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QTAG-MODULES HAVING ALL HIGH SUBMODULES RELATIVE CLOSURE

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Abstract. This article is concerned with the investigation of two questions. The first question is: What *QTAG*-modules M with $M^1 = 0$ are direct summands of all *QTAG*-modules containing them as high submodules? The second question is the following: What are the submodules whose relative closures are large submodules? In addition, a necessary and sufficient condition for a submodule to be h-pure-absolute summand is obtained.

Keywords: QTAG-modules; high submodules; relative closures; h-pure-absolute summands.

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1. Introduction and Background Material

The close association of abelian group theory and module theory has been extensively studied in the literature. As a generalization of the torsion abelian groups, the *TAG*-module was introduced by Singh [20] in 1976. Since then, this theory has enjoyed rapid development. The notion of *TAG*-modules is one of the most important tools in module theory, and many authors with an interest in module theory have made attempts to generalize the theory of abelian groups. Its importance lies behind the fact that this module can be applied in order to generalize torsion

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abelian group accurately. This kind of *TAG*-module has been widely investigated. For details on the abelian groups that behave like modules, refer to [4, 22].

Over an arbitrary (associative, unitary) ring R, a module M is called a *TAG*-module if it satisfies the following two conditions relating to uniserial modules.

(i) Every finitely generated submodule of any homomorphic image of M is a direct sum of uniserial modules.

(ii) Given any two uniserial submodules U_1 and U_2 of a homomorphic image of M, for any submodule N of U_1, any non-zero homomorphism $\phi : N \to U_2$ can be extended to a homomorphism $\psi : U_1 \to U_2$, provided that inequality $d(U_1/N) \leq d(U_2/\phi(N))$ holds.

In 1987, Singh [21] followed this up in his other work, "Abelian groups like modules" and introduced the notion of *QTAG*-module in a natural way arising from his investigation in [20]; this notion has been of interest in module theory ever since. The study was then followed by numerous developments on the topic. In particular, a lot of variations of the concept have been introduced and studied (see, for example, [1, 2, 8, 19] and the references cited therein). Unsurprisingly, a lot of these advancements are similar to the earlier developments made in the theory of torsion abelian groups. The present paper is a natural generalization of the research conducted in [3, 11, 14] and contributes to existing knowledge on the structure of *QTAG*-modules.

There is a rather natural question which arises in connection with the high submodules of the *QTAG*-modules. That is, what *QTAG*-modules M with $M^1 = 0$ are direct summands of all *QTAG*-modules containing them as high submodules? The purpose here is to investigate how the closures of high submodules as discussed in [2] are preserved under the direct summand and thereby to explore in detail some more specific properties of these high submodules. Besides, we shall also examine the characteristic properties of large submodules as studied in [19], especially when large submodules are relatively closed. We consider the following question. What are the submodules whose relative closures are large submodules? This question is partially answered in subsection (b) and it is proved that a high submodule of a large submodule is relatively closed in a high submodule of the *QTAG*-module itself. We also explore some of the results of h-pure-absolute summands as well as the inheritance of the *QTAG*-module structure under different types of high submodules. Note that *QTAG*-modules with an h-pure-absolute summand are also studied in [8] in another aspect. The goal of this paper is to provide a comprehensive study of the two questions stated above. The work is organized as follows. In the next section, we establish our main results which are stated in three different subsections. In the final section, we list some interesting left-open problems.

Some of the fundamental concepts used in this paper have already appeared in one of the previous works by the co-authors of [9], which is necessary for our successful presentation. Throughout our discussion, all rings below are assumed to be associative and with non-zero identity element; all modules are assumed to be unital *QTAG*-modules. A uniserial module M is a module over a ring R, whose submodules are totally ordered by inclusion. This means simply that for any two submodules S_1 and S_2 of M, either $S_1 \subseteq S_2$ or $S_2 \subseteq S_1$. An element $u \in M$ is uniform, if uR is a non-zero uniform (hence uniserial) module and for any module M with a unique decomposition series, the symbol $d(M)$ will denote its decomposition length. If u is a uniform element of M (i.e., $u \in M$), then $e(u)$ is called the exponent of u and $e(u) = d(uR)$. As usual, for such a module M, we set the height of u in M as $H_M(u) = \sup\{d(vR/uR) : v \in M,\ u \in vR$ and v uniform$\}$. For every non-negative integer t, $H_t(M) = \{u \in M |\ H_M(u) \geq t\}$ denotes the t-th copies of M which can be viewed as a submodule of M consisting of all elements of height at least t. The topology of M, which admits as a base of neighborhoods of zero, is known as the h-topology. This topology has the submodules $H_t(M)$ with $t = 0, 1, \ldots, \infty$. In this way, a submodule S of M is called the closure in M if $\overline{S} = \cap_{t=0}^{\infty}(S + H_t(M))$ and S is closed with respect to this topology provided that $\overline{S} = S$. For a module M, the letter M^1 will always denote in the sequel the submodule of M, containing elements of infinite height. Moreover, we denote by $Soc(M)$, the socle of M, i.e., the sum of all simple submodules of M. For any $t \geq 0$, $Soc^t(M)$ is defined inductively as follows: $Soc^0(M) = 0$ and $Soc^{t+1}(M)/Soc^t(M) = Soc(M/Soc^t(M))$.

We also add some additional background material from [7, 16]. The module M is termed h-divisible if $M = M^1 = \cap_{t=0}^{\infty} H_t(M)$, or equivalently, if $H_1(M) = M$. With this in hand, we say that a module M is h-reduced if it does not contain any h-divisible submodule. Moreover, the module M is defined to be bounded if $\exists\, t \geq 0$ such that $H_M(u) \leq t$ for some $u \in M$. A submodule S of M is named h-pure in M if for every non-negative integer t the equality $S \cap H_t(M) = H_t(S)$ holds. A submodule S_1 of M is said to be essential in M if $S_1 \cap S_2 = 0$ for every non-zero submodule S_2 of M. A submodule S of M is termed a basic submodule of M if S is an h-pure submodule of M, S is a direct sum of uniserial modules and M/S is h-divisible. A fully invariant submodule $S_1 \subseteq M$ is large if $S_1 + S_2 = M$, for every basic submodule S_2 of M.

It is well to note that various results for *TAG*-modules are also valid for *QTAG*-modules [17]. For a better understanding of the topic mentioned here, one must go through the papers [10, 13, 18]. Our notions and notations that we use in this paper are standard and essentially follow the classical books [5, 6]; for the specific ones, we refer the readers to [23].

2. Main Concepts and Results

The aim of the present section is to answer the stated above questions due to [11, 14] in some extra conditions on modules and, in particular, a condition on modules that every finitely generated submodule of any homomorphic image of the module is a direct sum of uniserial modules while the ring is associative with unity.

To develop the study, we shall examine here certain kinds of submodules of *QTAG*-modules. So, we start with the following subsection.

(a) High submodules.

The study of high submodules and its fascinating properties makes the theory of *QTAG*-modules more interesting. A submodule S of M is high [12], if it is maximal with respect to the property $S \cap H_\omega(M) = \{0\}$. There are numerous observations of high submodules more enlightening than the definition. If S is a high submodule of M, then S is h-pure in M and M/S is h-divisible. If all high submodules of M are the direct sum of uniserial modules, then M is said to be a Σ-module. If one high submodule of M is a direct sum of uniserial modules, then all high submodules of M are isomorphic and M is a Σ-module (see [2]). Though it has been stated in a variety of forms by a number of characterizations. Here, we follow a somewhat different path and explore a new problem of determining the *QTAG*-modules in light of high submodules.

Now, we state the following. If M is a *QTAG*-module and M' is a minimal h-divisible module containing M^1, we will say that A is the amalgamated sum of M and M' over M^1, that is, A is the *QTAG*-module generated by the elements of M and M' such that $M \cap M' = M^1$.

It is easy to see that

$$A/M = (M+M')/M \cong M'/(M\cap M') = M'/M^1,$$

and similarly that $A/M' \cong M/M^1$.

With the aid of the above discussion, we are now ready to formulate the following lemma.

Lemma 2.1. *Let M be a QTAG-module and let M' be a minimal h-divisible module containing M^1. If A is the amalgamated sum of M and M' over M^1 and S is a high submodule of M, then the following holds.*
(i) *M is an h-pure submodule of A.*
(ii) *$A = S+M'$ such that $S \cong M/M^1$.*
(iii) *$S\cap M$ is a high submodule of M.*
(iv) *$A = S+M$.*
(v) *M is a subdirect sum of S and M'.*

Proof. (i) Suppose $x\in M, y\in A$ and $k\in Z^+$; if $ky = x$, then $y = z+a$ with $z\in M$ and $a\in M'$. Therefore, $ka = x-kz$ is a uniform element of $M\cap M' = M^1$ and hence there exists an element $b\in M$ such that $kb = x-kz$. This, in turn, implies that $x = k(z+b)$, and therefore M is h-pure in A, as expected.
(ii) This is immediate, since h-divisible submodules are always direct summands and, as observed above, we have $A/M' \cong M/M^1$.
(iii) In order to show that $S\cap M$ is high in M, it suffices to show that $[(S\cap M)+xR]\cap M^1 \neq 0$, for every $x\in M\backslash S$. In order to do this, among all uniform elements in $M\backslash S$, choose x such that $x = y+z$ with $y\in S, z\in M'$ and $z\neq 0$. Since M' is the minimal h-divisible module containing M^1, it follows that $0\neq kz\in M'$ for some integer k. Therefore, $ky = k(x-z)\in S\cap M$. Hence $kz = kx-ky$ is a non-zero element of $\left[(S\cap M)+xR\right]\cap M'$.
(iv) Since $Soc(M^1) = Soc(M')$, we have $Soc(A)\subseteq S+M$. Without loss of generality, we assume that $Soc^k(A)\subseteq S+M$. In order to show that $Soc^{k+1}(A)\subseteq S+M$, we need only consider the uniform elements in $Soc^{k+1}(M')$. Note that $x'\in M^1$ for some $x\in Soc^{k+1}(M)$ and $d(xR/x'R) = k$, then $x-y\in Soc^k(A)\subseteq S+M$, for some $y\in M$. It follows that $x\in S+M$, and so $Soc^{k+1}(A)\subseteq S+M$. Hence, $A\subseteq S+M$.

What remains to show is $M' \subseteq S+M$. In fact, if $x \in M'$ such that $x \neq 0$, then $0 \neq kx \in M'$ for some integer k. Therefore, there exists a uniform element $y \in M$ such that $x-y \in Soc(A) \subseteq S+M$, and consequently $x \in S+M$, as required.

(v) Since $A = S+M = M'+M$, for each $x \in S$ there exists $y \in M$ and $z \in M'$ such that $y = x+z$, and similarly, for each $z \in M'$ there exists $y \in M$ and $x \in S$ such that $y = x+z$. Thus, M is a subdirect sum of S and M'. □

As an immediate consequence, we have the following corollary.

Corollary 2.1. *Let M be a QTAG-module, and let M' be a minimal h-divisible module containing the module P. If Q is the module without elements of infinite height having an h-pure submodule R such that $Q/R \cong M'/P$, then any subdirect sum M of Q and M' with kernels P and R is an h-pure submodule of $Q+M'$ such that*

(i) *$M^1 = P$,*

(ii) *$M/M^1 \cong Q$, and*

(iii) *R is a high submodule of M.*

All we have to do is check the validity of the following first main attainment.

Theorem 2.1. *If M is a QTAG-module with $H_\omega(M) = 0$, then M is a direct summand of every QTAG-module containing it as a high submodule if and only if $\overline{M}$ is closed.*

Proof. Suppose that $\overline{M}$ is closed. Consider Q as a $QTAG$-module such that $Q \supset M$ as a high submodule with $Q/M = M'$, a minimal h-divisible module. Then Q can be represented as a subdirect sum of S and M' such that $S \cong Q/Q^1$ and $M = S \cap Q$, where S is high in M. Therefore, $M'/Q^1 \cong S/M \cong \left(S/\overline{M}\right)/\left(M/\overline{M}\right)$. Since $M/\overline{M}$ is an h-pure submodule of $S/\overline{M}$ and M'/Q^1 is closed, $S/\overline{M} = Q'/\overline{M} + M/\overline{M}$ for some closed $QTAG$-module Q'. Since $\overline{M}$ is h-pure in Q', $Q' = N + \overline{M}$ such that $N \cong M'/Q^1$, where N is a submodule of $M+M'$. Thus, $S = N+M$. Finally, since M is a direct summand of S, it is fairly to see that M is a direct summand of Q.

Suppose now that $\overline{M}$ is not closed. Let C be the closure of M. Then M is an h-pure submodule of C such that C/M is h-divisible. Since $\overline{C}$ is closed, $(M+\overline{C})/M$ is the closed submodule of C/M. Let $S = M+\overline{C}$, then $M'/D \cong S/M$ such that $D \supseteq M'$ and D is a direct sum of uniserial modules. If Q is a subdirect sum of S and M' with kernels M and D, then Q is an h-reduced module having M as a high submodule. The proof is over. □

Next, we concentrate on the following theorem.

Theorem 2.2. *Let S be a high submodule of a QTAG-module M such that $\overline{S}$ is a summand of S. Then $\overline{(M/M^1)}$ is a summand of M/M^1.*

Proof. Assume that $S = S_1 + S_2$ is a high submodule of M, where S_1 is closed and S_2 is not closed. Knowing this, we see that there is a minimal h-divisible module M' containing M^1 such that $A = S + M'$, where A is the amalgamated sum of M and M' over M^1. After this, let us assume that $M \cap S = S_1 + S_2$, then

$$S/(S_1 + S_2) = S/(S \cap M) \cong (S + M)/M = A/M \cong M'/M^1.$$

Since M'/M^1 is closed and $(S_1 + S_2)/S_1$ is an h-pure submodule of S/S_1, then

$$S/S_1 = N/S_1 + (S_1 + S_2)/S_1,$$

for some submodules N of S. Hence $S = N + S_2$. Since N/S_1 is closed, N is a closed module. Consequently, $\overline{S}$ is a summand of S and $S \cong M/M^1$, as expected. □

The following corollary is of some interest.

Corollary 2.2. *If $S = S_1 + S_2$ is a high submodule of M such that S_1 is closed and S_2 is not closed, then $M = N + S_2$ with N/S_1 is h-divisible, for some submodules N of $M + M'$, and a minimal h-divisible module M' containing M^1.*

Proof. Assume that $A = S + M'$ is an amalgamated sum over M^1 such that $S_3 \cap M = S$, for some high submodules S_3 of M. Then $A = (S_4 + S_2) + M'$ for some submodules S_4 of S. Therefore, $M = N + S_2$, where $N = M \cap (S_4 + M')$. Henceforth, we see that

$$N/S_1 \cong M/(S_1 + S_2) = M/(S_3 \cap M) \cong M',$$

and we are finished. □

We will now argue the following theorem.

Theorem 2.3. *If M is a Σ-module, then every high submodule of M is an endomorphic image of M.*

Proof. Assume $S = S_1 + S_2$ is a high submodule of M where S_1 is closed and S_2 is not closed. Then we have $M/M^1 \cong S_3 + S_2$ such that S_3 is closed and $S_3 \supseteq S_4$, an h-pure submodule of M. If S_4 is a direct sum of uniserial modules, then S_4 is a basic submodule of M. Hence, S_4 is an endomorphic image of S_3. Consequently, it follows that S is an endomorphic image of M. □

We are now ready to give our desired example.

Example 2.1. The restrictions in Theorem 2.3 are necessary. Indeed, let $S = \Sigma_{t=1}^{\infty} S_t$ is a submodule of a *QTAG*-module M such that S_t is a direct sum of uniserial modules of exponent t. Then S is an h-pure submodule of the closure $\overline{S}$ of S in M and $\overline{S}/S$ is isomorphic to $2^{\aleph_0}$ copies of S_t. Setting $P = \Sigma_{\gamma\in\Gamma}\{x_\gamma\}$, where $\{x_\gamma\} \cong S_t$ for each γ and $g(\Gamma) = 2^{\aleph_0}$. Hence, according to Corollary 2.1, one may see that $M'/P \cong \overline{S}/S$ where M' is a minimal h-divisible module containing P. Furthermore, if M is a subdirect sum of $\overline{S}$ and M' with kernels $\overline{S}$ and P, then M is a Σ-module such that $M/M^1 \cong \overline{S}$. We are finished.

(b) Relative closures.

In this brief subsection, we begin with the following useful concept.

Definition 2.1. *A submodule S_1 of a QTAG-module M is called the relative closure of S_2 in M if and only if $S_1/S_2 = (M/S_2)^1$.*

Remark 2.1. *It is easy to see that a submodule $S \subseteq M$ is relatively closed under the h-topology if and only if $(M/S)^1 = 0$.*

The following proposition is elementary, but, however, is useful.

Proposition 2.1. *Suppose M is a QTAG-module with a large submodule L of M. Then L is the relative closure of every submodule S of L such that L/S is h-divisible.*

Proof. Note that if L/S is h-divisible, it immediately follows that

$$M/S = N/S \oplus L/S$$

for some submodules N of M. Now, appealing to [19, Corollary 9], we get $(N/S)^1 = 0$. Henceforth, according to [19, Corollary 10], we may write $(M/S)^1 = L/S$, and we are done. □

A valuable consequence is the following.

Corollary 2.3. *Suppose M is a QTAG-module with a large submodule L of M. Then L is the relative closure of $L \cap B$ for every basic submodule B of M.*

Proof. This follows immediately from [16] and Proposition 2.1. □

We are now in a position to state and prove the second main result of this article.

Theorem 2.4. *Let L be a large submodule of a QTAG-module such that $M^1 \neq 0$. Then a high submodule S of L is relatively closed in a high submodule of M.*

Proof. Let S be a high submodule of L, then L/S is h-divisible and we have

$$M/S = L/S \oplus N/S,$$

for some submodules N of M. Now, if we are appealing to [19, Corollary 10], it is easy to see that $N \cap M^1 = 0$. In order to complete the proof, it is sufficient to show that $(N + aR) \cap M^1 \neq 0$ for every $a \in L, a \notin N$. Suppose on contrary that $(N + aR) \cap M^1 = 0$. Then we have $(S + aR) \cap L^1 = 0$, a contradiction. Hence, it consequently follows that N is a high submodule of M. Since $(N/S)^1 = L/S$, we have $(N/S)^1 = 0$, and the result follows. □

The last proof actually shows the following.

Corollary 2.4. *If L is a large submodule of a QTAG-module M and S is a high submodule of L such that S is countably generated, then S is relatively closed in a maximal basic submodule of M.*

The next is well to be documented.

Corollary 2.5. *Let L be a large submodule of a QTAG-module M and L contain a countably generated high submodule S. Then L and M are Σ-modules.*

Proof. If S is a countably generated high submodule of L, then clearly S is a maximal basic submodule of L and hence by [12], L is a Σ-module. One seeing readily in view of Corollary 2.4 that M is a Σ-module. □

The following assertion relates the above concept to our investigation.

Theorem 2.5. *Let L be a large submodule of a QTAG-module M such that $M^1 \neq 0$. Then M is a Σ-module if and only if L is a Σ-module.*

Proof. If M is a Σ-module and L is large in M, then $L^1 = L \cap M^1$. Embed a high submodule S of L in a high submodule K of M. Since M is a Σ-module, K is a direct sum of uniserial modules and hence so is S. Therefore, it follows from [2, Theorem 2.3] that L is a Σ-module.

Conversely, if L is a Σ-module and S is a high submodule of L. Then S is a direct sum of uniserial modules and

$$M/S = L/S \oplus N/S,$$

where N (as in Theorem 2.4) is a high submodule of M. Knowing this, with the aid of [19, Lemma 2], we get N to be the maximal basic submodule of M. Hence M is a Σ-module. □

(c) h-pure-absolute summands.

The definitions of the investigated classical sorts of some submodules of the *QTAG*-modules can be found in [15] and [8], respectively. Nevertheless, for the completeness of the exposition, we shall recall a part of them once again.

We say that a submodule S_1 of M is S_2-high, if $S_1 \cap S_2 = 0$ and S_1 is maximal with respect to this intersection, that is, it is not properly contained in any different submodule of M having the same property. It is self-evident that all S_2-high submodules are bounded if and only if for every h-pure submodule S_1 of an h-reduced *QTAG*-module M containing S_2, M/S_1 is a direct sum of bounded modules.

Likewise, a submodule T of a *QTAG*-module M is h-pure-S-high in M if it is maximal among the h-pure submodules disjoint from S for some submodules S of M. It was seen that all h-pure-S-high submodules of M are S-high in M.

In addition, a submodule S of M is said to be an h-pure-absolute summand if for every h-pure-S-high submodule T of M, $M = S \oplus T$. It is well-known that all h-pure-absolute summands are absolute summands for some submodules S of M.

Therefore, we come to the following theorem.

Theorem 2.6. *Suppose $\{S_1, S_2\}$ is a pair of submodules of a QTAG-module M such that $S_1 \cap S_2 = 0$. Then S_1 is an h-pure-absolute summand of M if and only if S_2 is h-pure in M and $S_1 \cap S_2 = 0$ implies $S_1 + S_2$ is h-pure in M.*

Proof. Let S_1 be an h-pure-absolute summand of M, there is an h-pure-S_1-high submodule T such that $S_2 \subseteq T$ and $M = S_1 \oplus T$. Now,

$$(S_1+S_2)+H_k(M) = (S_1+S_2)\cap(H_k(S_1)+H_k(T)) = H_k(S_1)+(S_1+S_2)\cap H_k(T).$$

It is easy to see that $(S_1+S_2)\cap H_k(T) = H_k(S_2)$. Hence, S_1+S_2 is an h-pure submodule of M.

Conversely, let T be an h-pure-S_1-high submodule of M. Then by [1, Corollary 2.6], T is S_1-high in M. Hence S_1+T is an essential submodule of M and $Soc(S_1+T) = Soc(M)$. This, in turn, implies that S_1+T is h-pure in M, and so $M = S_1 \oplus T$. Consequently, S_1 is an h-pure-absolute summand of M. □

The next result exhibits a family of submodules of a *QTAG*-module which are h-pure-absolute summands.

Theorem 2.7. *Let S be a submodule of a QTAG-module M and suppose $S = \oplus_\lambda Soc(H_\lambda(M))$. Then S is an h-pure-absolute summand of M.*

Proof. If there are no h-pure-S-high submodules, then S is an h-pure-absolute summand. Now suppose that T is an h-pure-S-high submodule of M. We assume that $\overline{M} = S \oplus N$, where $\overline{M}$ is the closure of M and N is any submodule of M. First, we show that T contains N. Then, it is enough to show that $Soc(H_\lambda(M)) \subset T$. Let $x \in Soc(H_\lambda(M))$. Then we have $\langle T, xR\rangle \cap S \neq 0$ for $x \notin T$. Therefore, there exists $k \in Z^+, y \in T$ and $z \in S$ such that $z \neq 0$ and $kx+y = z$. Clearly $kx \neq 0$. Let $e(kx) = \gamma$, with $\gamma \geq 1$. Then we have $H_\gamma(kx)+H_\gamma(y) = H_\gamma(z) \in S\cap T = 0$. Therefore $H_\gamma(z) = 0$ and so $z = 0$, a contradiction. Hence, we get $x \in T$ and $Soc(H_\lambda(M)) \subseteq T$.

Next, suppose $a \in M$ and $a \notin T$. Then $\langle aR, T\rangle \cap S \neq 0$. Therefore, there exists $t \in Z^+, b \in T$ and $c \in S$ such that $ta+b=c\neq 0$. Let $e(c)=n$. Then $nta+nb=nc=0$, and so $nta=-nb\in T$. Since T is h-pure in $M, \exists\ u \in T$ such that $nt(a-u)R=0$. Hence $(a-u)R$ is closed and thus $a-u=v+w$, where $v\in S$ and $w\in N$. Thus we have $a=(u+w)+v\in T\oplus S$, and S is an h-pure-absolute summand of M. □

Now, we give a necessary and sufficient condition for a $QTAG$-module to contain a submodule S for which no S-high submodule is h-pure.

Theorem 2.8. *Let M be a QTAG-module. Then there exists a submodule S of M for which no S-high submodule is h-pure if and only if $\overline{M}$ is not a direct summand of M.*

Proof. The sufficiency follows directly from Theorem 2.7.

As for the necessity, suppose that S is a submodule of M for which no S-high submodule is h-pure. Consider a submodule S_1 such that $S_1/\overline{S}$ is $(S/\overline{S})$-high in $M/\overline{S}$. Then S_1 is an h-pure submodule of M containing $\overline{M}$, and it is plainly observed that $M=\overline{M}\oplus S_2$, for some submodules S_2 of M. Thus, $S_1=\overline{M}\oplus(S_1\cap S_2)$. Since $S_1=\overline{M}\oplus S_3$, we have that S_3 is $\overline{M}$-high in S_1, and so S_3 is h-pure in M. Henceforth, according to [8, Theorem 1], there exists an h-pure-$\overline{S}$-high submodule S_4 of $\overline{M}$ such that $S_3\oplus S_4$ is an h-pure submodule of M. This, in turn, implies that $S_3\oplus S_4$ is an S-high submodule of M. This is a contradiction. Hence, $\overline{S_1}$ is not a direct summand of S_1, and thus proves our assertion after all. □

Finally, we are able to demonstrate the truthfulness of the following theorem.

Theorem 2.9. *Suppose $\{S_1,S_2\}$ is a pair of submodules of a QTAG-module M such that $S_2=\oplus_\lambda Soc(H_\lambda(M))$. Then there exists a unique S_1-high submodule if and only if $S_1=0$ or S_1 is an essential submodule of the S_2-high submodule of M.*

Proof. Let S_3 be an S_2-high submodule of M, and let S_4 be an essential submodule of S_3. Then, it is readily seen that S_4-high submodules are precisely S_3-high submodules. We then turn to claim if S_5 is S_3-high in M then $S_5=S_2$. Note that since S_2 is S_3-high in M, it suffices to show that $S_2\subset S_5$. In fact, we claim that $Soc(H_k(M))\subset S_5$ for each $k\in Z^+$. Let $x\in Soc(H_k(M))$ and

$x \notin S_5$, then there exist $y \in S_5$, $z \in S_3$ such that $kx + y = z \neq 0$; but if $e(kx) = \alpha$ and $\alpha \geq 1$, then $d(yR/zR) = \alpha$. This, in turn, implies that $z = 0$, a contradiction. Therefore, $S_5 = S_2$.

Suppose now that S_1 has a unique S_1-high submodule S_4 and S_1 is not essential in M, where $S_1 \neq 0$. Observe that $Soc(S_1) \oplus Soc(S_4) = Soc(M)$. We show that either $Soc(S_1) = 0$ or $Soc(S_4) = 0$. In fact, if $Soc(S_1) \neq 0$ and $Soc(S_4) \neq 0$, then there exist $x \in Soc(S_1)$ and $a \in Soc(S_4)$ and since $\langle aR \rangle \cap S_1 = 0 = S_1 \cap \langle (x+a)R \rangle$, a and $x+a$ are both in S_4. This is a contradiction. We show now that $Soc(S_4) \neq 0$ implies $Soc(H_k(M)) \subset S_4$. Note that S_4 is necessarily a closed submodule of M and from [8, Corollary 1], S_4 is an h-pure submodule of M. Now we let $Soc(S_4) = Soc(M)$, and let b be any uniform element of $Soc(H_k(M))$ such that $e(b) = k$. Then $b' \in Soc(M) = Soc(S_4)$ where $d(bR/b'R) = k-1$. From the h-purity of S_4 in M, we have $b' = c'$ such that $d(bR/b'R) = d(cR/c'R) = k-1$ for some $c \in S_4$. Therefore, $b - c \in S_4$, and so $b \in S_4$. Since $S_4 \supset \oplus_\lambda (H_\lambda(M)) = S_2$ and S_4 are closed, we get $S_4 = S_2$. Thus, if S_3 is the S_4-high submodule of M containing S_1, we see that S_1 is an essential submodule of S_3, as desired. □

3. Conclusion and Open Problems

Our work has advanced the topic of torsion abelian group and contributed significant new information to module theory in general. The results that are provided here shed light on the particular properties of certain submodules and lay the groundwork for more general applications. By analyzing the properties of the torsion abelian groups and extending them through the concept of *QTAG*-modules, we have developed a comprehensive framework for understanding the behavior of high submdules, relative closures and h-pure-absolute summands within the context of *QTAG*-module theory. Various properties and findings concerning these types of concepts are elucidated. This research opens new avenues for further exploration and application of torsion abelian groups in various algebraic structures.

We terminate the investigation with some queries which are left-open yet.

Problem 3.1. *Whether or not the first question has a positive solution for the subclasses of modules: (i) h-pure-S-high submodules and (ii) h-pure-absolute summands.*

Problem 3.2. *If S_1 and S_2 are relative closures, is $S_1 \oplus S_2$ as well?*

Problem 3.3. *Does it follow that Theorem 2.4 remains true without the restriction $M^1 \neq 0$?*

Problem 3.4. *If L is a large submodule of a Σ-module M and S_1 is a relative closure of L, under what conditions there exists a relative closure S_2 of M such that $Soc(S_1) = Soc(S_2)$?*

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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