \leq \boldsymbol{r}(b)$.

So that, $a \leq b$ implies that $\boldsymbol{d}(a) \leq \boldsymbol{d}(b)$ and $\boldsymbol{r}(a) \leq \boldsymbol{r}(b)$.

For each $f \in \boldsymbol{C}(\boldsymbol{S_1})_o$ and $a \in \boldsymbol{C}(\boldsymbol{S_1})$ with $f \leq \boldsymbol{r}(a)$, we take $a|f = af$, the correstriction of a to f. af satisfies the required properties as follows:

$af \leq a$ and $a^{-1}af \leq a^{-1}a$. Therefore,$f \leq a^{-1}a = \boldsymbol{r}(a)$. $\boldsymbol{r}(af) = (af)^{-1}(af) = fa^{-1}af = a^{-1}af = f$.

For the uniqueness, let g be another value for $a|f$. This implies that $g \leq a$ and $\boldsymbol{r}(g) = f$. That is, $g^{-1}g = f$.

$g \leq a \Rightarrow g = ae$ for some $e \in \boldsymbol{C}(\boldsymbol{S_1})_o$. So that $g^{-1} = ea^{-1}$.

Therefore, $g^{-1}g = ea^{-1}ae = a^{-1}ae$. This implies that, $g^{-1}g \leq e$.

So that, $eg^{-1}g = g^{-1}g$. So with$g = ae, g = gg^{-1}g = aeg^{-1}g = ag^{-1}g$, and with $g^{-1}g = f$, we have $g = ag^{-1}g = af$.

Similarly, if we choose fa for the restriction of f to a, the desired properties will be satisfied.

Thus, $\boldsymbol{C}(\boldsymbol{S_1})$ is an ordered category.

For $\forall e, f \in \boldsymbol{C}(\boldsymbol{S})_o$, $ef \leq e$ and $ef \leq f$. So that $ef \leq e \wedge f$. Assuming $g = e \wedge f$, $g \in \boldsymbol{C}(\boldsymbol{S_1})$, then we have $g \leq e$, $g \leq f$ and therefore, $e \wedge f = g = g^2 \leq e \wedge f = g$. Hence, $e \wedge f \in \boldsymbol{C}(\boldsymbol{S_1})_o$

Hence, $\boldsymbol{C}(\boldsymbol{S_1})$ is an inductive category.

Corollary 3.2.3: Let S_1 be an inverse semigroup with the natural partial order $\leq$. Then $(S_1, \cdot, \leq) = \boldsymbol{C}(\boldsymbol{S_1})$ is an inductive category with $\boldsymbol{C}(\boldsymbol{S_1})_o = E(S_1), \boldsymbol{d}(a) = aa^{-1}, \boldsymbol{r}(a) = a^{-1}a, \forall a \in S_1$.

3.3. Construction of a category from an ample semigroup

In a very similar fashion as that of inverse semigroup, we construct a category from an ample semigroup as follows:

Let S be an ample semigroup and define a product in S by

$$a \cdot b = \begin{cases} ab & if \;\; a^* = b^\dagger \\ \text{undefined}, & \text{otherwise} \end{cases} \qquad a, b \in S_1 \qquad (3.3.0)$$

Theorem 3.3.1: Let S be an ample semigroup with the natural partial order $\leq$. Then $(S, \cdot, \leq) = \boldsymbol{C(S)}$ is a category with $\boldsymbol{C(S)}_o = E(S), \boldsymbol{d}(a) = a^\dagger$, $\boldsymbol{r}(a) = a^*$, $\forall a \in S$, where $'' \cdot ''$ is the product defined in (ii) right above.

Proof. Assuming e is an identity in $(S,\cdot)$ such that $\exists e \cdot x$ for $x \in S$. Then $e = x^\dagger$. Similarly, if f is an identity in $(S,\cdot)$ such that $\exists x \cdot f$ for $x \in S$. Then $f = x^*$. Thus, idempotents in S are the identities in $(S,\cdot)$. $x^\dagger \cdot x$ exists since $(x^\dagger)^* = x^\dagger$. Of course, $x^\dagger \cdot x = x$ and by uniqueness of $\boldsymbol{d}(x)$, $x^\dagger = \boldsymbol{d}(x)$. Similarly, $x^* = \boldsymbol{r}(x)$.

Now, suppose $\exists x \cdot (y \cdot z)$. That is $x^* = (yz)^\dagger$ and $y^* = z^\dagger$. So that $x^* = (yz^\dagger)^\dagger = (yy^*)^\dagger = y^\dagger$. So $\exists x \cdot (y \cdot z) \Rightarrow y^* = z^\dagger; x^* = y^\dagger$. But $(xy)^* = (x^*y)^* = (y^\dagger y)^* = y^* = z^\dagger$. So that $\exists x \cdot (y \cdot z) \Leftrightarrow (xy)^* = z^\dagger; x^* = y^\dagger \Leftrightarrow \exists (x \cdot y) \cdot z$. Hence, $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.

Moreover, $\exists x \cdot (y \cdot z) \Rightarrow x^* = y^\dagger; y^* = z^\dagger \Rightarrow \exists\, x \cdot y\,; \exists y \cdot z$.

Hence, $(S,\cdot)$ is a category. We denote by $\boldsymbol{C(S)}$ this category associated with type A semigroup S, and the set of identities of $\boldsymbol{C(S)}$ by $\boldsymbol{C(S)}_o$

Proposition 3.3.2. $\boldsymbol{C(S)}$ is inductive

Proof. Let $a, b, c, d \in \boldsymbol{C(S)}$ with $a \leq c, b \leq d, \exists a \cdot b, \exists c \cdot d$. Then for some $e, f \in E(S), a = ec$ and $b = fd$. So that $ab = ecfd = e(cf)^\dagger cd \leq cd$. Thus, $a \cdot b \leq c \cdot d$.

Assuming, $a \leq b$, then $a = eb$, and this implies that $a^\dagger = (eb)^\dagger = (eb^\dagger)^\dagger = eb^\dagger \leq b^\dagger$. So that $\boldsymbol{d}(a) \leq \boldsymbol{d}(b)$.

$a \leq b$ equally implies that $a = bf, f \in E(S)$. So that $a^* = (bf)^* = (b^*f)^* = b^*f \leq b^*$ and therefore, $\boldsymbol{r}(a) \leq \boldsymbol{r}(b)$.

For each $f \in \boldsymbol{C(S)}_o$ and $a \in \boldsymbol{C}$ with $f \leq \boldsymbol{r}(a)$, we take $a|f = af$, the correstriction of a to f and we note that af satisfies the required properties as follows: $af \leq a. f \leq \boldsymbol{r}(a) = a^*$, so that $\boldsymbol{r}(af) = (af)^* = (a^*f)^* = a^*f = f$. To confirm uniqueness, assuming there is another value g for $a|f$. This implies that $g \leq a$ and $\boldsymbol{r}(g) = g^* = f$. Then, $g \leq a \Rightarrow g = ag^* = af$. Similarly, if fa is chosen for the restriction of f to a, the desired properties will be satisfied. Hence, the $\boldsymbol{C(S)}$ is an ordered category.

Just as in $\boldsymbol{C(S_1)}$, For $\forall e, f \in \boldsymbol{C(S)}_o$, $e \wedge f \in \boldsymbol{C(S)}_o$.

Thus, $\boldsymbol{C(S)}$ is inductive.

Corollary 3.3.3: Let S be a type A semigroup with the natural partial order $\leq$. Then $(S, \cdot, \leq) = \boldsymbol{C}(\boldsymbol{S})$ is an inductive category with $\boldsymbol{C}(\boldsymbol{S})_o = E(S), \boldsymbol{d}(a) = a^\dagger, \boldsymbol{r}(a) = a^*, \forall a \in S$.

Theorem 3.3.4. $\eta \coloneqq \boldsymbol{C}(\boldsymbol{\phi}): \boldsymbol{C}(\boldsymbol{S}) \to \boldsymbol{C}(\boldsymbol{S_1})$ is an inductive functor.

Proof. Essentially, we mean to show that $\phi: (S, \cdot, \leq) \to (S_1, \cdot, \leq)$ is an inductive functor.

Suppose $\exists a \cdot b$ in. Then $a^* = b^\dagger$ and $a \cdot b = ab$. Since ϕ is a homomorphism, $a^* = b^\dagger \Rightarrow a^*\phi = b^\dagger\phi \Rightarrow (a\phi)^* = (b\phi)^\dagger \Rightarrow \exists a\phi \cdot b\phi$ in S_1

It further follows that $(a \cdot b)\phi = (ab)\phi = a\phi b\phi = a\phi \cdot b\phi$ in S_1 [since $(a\phi)^* = (b\phi)^\dagger$]

Thus, $\phi: (S, \cdot, \leq) \to (S_1, \cdot, \leq)$ is a functor.

To show that the functor is inductive, recall that $\forall e, f \in \boldsymbol{C}_o$, then $e \wedge f = ef$. so that

$$(e \wedge f)\phi = (ef)\phi = e\phi f\phi = e\phi \wedge f\phi.$$

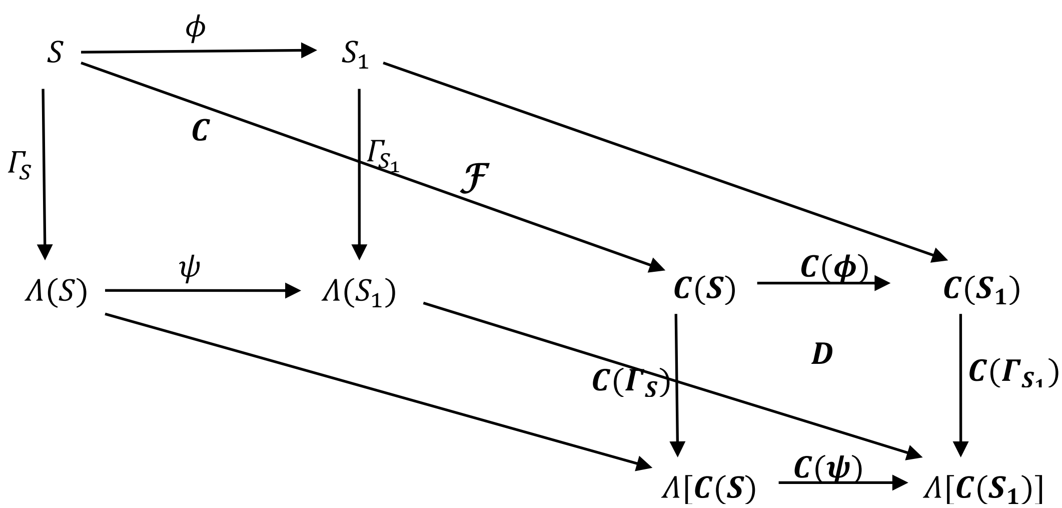
Finally, we show that the functor is order preserving

Assuming $a \leq b$. This implies that $a = e \cdot b$ for some $e \in \boldsymbol{C}_o$.

So that $a\phi = (e \cdot b)\phi = e\phi \cdot b\phi \Rightarrow a\phi \leq b\phi$.

Evidently, η is an ordered inductive functor.

It is important to note that $\boldsymbol{d}(a)\phi = a^\dagger\phi = (a\phi)^\dagger = \boldsymbol{d}(a\phi)$ and $\boldsymbol{r}(a)\phi = a^*\phi = (a\phi)^* = \boldsymbol{r}(a\phi)$.



$\boldsymbol{\mathcal{F}} : S \to \boldsymbol{C}(\boldsymbol{S})$ is the map $(a, b) \in S \times S \dot{\to} a \cdot b$. As given, $a \cdot b = ab$ (if $a^* = b^\dagger$).

$(a, b) \cdot = (a \cdot b) = ab = (a) \cdot (b)$. So $'' \cdot ''$ is the morphism and $\boldsymbol{\mathcal{F}} \in$ hom $[S, \boldsymbol{C}(\boldsymbol{S})]$

As we have seen in the construction of $\boldsymbol{C}(\boldsymbol{S})$ from S that:

$$E(S) \to \boldsymbol{C}_o, \quad a^* \in S \to \boldsymbol{r}(x) \in \boldsymbol{C}(\boldsymbol{S}), \quad a^\dagger \in S \to \boldsymbol{d}(x) \in \boldsymbol{C}(\boldsymbol{S}),$$

Notice that $x\,\mathcal{L}^*(S)y \Leftrightarrow\ x^* = y^* \Leftrightarrow \boldsymbol{r}(x) = \boldsymbol{r}(y)$. Therefore $x\mathcal{L}^*(S)y$ if and only if $\boldsymbol{r}(x) = \boldsymbol{r}(y)$ in $\boldsymbol{C}(\boldsymbol{S})$.

Similarly, $x\,\mathcal{R}^*(S)y$ if and only if $\boldsymbol{d}(x) = \boldsymbol{d}(y)$ in $\boldsymbol{C}(\boldsymbol{S})$.

Thus, the structure in $\boldsymbol{C}$ is not lost in $\boldsymbol{D}$.

F is a morphism between two categories. Hence **F** is a functor. As a functor, **F** preserves, in category $\boldsymbol{D}$, the identities and the isomorphisms in category $\boldsymbol{C}$. Moreover, **F** carries the commutative quality of diagram $\boldsymbol{C}$ to $\boldsymbol{D}$.

Thus, $\boldsymbol{D}$-diagram commutes.

Proposition 3.3.5. $\boldsymbol{C}(\boldsymbol{S})$ is cancellative (with respect to $''\ \cdot\ ''$)

Proof. Assuming $\exists x \cdot z, \exists y \cdot z$ and let $x \cdot z = y \cdot z$. By definition of the product, $x \cdot z\ \Rightarrow x^* = z^\dagger$ and $y \cdot z\ \Rightarrow y^* = z^\dagger$. Therefore, $x^* = y^*$. That is, $\boldsymbol{r}(x) = \boldsymbol{r}(y)$. But, $x \cdot z = y \cdot z$ implies that $\boldsymbol{d}(x \cdot z) = \boldsymbol{d}(y \cdot z)$ and this gives $\boldsymbol{d}(x) = \boldsymbol{d}(y)$. So we have $\boldsymbol{r}(x) = \boldsymbol{r}(y)$ and $\boldsymbol{d}(x) = \boldsymbol{d}(y)$. Therefore, $x = y$. Similarly, if $\exists z \cdot x, \exists z \cdot y$ and $z \cdot x = z \cdot y$, then $x = y$.

Theorem 3.3.6. $[\boldsymbol{C}(\boldsymbol{S}),\otimes]$ is a type A semigroup, where $\boldsymbol{C}(\boldsymbol{S})$ is the inductive category above.

Proof. Obviously,$[\boldsymbol{C}(\boldsymbol{S}),\otimes]$ is a semigroup. For each $a \in \boldsymbol{C}(\boldsymbol{S})$, we define $a^\dagger = \boldsymbol{d}(a)$ and $a^* = \boldsymbol{r}(a)$.

$$a \otimes a^* = a|\boldsymbol{r}(a) \wedge a^*$$

$$\Rightarrow a \otimes a^* = a|a^* \wedge a^* = a|a^* = a$$

Thus, with respect to $\otimes$, a^* is a right identity for a. Similarly, $a^\dagger$ is a left identity for a with $\otimes$.

$$a^* \otimes a^* = a^*|a^* \wedge a^* = a^* \text{ and } a^\dagger \otimes a^\dagger = a^\dagger$$

That is, for each $a \in [\boldsymbol{C}(\boldsymbol{S}),\otimes]$, $a^*$and $a^\dagger$ are idempotents, which implies that $[\boldsymbol{C}(\boldsymbol{S}),\otimes]$ is full in $\boldsymbol{C}(\boldsymbol{S})$.

Let e, f be idempotents in $[\boldsymbol{C}(\boldsymbol{S}),\otimes]$

$$e \otimes f = (e|e \wedge f) \cdot (e \wedge f|f)$$

$e \wedge f \leq e, e \wedge f \leq f$. Therefore, $e|e \wedge f = e \wedge f$ and $e \wedge f|f = e \wedge f$ by (5) in section 3.5.

Therefore, $e \otimes f = (e \wedge f) \cdot (e \wedge f) = e \wedge f$. Similarly $f \otimes e = f \wedge e = e \wedge f$. Thus, the idempotents in $[\boldsymbol{C}(\boldsymbol{S}),\otimes]$ commute.

Next, we show that $a^* = \boldsymbol{r}(a)$ is the unique idempotent in$\mathcal{L}_a^*$.

Let $x, y \in \boldsymbol{C}(\boldsymbol{S})^{\boldsymbol{1}}$ and assuming $a \otimes x = a \otimes y$

$a \otimes x = a \otimes y$ implies that $(a \otimes x)^\dagger = (a \otimes y)^\dagger$

That is, $[(a|a^* \wedge x^\dagger) \cdot (a^* \wedge x^\dagger|x)]^\dagger = [(a|a^* \wedge y^\dagger) \cdot (a^* \wedge y^\dagger|y)]^\dagger$

That is, $\boldsymbol{d}[(a|a^* \wedge x^\dagger) \cdot (a^* \wedge x^\dagger|x)] = \boldsymbol{d}[(a|a^* \wedge y^\dagger) \cdot (a^* \wedge y^\dagger|y)]$ and we have that

$\boldsymbol{d}(a|a^* \wedge x^\dagger) = \boldsymbol{d}(a|a^* \wedge y^\dagger)$.

Notice that $a^* \wedge x^\dagger, a^* \wedge y^\dagger \in \boldsymbol{C}(\boldsymbol{S})_o$ and by the definition of corestriction, $a|a^* \wedge x^\dagger \leq a$ and $a|a^* \wedge y^\dagger \leq a$. So we have:

$a|a^* \wedge x^\dagger \leq a$, $a|a^* \wedge y^\dagger \leq a$ and $\boldsymbol{d}(a|a^* \wedge x^\dagger) = \boldsymbol{d}(a|a^* \wedge y^\dagger)$

Therefore, $a|a^* \wedge x^\dagger = a|a^* \wedge y^\dagger$.

Now, $a \otimes x = [(a|a^* \wedge x^\dagger) \cdot (a^* \wedge x^\dagger|x)] = [(a|a^* \wedge y^\dagger) \cdot (a^* \wedge x^\dagger|x)]$

By assumption, $a \otimes x = a \otimes y = [(a|a^* \wedge y^\dagger) \cdot (a^* \wedge y^\dagger|y)]$

That is, $[(a|a^* \wedge y^\dagger) \cdot (a^* \wedge x^\dagger|x)] = [(a|a^* \wedge y^\dagger) \cdot (a^* \wedge y^\dagger|y)]$ and by cancellation,

$a^* \wedge x^\dagger|x = a^* \wedge y^\dagger|y$. Hence, by (2.7.7), $a^* \otimes x = a^* \wedge x^\dagger|x = a^* \wedge y^\dagger|y = a^* \otimes y$

Thus, $\forall a \in \boldsymbol{C}(\boldsymbol{S}), (a, a^*) \in \mathcal{L}^*[\boldsymbol{C}(\boldsymbol{S}),\otimes]$. Similarly, $(a, a^\dagger) \in \mathcal{R}^*[\boldsymbol{C}(\boldsymbol{S}),\otimes]$.

Since idempotents in $[\boldsymbol{C}(\boldsymbol{S}),\otimes]$ commute, a^* is the unique idempotent in $\mathcal{L}_a^*[\boldsymbol{C}(\boldsymbol{S}),\otimes]$ and $a^\dagger$ the unique idempotent in $\mathcal{R}_a^*[\boldsymbol{C}(\boldsymbol{S}),\otimes]$.

Now, we show that $a \otimes e = (a \otimes e)^\dagger \otimes a$ and $e \otimes a = a \otimes (e \otimes a)^*$

By (2.7.7), $(a \otimes e)^\dagger \otimes a = [(a \otimes e)^\dagger \wedge a^\dagger]|a = [(a|a^* \wedge e)^\dagger \wedge a^\dagger]|a$

Since $a^* \wedge e \in \boldsymbol{C}(\boldsymbol{S})_o$, by definition of corestriction , $a|a^* \wedge e \leq a$. So that $(a|a^* \wedge e)^\dagger \leq a^\dagger$.

Therefore, $(a \otimes e)^\dagger \otimes a = [(a|a^* \wedge e)^\dagger \wedge a^\dagger]|a = (a|a^* \wedge e)^\dagger|a$.

Since $a|a^* \wedge e \leq a, (a|a^* \wedge e)^\dagger|a = \boldsymbol{d}(a|a^* \wedge e)|a = a|a^* \wedge e$. So that

$$[(a \otimes e)^\dagger \otimes a]^\dagger = [(a|a^* \wedge e)^\dagger|a]^\dagger = [a|a^* \wedge e]^\dagger = (a \otimes e)^\dagger.$$

So we have: $a \otimes e = a|a^* \wedge e \leq a, (a \otimes e)^\dagger \otimes a = (a|a^* \wedge e)^\dagger|a \leq a$

And $\boldsymbol{d}[(a \otimes e)^\dagger \otimes a] = [(a \otimes e)^\dagger \otimes a]^\dagger = (a \otimes e)^\dagger = \boldsymbol{d}(a \otimes e)$.

So that, by (2.7.3), $(a \otimes e)^\dagger \otimes a = a \otimes e$. Similarly, $e \otimes a = a \otimes (e \otimes a)^*$.

We therefore conclude that $[\boldsymbol{C}(\boldsymbol{S}),\otimes]$ is an ample semigroup.

We denote by $\boldsymbol{S}[\boldsymbol{C}(\boldsymbol{S})]$ this ample semigroup constructed from the inductive category $\boldsymbol{C}(\boldsymbol{S})$**.**

Having constructed an ample semigroup from an inductive category, doing the same for an inverse semigroup is straightforward and that is what we wish to do next.

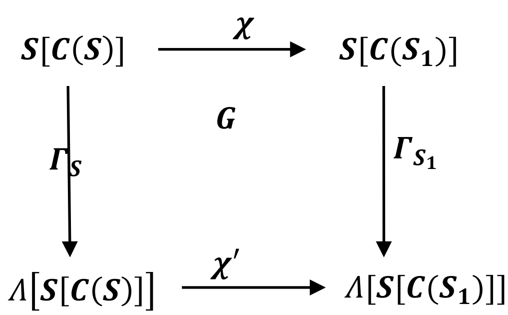
Theorem 3.3.7. $[\boldsymbol{C}(\boldsymbol{S_1}),\otimes]$ is an inverse semigroup, where $\boldsymbol{C}(\boldsymbol{S_1})$ is the inductive category above

Proof. We put $aa^{-1} = \boldsymbol{d}(a)$ and $a^{-1}a = \boldsymbol{r}(a)$. For each $a \in \boldsymbol{C}(\boldsymbol{S_1})$.
$aa^{-1} \otimes aa^{-1} = aa^{-1}$ and $a^{-1}a \otimes a^{-1}a = a^{-1}a$. So, aa^{-1} and $a^{-1}a$ are idempotents in $[\boldsymbol{C}(\boldsymbol{S_1}),\otimes]$. As shown in type A case, idempotents in $[\boldsymbol{C}(\boldsymbol{S_1}),\otimes]$ commute. What remains is to simply show that $[\boldsymbol{C}(\boldsymbol{S_1}),\otimes]$ is regular. For $a, b \in \boldsymbol{C}(\boldsymbol{S_1})$, if $\exists\ a \cdot b$, $\boldsymbol{r}(a) = \boldsymbol{d}(b)$.
So that, $a \otimes b = [a|\boldsymbol{r}(a)] \cdot [\boldsymbol{d}(b)|b] = a \cdot b$ by (2.7.4). Thus, if $a \cdot b$ is defined in $\boldsymbol{C}(\boldsymbol{S_1})$, $'' \cdot ''$ and $\otimes$ coincide.
Let x^{-1} be the inverse of x in $\boldsymbol{C}(\boldsymbol{S_1})$ (note that the underlying set in $\boldsymbol{C}(\boldsymbol{S_1})$ is the inverse semigroup $\boldsymbol{S_1}$)

$$x \otimes x^{-1} \otimes x = (x \cdot x^{-1}) \otimes x = \boldsymbol{d}(x) \otimes x = \boldsymbol{d}(x) \cdot x = x.$$

Similarly, $x^{-1} \otimes x \otimes x^{-1} = x^{-1}$. Thus, $[\boldsymbol{C}(\boldsymbol{S_1}),\otimes]$ is regular and the idempotents commute.
We denote by $\boldsymbol{S}[\boldsymbol{C}(\boldsymbol{S_1})]$ this inverse semigroup constructed from the inductive category $\boldsymbol{C}(\boldsymbol{S_1})$.

Lemma 3.3.8 [15] Let $\phi: \boldsymbol{C} \to \boldsymbol{D}$ be an ordered functor between ordered categories $\boldsymbol{C}$ and $\boldsymbol{D}$. If $f \in \boldsymbol{C}_o$ is such that $f \leq \boldsymbol{r}(a)$ for some $a \in \boldsymbol{C}$, then $(a|f)\phi = a\phi|f\phi$. Similarly, if $f \leq \boldsymbol{d}(a)$, then $(f|a)\phi = f\phi|a\phi$
Theorem 3.3.9. $\chi : \boldsymbol{S}[\boldsymbol{C}(\boldsymbol{S})] \to \boldsymbol{S_1}[\boldsymbol{C}(\boldsymbol{S_1})]$ is a functor



Proof. This implies showing that $\eta: [\boldsymbol{C}(\boldsymbol{S}),\otimes] \to [\boldsymbol{C}(\boldsymbol{S_1}),\otimes]$is a functor with respect to $\otimes$ which amounts to showing that ϕ is a functor under $\otimes$.
Since $\boldsymbol{C}(\boldsymbol{S})$ and $\boldsymbol{C}(\boldsymbol{S_1})$ are both inductive categories, $\forall a, b \in \boldsymbol{C}(\boldsymbol{S})$, $a \otimes b$ will always exist in $\boldsymbol{C}(\boldsymbol{S})$ while $a\phi \otimes b\phi$ and $(a \otimes b)\phi$ will always exist in $\boldsymbol{C}(\boldsymbol{S_1})$. This is evident from the fact that in an inductive category, the meet of every pair of elements must exist, and consequently, the corestriction and restriction in the product $\otimes$ must exist.
Now, we simply need to show that $a\phi \otimes b\phi = (a \otimes b)\phi$.

$$a\phi \otimes b\phi = [a\phi|\boldsymbol{r}(a\phi) \wedge \boldsymbol{d}(b\phi)] \cdot [\boldsymbol{r}(a\phi) \wedge \boldsymbol{d}(b\phi)|b\phi]$$
$$= [a\phi|\boldsymbol{r}(a)\phi \wedge \boldsymbol{d}(b)\phi] \cdot [\boldsymbol{r}(a)\phi \wedge \boldsymbol{d}(b)\phi|b\phi]$$

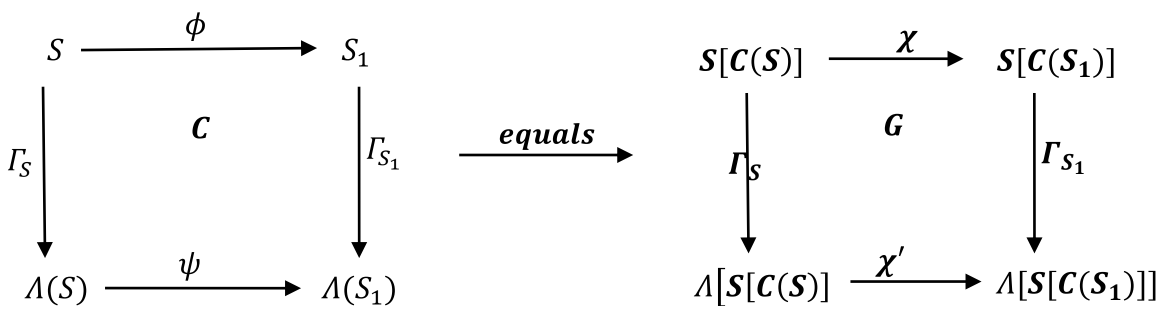
$= [a\phi|(\boldsymbol{r}(a)\wedge \boldsymbol{d}(b))\phi] \cdot [(\boldsymbol{r}(a)\wedge \boldsymbol{d}(b))\phi|b\phi]$ since ϕ is a morphism

$= [a|\boldsymbol{r}(a)\wedge \boldsymbol{d}(b)]\phi \cdot [\boldsymbol{r}(a)\wedge \boldsymbol{d}(b)|b]$

$= \big[[a|\boldsymbol{r}(a)\wedge \boldsymbol{d}(b)] \cdot [\boldsymbol{r}(a)\wedge \boldsymbol{d}(b)|b]\big]\phi$ since ϕ is a functor under $'' \cdot ''$

$= (a \otimes b)\phi$

Theorem 3.3.10



Proof. We simply need to show that $\boldsymbol{S}[\boldsymbol{C}(\boldsymbol{S})] = \boldsymbol{S}$ and $\boldsymbol{S}[\boldsymbol{C}(\boldsymbol{S_1})] = \boldsymbol{S_1}$, and since $\chi : \boldsymbol{S}[\boldsymbol{C}(\boldsymbol{S})] \rightarrow \boldsymbol{S}[\boldsymbol{C}(\boldsymbol{S_1})]$ as a functor amounts to ϕ being a functor under $\otimes$, every fact will follow, through the coincidence of $\otimes$ and the product in $\boldsymbol{S}$.

$\boldsymbol{C}(\boldsymbol{S})$ is an inductive category under $'' \cdot ''$ in which $(\forall a, b \in \boldsymbol{C}(\boldsymbol{S}))(\forall e, f \in \boldsymbol{C}(\boldsymbol{S})_o), e|a = ea, a|e = ae$ and $e \wedge f = ef$. The underlying set in $\boldsymbol{S}[\boldsymbol{C}(\boldsymbol{S})]$ is $\boldsymbol{S}$ and $a \otimes b = [a|\boldsymbol{r}(a)\wedge \boldsymbol{d}(b)] \cdot [\boldsymbol{r}(a)\wedge \boldsymbol{d}(b)|b]$.

Since $e \wedge f = ef \forall e, f \in \boldsymbol{C}(\boldsymbol{S})_o$, then $a \otimes b = [a|\boldsymbol{r}(a)\boldsymbol{d}(b)] \cdot [\boldsymbol{r}(a)\boldsymbol{d}(b)|b]$

and since $e|a = ea, a|e = ae$, we have

$$a \otimes b = [a|\boldsymbol{r}(a)\boldsymbol{d}(b)] \cdot [\boldsymbol{r}(a)\boldsymbol{d}(b)|b] = (aa^*b^\dagger) \cdot (a^*b^\dagger b) = (ab^\dagger) \cdot (a^*b)$$

and since $(ab^\dagger)^* = (a^*b^\dagger)^* = a^*b^\dagger = (a^*b)^\dagger$, then $a \otimes b = (ab^\dagger) \cdot (a^*b) = ab^\dagger a^*b = ab$

Thus, the operations of $\boldsymbol{S}[\boldsymbol{C}(\boldsymbol{S})]$ and $\boldsymbol{S}$ coincide, and since they have the same underlying set, $\boldsymbol{S}[\boldsymbol{C}(\boldsymbol{S})] = \boldsymbol{S}$.

Similarly, the operations of $\boldsymbol{S}[\boldsymbol{C}(\boldsymbol{S_1})]$ and $\boldsymbol{S_1}$ coincide, and $\boldsymbol{S}[\boldsymbol{C}(\boldsymbol{S_1})] = \boldsymbol{S_1}$.

CONFLICT OF INTEREST

The authors declare that there is no conflict of interest.

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