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ON EPIMORPHISMS AND SEMIGROUP IDENTITIES

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Abstract. Khan and Shah associated two natural numbers with a seminormal identity. Using these natural numbers, we further enlarge the class of homotypical identities of which both sides contain repeated variables which are preserved under epis in conjunction with a seminormal permutation identity.

Keywords: Epimorphism, dominion, saturated, permutative identity, semicommutative identity, seminormal identity.

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1. Introduction

Let U and S be any semigroups with U a subsemigroup of S. Following Isbell [5], we say that U dominates an element d of S if for every semigroup T and for all homomorphisms $\alpha, \beta : S \to T$, $u\alpha = u\beta$ for all $u \in U$ implies $d\alpha = d\beta$. The set of all elements of S dominated by U is called the *dominion* of U in S, and we denote it by $Dom(U, S)$. It may easily be seen that $Dom(U, S)$ is a subsemigroup of S containing U. A semigroup U is said to be *saturated* if $Dom(U, S) \neq S$ for every properly containing semigroup S, and *epimorphically embedded* or *dense* in S if $Dom(U, S) = S$.

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A morphism $\alpha : S \to T$ in the category of all semigroups is called an *epimorphism* (*epi* for short) if for all morphisms β, γ, $\alpha\beta = \alpha\gamma$ implies $\beta = \gamma$. Every onto morphism is epi, but the converse is not true in general. It may be easily checked that $\alpha : S \to T$ is epi if and only if the inclusion map $i : S\alpha \to T$ is epi and the inclusion map $i : U \to S$ is epi if and only if $Dom(U, S) = S$. A variety $\mathcal{V}$ of semigroups is said to be *saturated* if all its members are saturated and *epimorphically closed* or *closed under epis* if whenever $S \in V$ and $\varphi : S \to T$ is epi in the category of all semigroups, then $T \in \mathcal{V}$ or equivalently whenever $U \in \mathcal{V}$ and $Dom(U, S) = S$, then $S \in \mathcal{V}$.

An identity μ is said to be preserved under epis in conjunction with an identity τ if whenever S satisfies τ and μ, and $\varphi : S \to T$ is an epimorphism in the category of all semigroups, then T also satisfies τ and μ; or equivalently, whenever U satisfies τ and μ and $Dom(U, S) = S$, then S also satisfies τ and μ.

An identity of the form

$$x_1 x_2 \cdots x_n \;=\; x_{i_1} x_{i_2} \cdots x_{i_n} \; (n \geq 2), \tag{1}$$

is called a permutation identity, where i is any permutation of the set $\{1, 2, 3, \ldots, n\}$ and i_k, for each k $(1 \leq k \leq n)$, is the image of k under the permutation i. A permutation identity of the form (1) is said to be nontrivial if the permutation i is different from the identity permutation. Further, a nontrivial permutation identity of the form (1) is said to be *left semicommutative* if $i_1 \neq 1$, *right semicommutative* if $i_n \neq n$, and *seminormal* if $i_1 = 1$ and $i_n = n$. Clearly, every nontrivial permutation identity is either *left semicommutative*, *right semicommutative*, or *seminormal*. A semigroup S satisfying a nontrivial permutation identity is said to be permutative, and a variety $\mathcal{V}$ of semigroups is said to be permutative if it admits a nontrivial permutation identity. For example, some of the well known permutation identities are:

$$\begin{array}{lcll} x_1x_2 & = & x_2x_1 & \text{[commutativity];} \\ x_1x_2x_3 & = & x_1x_3x_2 & \text{[left normality];} \\ x_1x_2x_3 & = & x_2x_1x_3 & \text{[right normality];} \\ x_1x_2x_3x_4 & = & x_1x_3x_2x_4 & \text{[normality].} \end{array}$$

An identity $u = v$ is said to be *preserved under epis* if for all semigroups U and S with U dense in S, if U satisfies $u = v$, then S also satisfies $u = v$.

We now consider three conditions that a semigroup identity ϕ may satisfy:

(A) ϕ is preserved under epis;

(B) each variety admitting ϕ is epimorphically closed;

(C) each variety admitting ϕ is saturated.

Condition (C) clearly implies (B), which in turn implies (A), but the reverse implications are not true in general. In [2], Higgins gave a necessary condition for a semigroup identity to satisfy condition (A): a semigroup identity is preserved under epis only if one of its sides contains no repeated variable. Isbell [5] showed that the dominion of a commutative semigroup is commutative, which yields that commutativity satisfies condition (A). Khan [6, 9] generalised this result in two directions by showing that commutativity satisfies condition (B), and that all permutation identities satisfy condition (A). In [9], Khan further showed, jointly with Higgins, that left and right semicommutative identities satisfy condition (B). Therefore, it is natural to try to determine all those semigroup identities that satisfy condition (A) in conjunction with seminormal identities. Khan [7, 8] showed that all semigroup identities in which both sides do not contain repeated variables satisfy condition (A) in conjunction with any nontrivial permutation identity, while Higgins [3] has shown that seminormal permutation identities do not satisfy condition (B) by showing that the identity $xyx = yxy$ does not satisfy (A) in conjunction with the normality identity.

It is well known that all subvarieties of a saturated variety are saturated, but the same is not true for epimorphically closed varieties in general. Thus, determining all semigroup identities whose both sides contain repeated variables and satisfy condition (A) in conjunction with any seminormal identity becomes much more difficult, but interesting.

In [10], Khan and Shah obtained some partial results towards this goal, by establishing some sufficient conditions for such identities to lie in this class. In the present paper, we further enlarge the class of homotypical identities containing repeated variables on both sides that are preserved under epis in conjunction with a seminormal identity by generalizing results Theorem 1.5, 2.7 and 3.1 of [10]. However, a full determination of all semigroup identities that satisfy condition (A) in conjunction with a seminormal identity remains an open problem.

2. Preliminaries

Now, we quote some results that will be used in rest of the paper. Our notation will be standard and we refer the reader to Clifford and Preston [1] and Howie [4] for any unexplained symbols and terminology. Further in what follows, we will often speak of a semigroup *"satisfying (1)"* to mean that the semigroup in question satisfies an identity of that type.

Result 2.1 ([8], Proposition 3.1). Let S be any permutative semigroup satisfying (1) with $n \geq 3$.

(i) For each $g \in \{2,\ 3, \ldots, n\}$ such that $x_{g-1}x_g$ is not a subword of $x_{i_1}x_{i_2}\cdots x_{i_n}$, S also satisfies the permutation identity

$$x_1x_2\cdots x_{g-1}xyx_g\cdots x_n = x_1x_2\cdots x_{g-1}yxx_g\cdots x_n.$$

(ii) If $x_1 \neq x_{i_1}$, then S also satisfies the permutation identity

$$xyx_1x_2\cdots x_n = yxx_1x_2\cdots x_n.$$

In the following result and elsewhere in the paper $S^{(m)}$, for any positive integer m and semigroup S, will denote the set of all m-fold products of elements of S.

Result 2.2 ([8], Proposition 6.3). Let S be any semigroup satisfying (1) with $n \geq 3$. Then for each $g \in \{2, 3, \ldots, n\}$ such that $x_{g-1}x_g$ is not a subword of $x_{i_1}x_{i_2}\cdots x_{i_n}$, for all $r \geq g-1$, $s \geq n-g+1$ and for all $u \in S^{(r)}, v \in S^{(s)}$, we have

$$ux_1x_2v = ux_2x_1v, \qquad \text{for all } x_1, x_2 \in S.$$

In particular, $S^{(k)}$ satisfies the normality identity for all $k \geq \max(g-1, n-g+1)$.

Result 2.3 ([9], Theorem 3.1). All permutation identities are preserved under epis.

A most useful characterization of semigroup dominions is provided by Isbell's Zigzag Theorem.

Result 2.4 ([5, Theorem 2.3] or [4, Theorem VII.2.13]). Let U be a subsemigroup of a semigroup S and let $d \in S$. Then $d \in Dom(U,S)$ if and only if $d \in U$ or there exists a series of factorizations of d as follows:

$$d = a_0t_1 = y_1a_1t_1 = y_1a_2t_2 = y_2a_3t_2 = \cdots = y_ma_{2m-1}t_m = y_ma_{2m}, \qquad (2)$$

where $m \geq 1$, $a_i \in U$ $(i = 0, 1, \ldots, 2m)$, $y_i, t_i \in S$ $(i = 1, 2, \ldots, m)$, and

$$a_0 = y_1a_1, \qquad a_{2m-1}t_m = a_{2m},$$

$$a_{2i-1}t_i = a_{2i}t_{i+1}, \qquad y_ia_{2i} = y_{i+1}a_{2i+1} \qquad (1 \leq i \leq m-1).$$

Such a series of factorization is called a *zigzag* in S over U with value d, length m, and spine $a_0, a_1, \ldots, a_{2m}$. We refer to the equations in Result 2.4 as *the zigzag equations.*

Result 2.5 ([8], Result 3). Let U be any subsemigroup of a semigroup S and let $d \in Dom(U,S) \setminus U$. If (2) is a zigzag of minimal length m over U with value d, then $y_j, t_j \in S \setminus U$ for all $j = 1, 2, \ldots, m$.

In the following results, let U and S be any semigroups with U dense in S.

Result 2.6 ([8], Result 4). For any $d \in S \setminus U$ and k any positive integer, if (2) is a zigzag of minimal length over U with value d, then there exist $b_1, b_2, \ldots, b_k \in U$ and $d_k \in S \setminus U$ such that $d = b_1b_2 \cdots b_kd_k$.

Result 2.7 ([8], Corollary 4.2). If U be permutative, then

$$sx_1x_2 \cdots x_kt = sx_{j_1}x_{j_2} \cdots x_{j_k}t,$$

for all $x_1, x_2, \ldots, x_k \in S$, $s, t \in S \setminus U$ and any permutation j of the set $\{1, 2, \ldots, k\}$.

Result 2.8 ([10], Corollary 2.8). For any $d \in S$ and positive integer k, if $d = b_1b_2 \cdots b_kd_k$ for some $b_1, b_2, \ldots, b_k \in U$ and $d_k \in S \setminus U$ such that $b_1 = \acute{y}_1c_1$ for some $\acute{y}_1$ in $S \setminus U$, $c_1 \in U$, then $d^p = b_1^p b_2^p \cdots b_k^p d_k^p$ for any positive integer p.

The symmetrical statement in the following result does not appear in the original, but is immediate.

Result 2.9 ([9], Proposition 4.6). Assume that U is permutative. If $d \in S \setminus U$ and (2) is a zigzag of length m over U with value d such that $y_1 \in S \setminus U$, then $d^k = a_0^k t_1^k$ for each positive integer k; in particular, the conclusion holds if (2) is of minimal length. Symmetrically, if $d \in S \setminus U$ and (2) is a zigzag of length m over U with value d such that $t_m \in S \setminus U$, then $d^k = y_m^k a_{2m}^k$ for each positive integer k; in particular, the conclusion holds if (2) is of minimal length.

3. Main results

The following proposition easily follows from Result 2.2.

Proposition 3.1. Let S be any permutative semigroup satisfying (1) with $n \geq 3$. Then for each $g \in \{2, 3, \ldots, n\}$ such that $x_{g-1}x_g$ is not a subword of $x_{i_1}x_{i_2}\cdots x_{i_n}$, for all $r \geq g-1,\ s \geq n-g+1$, and for all $u \in S^{(r)},\ v \in S^{(s)}$, we have

$$ux_1x_2\cdots x_\ell v = ux_{\lambda_1}x_{\lambda_2}\cdots x_{\lambda_\ell}v,$$

for all $x_1, x_2, \ldots, x_\ell \in S$ $(\ell \geq 2)$, where λ is any permutation of the set $\{1, 2, \ldots, \ell\}$.

Now using Results 2.3, 2.6 and Proposition 3.1, we easily get the following.

Proposition 3.2. Let U be any semigroup satisfying (1) with $n \geq 3$ and let S be any semigroup containing U such that $Dom(U, S) = S$. Then for each $g \in \{2, 3, \ldots, n\}$ such that $x_{g-1}x_g$ is not a subword of $x_{i_1}x_{i_2}\cdots x_{i_n}$, for all $r \geq g-1$ and for all $u \in S^{(r)},\ v \in S \setminus U$

$$ux_1x_2\cdots x_\ell v = ux_{\lambda_1}x_{\lambda_2}\cdots x_{\lambda_\ell}v,$$

for all $x_1, x_2, \ldots, x_\ell \in S$ $(\ell \geq 2)$, where λ is any permutation of the set $\{1, 2, \ldots, \ell\}$.
Symmetrically, for all $s \geq h-1$ such that $x_{n-h}x_{n-(h-1)}$ is not a subword of $x_{i_1}x_{i_2}\cdots x_{i_n}$ and for all $v \in S^{(s)}, u \in S \setminus U$

$$ux_1x_2\cdots x_\ell v = ux_{\lambda_1}x_{\lambda_2}\cdots x_{\lambda_\ell}v,$$

for all $x_1, x_2, \ldots, x_\ell \in S$ ($\ell \geq 2$), where λ is any permutation of the set $\{1, 2, \ldots, \ell\}$.

Lemma 3.3. Let U be any permutative semigroup satisfying a seminormal identity which is dense in S. Let u and v be any words in $z_1, \ldots, z_\ell$ and let $p_1, \ldots, p_r,\ q_1, \ldots, q_s$ be any positive integers such that $p_1 \leq \cdots \leq p_r;\ q_s \leq \cdots \leq q_1 (r, s \geq 1)$. If U satisfies

$$x_1^{p_1}\cdots x_r^{p_r}u(z_1, \ldots, z_\ell)y_1^{q_1}\cdots y_s^{q_s} = x_1^{p_1}\cdots x_r^{p_r}v(z_1, \ldots, z_\ell)y_1^{q_1}\cdots y_s^{q_s}, \tag{3}$$

then the identity (3) is also satisfied for all $x_1, \ldots, x_r, y_1, \ldots, y_s \in S$ and $z_1, \ldots, z_\ell$ in U.

Proof. Take any semigroups U and S with U a subsemigroup of S such that $Dom(U, S) = S$. Since U satisfies (1), by Result 2.3, S also satisfies (1). Now we shall show that the identity (3) satisfied by U is also satisfied when $z_1, z_2, \ldots, z_\ell \in U$ and $x_1, x_2, \ldots, x_r, y_1, y_2, \ldots, y_s$ in S.

Case(i): First, take any $x_1, x_2, \ldots, x_r \in S$ and $y_1, \ldots, y_s, z_1, \ldots, z_\ell \in U$. If $x_1, x_2, \ldots, x_r$ in U, then (3) holds trivially. So assume without loss of generality that $x_1 \in S \setminus U$. Let (2) be a zigzag of minimal length m over U with value x_1. Letting $y = y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}$, we have

$$\begin{aligned}
& x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}u(z_1, z_2, \ldots, z_\ell)y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s} \\
= \ & y_m^{p_1}a_{2m}^{p_1}x_2^{p_2}\cdots x_r^{p_r}u(z_1, z_2, \ldots, z_\ell)y \text{ (by the zigzag equations and Result 2.9)} \\
= \ & y_m^{p_1}a_{2m}^{p_1}x_2^{p_2}\cdots x_r^{p_r}v(z_1, z_2, \ldots, z_\ell)y \text{ (as U satisfies (3))} \\
= \ & x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}v(z_1, z_2, \ldots, z_\ell)y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s} \\
& \text{(by the zigzag equations and Result 2.9 and as } y = y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s})
\end{aligned}$$

as required.

Next, we assume inductively that the result is true for all $x_1, \ldots, x_{k-1} \in S$ and $x_k, \ldots, x_r$ in U. We shall prove that the result is also true for all $x_1, \ldots, x_k \in S$ and $x_{k+1}, \ldots, x_r \in U$.

Again if $x_k \in U$, then the result follows by the inductive hypothesis. So assume that $x_k \in S \setminus U$. Let (2) be a zigzag of minimal length m in S over U with value x_k.
Now, letting $y = y_1^{q_1} y_2^{q_2} \cdots y_s^{q_s}$, we have

$$
\begin{aligned}
& x_1^{p_1} x_2^{p_2} \cdots x_r^{p_r} u(z_1, z_2, \ldots, z_\ell) y \\
= \;& x_1^{p_1} x_2^{p_2} \cdots x_{k-1}^{p_{k-1}} y_m^{p_k} a_{2m}^{p_k} x_{k+1}^{p_{k+1}} \cdots x_r^{p_r} u(z_1, z_2, \ldots, z_\ell) y \\
& \quad \text{(by Result 2.9 and zigzag equations)} \\
= \;& w y_m^{(m)^{p_k}} b_1^{(m)^{p_k}} \cdots b_{k-1}^{(m)\,p_k} a_{2m}^{p_k} x_{k+1}^{p_{k+1}} \cdots x_r^{p_r} u(z_1, z_2, \ldots, z_\ell) y \\
& \text{(by Results 2.6 and 2.7 for some } b_1^{(m)}, \ldots, b_{k-1}^{(m)} \in U \text{ and } y_m^{(m)} \in S \backslash U \text{ as} \\
& y_m \in S \backslash U \text{ and } a_{2m} = a_{2m-1} t_m \text{ with } t_m \in S \backslash U \text{ and where } w = x_1^{p_1} \cdots x_{k-1}^{p_{k-1}}) \\
= \;& w y_m^{(m)^{p_k}} v^{(m)} b_1^{(m)^{p_1}} \cdots b_{k-1}^{(m)\,p_{k-1}} a_{2m}^{p_k} x_{k+1}^{p_{k+1}} \cdots x_r^{p_r} u(z_1, z_2, \ldots, z_\ell) y \\
& \text{(by Result 2.7 as } y_m^{(m)}, t_m \in S \backslash U \text{ and where } v^{(m)} = b_1^{(m)^{p_k - p_1}} \cdots b_{k-1}^{(m)\,p_k - p_{k-1}}) \\
= \;& w y_m^{(m)^{p_k}} v^{(m)} b_1^{(m)^{p_1}} \cdots b_{k-1}^{(m)\,p_{k-1}} a_{2m}^{p_k} x_{k+1}^{p_{k+1}} \cdots x_r^{p_r} v(z_1, z_2, \ldots, z_\ell) y \\
& \quad \text{(as U satisfies (3))} \\
= \;& w y_m^{(m)^{p_k}} b_1^{(m)^{p_k}} \cdots b_{k-1}^{(m)\,p_k} a_{2m}^{p_k} x_{k+1}^{p_{k+1}} \cdots x_r^{p_r} v(z_1, z_2, \ldots, z_\ell) y \\
& \quad \text{(by Result 2.7 and the definition of } v^{(m)}) \\
= \;& x_1^{p_1} x_2^{p_2} \cdots x_{k-1}^{p_{k-1}} y_m^{p_k} a_{2m}^{p_k} x_{k+1}^{p_{k+1}} \cdots x_r^{p_r} v(z_1, z_2, \ldots, z_\ell) y \\
& \text{(as } y_m^{(m)^{p_k}} b_1^{(m)^{p_k}} \cdots b_{k-1}^{(m)\,p_k} = y_m^{p_k} \text{ and } w = x_1^{p_1} x_2^{p_2} \cdots x_{k-1}^{p_{k-1}}) \\
= \;& x_1^{p_1} x_2^{p_2} \cdots x_{k-1}^{p_{k-1}} x_k^{p_k} x_{k+1}^{p_{k+1}} \cdots x_r^{p_r} v(z_1, z_2, \ldots, z_\ell) y \\
& \text{(by Result 2.9 and zigzag equations)} \\
= \;& x_1^{p_1} x_2^{p_2} \cdots x_r^{p_r} v(z_1, z_2, \ldots, z_\ell) y_1^{q_1} y_2^{q_2} \cdots y_s^{q_s} \text{ (as } y = y_1^{q_1} y_2^{q_2} \cdots y_s^{q_s})
\end{aligned}
$$

as required.

Case(ii): Now, we show that (3) is satisfied for all $x_1, \ldots, x_r, y_1, \ldots, y_s \in S$ and $z_1, \ldots, z_\ell$ in U. Again, we can assume without loss of generality that y_1 in $S \setminus U$. Let (2) be a zigzag of minimal length m over U with value y_1. Now as the equalities (4) and (5) below follow by Results 2.6 and 2.7 for some $b_2^{(1)}, \ldots, b_s^{(1)}$ in U and $t_1^{(1)}$ in $S \setminus U$ as $y_1, t_1 \in S \setminus U$ and where $w^{(1)} = {b_2^{(1)}}^{q_1 - q_2} \cdots {b_s^{(1)}}^{q_1 - q_s}$ respectively. Letting $x = x_1^{p_1} x_2^{p_2} \cdots x_r^{p_r}$, we have

$$
\begin{aligned}
& x_1^{p_1} x_2^{p_2} \cdots x_r^{p_r} u(z_1, z_2, \ldots, z_\ell) y_1^{q_1} y_2^{q_2} \cdots y_s^{q_s} \\
= \;& x u(z_1, z_2, \ldots, z_\ell) a_0^{q_1} t_1^{q_1} y_2^{q_2} \cdots y_s^{q_s} \text{ (by the zigzag equations and Result 2.9)} \\
= \;& x u(z_1, z_2, \ldots, z_\ell) a_0^{q_1} {b_2^{(1)}}^{q_1} \cdots {b_s^{(1)}}^{q_1} {t_1^{(1)}}^{q_1} y_2^{q_2} \cdots y_s^{q_s} \qquad (4) \\
= \;& x u(z_1, z_2, \ldots, z_\ell) a_0^{q_1} {b_2^{(1)}}^{q_2} \cdots {b_s^{(1)}}^{q_s} w^{(1)} {t_1^{(1)}}^{q_1} y_2^{q_2} \cdots y_s^{q_s} \qquad (5) \\
= \;& x v(z_1, z_2, \ldots, z_\ell) a_0^{q_1} {b_2^{(1)}}^{q_2} \cdots {b_s^{(1)}}^{q_s} w^{(1)} {t_1^{(1)}}^{q_1} y_2^{q_2} \cdots y_s^{q_s} \text{ (as U satisfies (3))} \\
= \;& x v(z_1, z_2, \ldots, z_\ell) a_0^{q_1} {b_2^{(1)}}^{q_1} \cdots {b_s^{(1)}}^{q_1} {t_1^{(1)}}^{q_1} y_2^{q_2} \cdots y_s^{q_s} \\
& \text{(by Result 2.7 and definition of } w^{(1)}) \\
= \;& x v(z_1, z_2, \ldots, z_\ell) a_0^{q_1} t_1^{q_1} y_2^{q_2} \cdots y_s^{q_s} \text{ (by Result 2.7 as } {b_2^{(1)}}^{q_1} \cdots {b_s^{(1)}}^{q_1} {t_1^{(1)}}^{q_1} = t_1^{q_1}) \\
= \;& x_1^{p_1} x_2^{p_2} \cdots x_r^{p_r} v(z_1, z_2, \ldots, z_\ell) y_1^{q_1} y_2^{q_2} \cdots y_s^{q_s} \\
& \text{(by the zigzag equations and Result 2.9 and as } x = x_1^{p_1} x_2^{p_2} \cdots x_r^{p_r})
\end{aligned}
$$

Next, we assume inductively that the result is true for all $y_1, \ldots, y_{k-1} \in S$ and $y_k, \ldots, y_s$ in U. We shall prove that the result is also true for all $y_1, \ldots, y_{k-1}, y_k \in S$ and $y_{k+1}, \ldots, y_s$ in U. Again if $y_k \in U$, then the result follows by the inductive hypothesis. So assume that $y_k \in S \setminus U$. Let (2) be a zigzag of minimal length m in S over U with value y_k. Now as equalities (6) and (7) follow by Results 2.6 and 2.7 for some $b_{k+1}^{(1)}, \ldots b_s^{(1)}$ in U and $t_1^{(1)} \in S \setminus U$ as $y_1, t_1 \in S \setminus U$ and where $v = y_{k+1}^{q_{k+1}} \cdots y_s^{q_s}$, and by Result 2.7 as $a_0 = y_1 a_1,\ y_1, t_1^{(1)} \in S \setminus U$ and where $w^{(1)} = {b_{k+1}^{(1)}}^{q_k - q_{k+1}} \cdots {b_s^{(1)}}^{q_k - q_s}$ respectively, we have

$$x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}u(z_1,z_2,\ldots,z_\ell)y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}$$

$$= \quad xu(z_1,z_2,\ldots,z_\ell)y_1^{q_1}y_2^{q_2}\cdots y_{k-1}^{q_{k-1}}a_0^{q_k}t_1^{q_k}y_{k+1}^{q_{k+1}}\cdots y_s^{q_s}$$

(by Result 2.9 and zigzag equations)

$$= \quad xu(z_1,z_2,\ldots,z_\ell)y_1^{q_1}\cdots y_{k-1}^{q_{k-1}}a_0^{q_k}{b_{k+1}^{(1)}}^{q_k}\cdots {b_s^{(1)}}^{q_k}{t_1^{(1)}}^{q_k}v \tag{6}$$

$$= \quad xu(z_1,z_2,\ldots,z_\ell)y_1^{q_1}\cdots y_{k-1}^{q_{k-1}}a_0^{q_k}{b_{k+1}^{(1)}}^{q_{k+1}}\cdots {b_s^{(1)}}^{q_s}w^{(1)}{t_1^{(1)}}^{q_k}v \tag{7}$$

$$= \quad xv(z_1,z_2,\ldots,z_\ell)y_1^{q_1}\cdots y_{k-1}^{q_{k-1}}a_0^{q_k}{b_{k+1}^{(1)}}^{q_{k+1}}\cdots {b_s^{(1)}}^{q_s}w^{(1)}{t_1^{(1)}}^{q_k}v$$

(by the inductive hypothesis)

$$= \quad xv(z_1,z_2,\ldots,z_\ell)y_1^{q_1}\cdots y_{k-1}^{q_{k-1}}a_0^{q_k}{b_{k+1}^{(1)}}^{q_k}\cdots {b_s^{(1)}}^{q_k}{t_1^{(1)}}^{q_k}v$$

(by Result 2.7 and the definition of $w^{(1)}$)

$$= \quad xv(z_1,z_2,\ldots,z_\ell)y_1^{q_1}\cdots y_{k-1}^{q_{k-1}}a_0^{q_k}t_1^{q_k}y_{k+1}^{q_{k+1}}\cdots y_s^{q_s}$$

(by Result 2.7 as ${b_{k+1}^{(1)}}^{q_k}\cdots {b_s^{(1)}}^{q_k}{t_1^{(1)}}^{q_k} = t_1^{q_k}$ and the definition of v)

$$= \quad x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}v(z_1,z_2,\ldots,z_\ell)y_1^{q_1}y_2^{q_2}\cdots y_{k-1}^{q_{k-1}}y_k^{q_k}y_{k+1}^{q_{k+1}}\cdots y_s^{q_s}$$

(by Result 2.9 and zigzag equations and as $x = x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}$).

as required. This completes the proof of the lemma. □

The following corollary easily follows from Proposition 3.1 and Lemma 3.3.

Corollary 3.4. Let U be any permutative semigroup which is dense in S, and $t_1,t_2,\ldots,t_\ell$ be any positive integers. Let $p_1,p_2,\ldots,p_r,q_1,q_2,\ldots,q_s$ be any positive integers such that $p_1 \leq p_2 \leq \cdots \leq p_{r-1} \leq p_r$, $q_s \leq q_{s-1}\cdots \leq q_2 \leq q_1 (r,s \geq 0)$. If U satisfies

$$x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}z_1^{t_1}z_2^{t_2}\cdots z_\ell^{t_\ell}y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s} = x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}z_{j_1}^{t_{j_1}}z_{j_2}^{t_{j_2}}\cdots z_{j_\ell}^{t_{j_\ell}}y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}$$

where j is any permutation of the set $\{1, 2, \ldots, \ell\}$, then the above identity is also satisfied for all $x_1, \ldots, x_r,\ y_1, \ldots, y_s\ \in\ S$ and $z_1, \ldots, z_\ell \in U$.

Proposition 3.5: Let U be a permutative subsemigroup satisfying a seminormal permutation identity which is dense in S. Let u and v be any words in $w_1, \ldots, w_\ell$. If the identity

$$x_1^{p_1} \cdots x_r^{p_r} u(w_1, \ldots, w_\ell) y_1^{q_1} \cdots y_s^{q_s} = x_1^{p_1} \cdots x_r^{p_r} v(w_1, \ldots, w_\ell) y_1^{q_1} \cdots y_s^{q_s}$$

holds for all $x_1, \ldots, x_r, y_1, \ldots, y_s \in S$ and $w_1, \ldots, w_\ell \in U$, then the identity

$$x_1^{p_1} \cdots x_r^{p_r} u(w_1, \ldots, w_\ell) y^q = x_1^{p_1} \cdots x_r^{p_r} v(w_1, \ldots, w_\ell) y^q$$

$$[x^p u(w_1, \ldots, w_\ell) y_1^{q_1} \cdots y_s^{q_s} = x^p v(w_1, \ldots, w_\ell) y_1^{q_1} \cdots y_s^{q_s}]$$

also holds for all $y \in S \setminus U$; $x_1, \ldots, x_r \in S$; $w_1, \ldots, w_\ell \in U$ and positive integer $q \geq q_1$ [for all $x \in S \setminus U$; $y_1, \ldots, y_s \in S$; $w_1, \ldots, w_\ell \in U$ and positive integer $p \geq p_r$].

Proof. We have,

$x_1^{p_1} \cdots x_r^{p_r} u(w_1, w_2, \ldots, w_\ell) y^q$

$= \quad x_1^{p_1} \cdots x_r^{p_r} u(w_1, w_2, \ldots, w_\ell) y^{q_1} y^{q-q_1}$

$= \quad x_1^{p_1} \cdots x_r^{p_r} u(w_1, w_2, \ldots, w_\ell) a_1^{q_1} \cdots a_s^{q_1} {y'}^{q_1} y^{q-q_1}$
(by Results 2.6 and 2.8 for some $\mathrm{a}_1, \ldots, a_s \in U$ and
$y' \in S \backslash U$ as $a_1 = {z_1}' b_1$ for some ${z_1}' \in S \backslash U,\ b_1 \in U$)

$= \quad x_1^{p_1} \cdots x_r^{p_r} u(w_1, w_2, \ldots, w_\ell) a_1^{q_1} \cdots a_s^{q_s} w {y'}^{q_1} y^{q-q_1}$
(by Result 2.8 as $a_1 = {z_1}' b_1$ for some ${z_1}' \in S \backslash U,\ b_1 \in U$
and where $w = a_2^{q_1 - q_2} \cdots a_s^{q_1 - q_s}$)

$= \quad x_1^{p_1} \cdots x_r^{p_r} v(w_1, \ldots, w_\ell) a_1^{q_1} \cdots a_s^{q_s} w {y'}^{q_1} y^{q-q_1}$

$= \quad x_1^{p_1} \cdots x_r^{p_r} v(w_1, \ldots, w_\ell) a_1^{q_1} \cdots a_s^{q_1} {y'}^{q_1} y^{q-q_1}$ (by definition of w)

$$= x_1^{p_1}\cdots x_r^{p_r}v(w_1,\ldots,w_\ell)y^{q_1}y^{q-q_1}$$

(by Results 2.6 and 2.8 as $y^{q_1} = a_1^{q_1}\cdots a_s^{q_1}{y'}^{q_1}$)

$$= x_1^{p_1}\cdots x_r^{p_r}v(w_1,\ldots,w_\ell)y^{q}$$

as required. Dual statment may be proved on the similar lines. □

Following Khan and Shah [10], for any seminormal identity, let $g_0 = \min P$, the minimum of P, where

$$P = \{2 \leq g \leq n-2 : x_{g-1}x_g \text{ is not a subword of } x_{i_1}x_{i_2}\cdots x_{i_n}\};$$

and let $h_0 = \min Q$, where

$$Q = \{1 \leq h \leq n-g_0-1 : x_{n-h}x_{n-(h-1)} \text{ is not a subword of } x_{i_1}x_{i_2}\cdots x_{i_n}\}.$$

Throughout the paper $\mathbf{g_0}$ and $\mathbf{h_0}$ will stand as defined above; and $\mathbf{p_1, p_2 \ldots, p_r}$, $\mathbf{q_1, q_2 \ldots, q_s}$ will stand for any positive integers such that $\mathbf{p_1 + p_2 + \cdots + p_r \geq g_0 - 1}$, $\mathbf{q_1 + q_2 + \cdots + q_s \geq h_0 - 1}$, $\mathbf{p_1 \leq \cdots \leq p_{r-1} \leq p_r}$ and $\mathbf{q_s \leq \cdots \leq q_2 \leq q_1 (r, s \geq 1)}$ unless exclusively mentioned. Also, to avoid introduction of new symbols, we shall treat, wherever is appropriate, $\mathbf{x_1, x_2, \ldots, x_r, y_1, y_2, \ldots, y_s, w_1, w_2, \ldots, w_\ell}$,$\mathbf{z_1}$, $\mathbf{z_2, \ldots, z_p}$ etc. as variables as well as the members of a semigroup without explicit mention of distinction.

Further for any word u and any variable x of u, $|x|_u$ will denote the number of occurrences of x in the word u.

Theorem 3.6. Let u and v are any words in the variables $z_1, z_2, \ldots, z_\ell$ such that $|z_i|_u = |z_i|_v$ for all $i = 1, 2, \ldots, \ell$. Then all semigroup identities of the form

$$x_1^{p_1}\cdots x_r^{p_r}u(z_1,\ldots,z_\ell)y_1^{q_1}\cdots y_s^{q_s} = x_1^{p_1}\cdots x_r^{p_r}v(z_1,\ldots,z_\ell)y_1^{q_1}\cdots y_s^{q_s} \ (\ell \geq 3) \tag{8}$$

are preserved under epis in conjunction with (1).

Proof. Take any semigroups U and S with U as a subsemigroup of S such that U is dense in S. Let U satisfy (1) and (8). As U satisfies (1), by Result 2.3, S also satisfies (1). Now we shall prove that the identity (8) satisfied by U is also satisfied by S. Suppose that $x_1, x_2, \ldots, x_r, y_1, y_2, \ldots, y_s, z_1, z_2, \ldots, z_\ell \in S$. If all of $z_1, z_2, \ldots, z_\ell$ are in U, then, (8) is satisfied by lemma 3.3, So, we assume that not all of $z_1, z_2, \ldots, z_\ell$ are from U. Suppose that $z_j \in S \setminus U$, for some $j \in \{1, 2, \ldots, \ell\}$. By Result 2.4, $z_j = x'b = x'b'y'$ for some $x', y' \in S \setminus U$ and $b, b' \in U$ with $b = b'y'$. Now, as equalities (9) and (10) below hold respectively by Proposition 3.2 as $y' \in S \setminus U$ and $p_1 + \cdots + p_r \geq g_0 - 1$, and by Proposition 3.2 as $x' \in S \setminus U$, $q_1 + \cdots + q_s \geq h_0 - 1$ together with the fact that $|z_i|_u = |z_i|_v$ for all $i = 1, 2, \ldots, \ell$, we have

$$\begin{aligned}
& x_1^{p_1} x_2^{p_2} \cdots x_r^{p_r} u(z_1, z_2, \ldots, z_\ell) y_1^{q_1} y_2^{q_2} \cdots y_s^{q_s} \\
= \;& x_1^{p_1} x_2^{p_2} \cdots x_r^{p_r} u(z_1, z_2, \ldots, z_{j-1}, x'b'y', z_{j+1}, \ldots, z_\ell) y_1^{q_1} y_2^{q_2} \cdots y_s^{q_s} \\
= \;& x_1^{p_1} x_2^{p_2} \cdots x_r^{p_r} (x')^{|z_j|_u} u(z_1, z_2, \ldots, z_{j-1}, b'y', z_{j+1}, \ldots, z_\ell) y_1^{q_1} y_2^{q_2} \cdots y_s^{q_s} \qquad (9) \\
= \;& x_1^{p_1} x_2^{p_2} \cdots x_r^{p_r} (x')^{|z_j|_u} v(z_1, z_2, \ldots, z_{j-1}, b'y', z_{j+1}, \ldots, z_\ell) y_1^{q_1} y_2^{q_2} \cdots y_s^{q_s} \qquad (10) \\
= \;& x_1^{p_1} x_2^{p_2} \cdots x_r^{p_r} v(z_1, z_2, \ldots, z_{j-1}, x'b'y', z_{j+1}, \ldots, z_\ell) y_1^{q_1} y_2^{q_2} \cdots y_s^{q_s} \qquad (11) \\
= \;& x_1^{p_1} x_2^{p_2} \cdots x_r^{p_r} v(z_1, z_2, \ldots, z_{j-1}, z_j, z_{j+1}, \ldots, z_\ell) y_1^{q_1} y_2^{q_2} \cdots y_s^{q_s},
\end{aligned}$$

where the equality (11) above follows by Proposition 3.2 as $p_1 + \cdots + p_r \geq g_0 - 1$ and $y' \in S \setminus U$, as required. □

Corollary 3.7. All semigroup identities of the form

$$x_1^{p_1} \cdots x_r^{p_r} z_1^{t_1} \cdots z_\ell^{t_\ell} y_1^{q_1} \cdots y_s^{q_s} = x_1^{p_1} \cdots x_r^{p_r} {z_{j_1}}^{t_{j_1}} \cdots {z_{j_\ell}}^{t_{j_\ell}} y_1^{q_1} \cdots y_s^{q_s} \ (\ell \geq 3),$$

where j is any permutation of the set $\{1, 2, \ldots, \ell\}$ and $t_1, t_2, \ldots, t_\ell \geq 1$, are preserved under epis in conjunction with a seminormal identity.

Theorem 3.8. Let u and v be any words in the variables $z_1, \ldots, z_\ell$. If $\min\{|z_i|_u, |z_i|_v\} \geq \max\{p_r, q_1\}$ for all i in $\{1, 2, \ldots, \ell\}$, then all semigroup identities of the form

$$x_1^{p_1} \cdots x_r^{p_r} u(z_1, \ldots, z_\ell) y_1^{q_1} \cdots y_s^{q_s} = x_1^{p_1} \cdots x_r^{p_r} v(z_1, \ldots, z_\ell) y_1^{q_1} \cdots y_s^{q_s} \ (\ell \geq 3), \tag{12}$$

are preserved under epis in conjunction with (1).

The following lemmas, where U and S be any semigroups such that U is dense in S, will be required to complete the proof of Theorem 3.8; bracketed clauses yield the dual statements.

In the following, S^1 will denote the semigroup obtained from the semigroup S by adjoining an identity, if necessary; while the *length of a word* u, denoted by $\ell(u)$, is defined as the sum of the occurrences of all the variables appearing in u.

Lemma 3.8.1([10], Lemma 2.7.1). Let (1) be any seminormal identity, and let u, v and w be any words in the variables $x_1, x_2, \ldots, x_k$ $(k \geq 2)$ such that $\ell(u) \geq g_0 - 1$ and $\ell(v) \geq h_0 - 1$. Take any $a_1, a_2, \ldots, a_k \in U$ and $t_1, t_2, \ldots, t_k \in S^1$. If for each i such that $t_i \in S$, $a_i = y_i b_i$ $[a_i = b_i y_i]$ for some $y_i \in S \setminus U$ and $b_i \in S$ $(i = 1, 2, \ldots, k)$, then for any choice $d_1, d_2, \ldots, d_k$ for the variables $x_1, x_2, \ldots, x_k$ in S respectively

$$u(\tilde{d}) w(a_1 t_1, a_2 t_2, \ldots, a_k t_k) v(\tilde{d}) = u(\tilde{d}) w(a_1, a_2, \ldots, a_k) w(t_1, t_2, \ldots, t_k) v(\tilde{d})$$

$$[u(\tilde{d}) w(t_1 a_1, t_2 a_2, \ldots, t_k a_k) v(\tilde{d}) = u(\tilde{d}) w(t_1, t_2, \ldots, t_k) w(a_1, a_2, \ldots, a_k) v(\tilde{d})],$$

where $\tilde{d} = (d_1, d_2, \ldots, d_k)$.

For any word u, the *content* of u (necessarily finite) is the set of all variables appearing in the word u and is denoted by $C(u)$.

Lemma 3.8.2([10], Lemma 2.7.2). Let (1) be any seminormal identity and let u, v, w and w' be any words in the variables $x_1, x_2, \ldots, x_k$ such that $\ell(w) \geq g_0 - 1$, $\ell(w') \geq h_0 - 1$. Take any $d_1, d_2, \ldots, d_k$ in S for the variables $x_1, x_2, \ldots, x_k$ respectively. Let $x_j \in C(v)$ $[x_j \in C(u)]$ be such that $d_j \in S \setminus U$ for some $1 \leq j \leq k$. Then

$$w(\tilde{d}) u(\tilde{d}) v(\tilde{d}) w'(\tilde{d}) = w(\tilde{d}) v(\tilde{d}) u(\tilde{d}) w'(\tilde{d}),$$

where $\tilde{d} = (d_1, d_2, \ldots, d_k)$.

Lemma 3.8.3([10], Lemma 2.7.3). Let (1) be any seminormal identity and let u, v and w be any words in the variables $x_1, x_2, \ldots, x_k$ $(k \geq 2)$ such that $\ell(u) \geq g_0 - 1$ and $\ell(w) \geq h_0 - 1$. Take any $d_1, d_2, \ldots, d_k$ in S for the variables $x_1, x_2, \ldots, x_k$ respectively. If $x_j \in C(v)$, for some $1 \leq j \leq k$, be such that $d_j \in S \setminus U$, then

$$u(\tilde{d})v(\tilde{d})w(\tilde{d}) = u(\tilde{d})(d_j)^{|x_j|_v} v(\tilde{d}')w(\tilde{d})$$

in S^1(in fact the two products are equal in S), where

$$\tilde{d} = (d_1, d_2, \ldots, d_k)$$

and

$$\tilde{d}' = (d_1, d_2, \ldots, d_{j-1}, 1, d_{j+1}, \ldots, d_k),$$

for all $d_1, d_2, \ldots, d_k \in S$ (thus the product $v(\tilde{d}')$ is obtained from the product $v(\tilde{d})$ by ommitting all the occurences of the element d_j).

Proof of Theorem 3.8. Take any semigroups U and S with U dense in S, and assume that U satisfies (1) and (11). As U satisfies (1), by Result 2.4, S also satisfies (1). Now, we show that the identity (11) is also satisfied by S.

We shall prove the theorem in the case when $q_1 \geq p_r$, so $|z_i|_u \geq q_1$ and $|z_i|_v \geq q_1$ hold for all i; the proof when $p_r \geq q_1$ follows by dual arguments on similar lines.

So, take any $d_1, d_2, \ldots, d_\ell \in S$. If some $d_i \in U$, there is a zigzag in S^1 over U with value d_i, namely, $d_i = d_i \cdot 1 = 1 \cdot d_i \cdot 1 = 1 \cdot d_i$. And if $d_i \in S \setminus U$, then there is a zigzag over U in S, hence in S^1. Thus $d_1, d_2, \ldots, d_l$ all have zigzags over U in S^1 of some common length [6, Lemma 4.2], say

$$\begin{aligned}
&d_i = a_0^{(i)} t_1^{(i)}, \qquad a_0^{(i)} = y_1^{(i)} a_1^{(i)};\\
&y_k^{(i)} a_{2k}^{(i)} = y_{k+1}^{(i)} a_{2k+1}^{(i)}, \quad a_{2k-1}^{(i)} t_k^{(i)} = a_{2k}^{(i)} t_{k+1}^{(i)} \ (1 \leq i \leq \ell,\ 1 \leq k \leq m-1);\\
&a_{2m-1}^{(i)} t_m^{(i)} = a_{2m}^{(i)}, \qquad y_m^{(i)} a_{2m}^{(i)} = d_i;
\end{aligned} \tag{13}$$

where $a_j^{(i)} \in U$ $(i = 1, 2, \ldots, \ell;\ j = 0, 1, \ldots, 2m)$ and $t_q^{(i)},\ y_q^{(i)} \in S^1$ for $i = 1, 2, \ldots, \ell$ and $q = 1, 2, \ldots, m$. Further, for each $d_i \in S \setminus U$, we may assume that $t_q^{(i)}$ and $y_q^{(i)}$ are in $S \setminus U$ [6, Lemma 4.2].

Let $\tilde{z} = (z_1, z_2, \ldots, z_\ell)$. With this notation, as in [6], identity (12) becomes

$$x_1^{p_1} x_2^{p_2} \cdots x_r^{p_r} u(\tilde{z}) y_1^{q_1} y_2^{q_2} \cdots y_s^{q_s} = x_1^{p_1} x_2^{p_2} \cdots x_r^{p_r} v(\tilde{z}) y_1^{q_1} y_2^{q_2} \cdots y_s^{q_s}.$$

Also, let

$$\begin{aligned}
\tilde{d} &= (d_1, d_2, \ldots, d_\ell);\\
\tilde{a}_k &= (a_k^{(1)}, a_k^{(2)}, \ldots, a_k^{(\ell)}) \quad (k = 0, 1, \ldots, 2m);\\
\tilde{t}_q &= (t_q^{(1)}, t_q^{(2)}, \ldots, t_q^{(\ell)}) \quad (q = 1, 2, \ldots, m);\\
\tilde{y}_q &= (y_q^{(1)}, y_q^{(2)}, \ldots, y_q^{(\ell)}) \quad (q = 1, 2, \ldots, m).
\end{aligned}$$

We, now, wish to show that

$$x_1^{p_1} \cdots x_r^{p_r} u(\tilde{d}) y_1^{q_1} \cdots y_s^{q_s} = x_1^{p_1} \cdots x_r^{p_r} v(\tilde{d}) y_1^{q_1} \cdots y_s^{q_s}.$$

By [6, Lemma 4.3], $\tilde{d} \in S^{[\ell]}$ is in the dominion of $U^{[\ell]}$ in $(S^1)^{[\ell]}$, where $T^{[\gamma]}$, for any semigroup T and for any integer $\gamma \geq 2$, denotes the cartesian product of the γ-copies of T. In fact, $\tilde{d}$ has a zigzag over $U^{[\ell]}$ in $(S^1)^{[\ell]}$ of length m as follows:

$$\begin{aligned}
&\tilde{d} = \tilde{a}_0 \tilde{t}_1, && \tilde{a}_0 = \tilde{y}_1 \tilde{a}_1;\\
&\tilde{y}_k \tilde{a}_{2k} = \tilde{y}_{k+1} \tilde{a}_{2k+1}, && \tilde{a}_{2k-1} \tilde{t}_k = \tilde{a}_{2k} \tilde{t}_{k+1}\ (1 \leq k \leq m-1,\ 1 \leq i \leq m-1);\\
&\tilde{a}_{2m-1} \tilde{t}_m = \tilde{a}_{2m}, && \tilde{y}_m \tilde{a}_{2m} = \tilde{d};
\end{aligned} \tag{14}$$

where $\tilde{a}_t \in U^{[\ell]}$ $(t = 0, 1, \ldots, 2m)$ and $\tilde{y}_q,\ \tilde{t}_q\ \in (S^1)^{[\ell]}$ $(q = 1, 2, \ldots, m)$.

So, suppose that $x_1, \ldots, x_r, y_1, \ldots, y_s,\ d_1, \ldots, d_\ell \in S$. If all d_i that occur in the word u are from U, then, by lemma 3.3,

$$x_1^{p_1} \cdots x_r^{p_r} u(\tilde{d}) y_1^{q_1} \cdots y_s^{q_s} = x_1^{p_1} \cdots x_r^{p_r} v(\tilde{d}) y_1^{q_1} \cdots y_s^{q_s},$$

as required. Hence, we may assume that there exists at least one d_j $(1 \leq j \leq \ell)$, say, such that $d_j \in S \setminus U$. Then $t_i^{(j)}, y_i^{(j)} \in S \setminus U$ for all $i = 1, 2, \ldots, m$. Letting $x = x_1^{p_1} x_2^{p_2} \cdots x_r^{p_r}$ and $y = y_1^{q_1} y_2^{q_2} \cdots y_s^{q_s}$, we have

$$
\begin{aligned}
& x_1^{p_1} \cdots x_r^{p_r} u(\tilde{d}) y_1^{q_1} \cdots y_s^{q_s} \\
= \;& x u(\tilde{a}_0 \tilde{t}_1) y (\text{from equations (14)}) \\
= \;& x u(\tilde{a}_0) u(\tilde{t}_1) y \text{ (by lemma 3.8.1)} \\
= \;& x u(\tilde{a}_0) (t_1^{(j)})^{|z_j|_u} u(\tilde{t}_1') y \text{ (by lemma 3.8.3 as } t_1^{(j)} \in S \setminus U) \\
= \;& x v(\tilde{a}_0) (t_1^{(j)})^{|z_j|_u} u(\tilde{t}_1') y \text{ (by Proposition 3.5 as } |z_j|_u \geq q_1 \text{ and } t_1^{(j)} \in S \setminus U) \\
= \;& x v(\tilde{y}_1 \tilde{a}_1) u(\tilde{t}_1) y \text{ (by equations (14) and Proposition 3.1 as } t_1^{(j)} \in S \setminus U) \\
= \;& x v(\tilde{y}_1) v(\tilde{a}_1) u(\tilde{t}_1) y \text{ (by dual of lemma 3.8.1 as } y_1^{(j)} \in S \setminus U) \\
= \;& x v(\tilde{a}_1) v(\tilde{y}_1) u(\tilde{t}_1) y \text{ (by lemma 3.8.2 as } y_1^{(j)} \in S \setminus U) \\
= \;& x v(\tilde{a}_1) (y_1^{(j)})^{|z_j|_v} v(\tilde{y}_1') u(\tilde{t}_1) y \text{ (by lemma 3.8.3 as } y_1^{(j)} \in S \setminus U) \\
= \;& x u(\tilde{a}_1) (y_1^{(j)})^{|z_j|_v} v(\tilde{y}_1') u(\tilde{t}_1) y \text{ (by Proposition 3.5 as } |z_j|_v \geq q_1 \text{ and } y_1^{(j)} \in S \setminus U) \\
= \;& x u(\tilde{a}_1) v(\tilde{y}_1) u(\tilde{t}_1) y \text{ (by Lemmas 3.8.3 as } y_1^{(j)} \in S \setminus U) \\
= \;& x v(\tilde{y}_1) u(\tilde{a}_1 \tilde{t}_1) y \text{ (by Lemmas 3.8.1 and 3.8.2)} \\
= \;& x v(\tilde{y}_1) u(\tilde{a}_2 \tilde{t}_2) y \text{ (from equations (14)).}
\end{aligned}
$$

Continuing in this way, we obtain

$$
\begin{aligned}
&x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}u(\tilde{d})y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}\\
&= xv(\tilde{y}_1)u(\tilde{a}_2\tilde{t}_2)y\\
&\vdots\\
&= xv(\tilde{y}_{m-1})u(\tilde{a}_{2m-2}\tilde{t}_m)y\\
&= xv(\tilde{y}_{m-1})u(\tilde{a}_{2m-2})u(\tilde{t}_m)y \text{ (by lemma 3.8.1)}\\
&= xu(\tilde{a}_{2m-2})v(\tilde{y}_{m-1})u(\tilde{t}_m)y \text{ (by lemma 3.8.2 as } y_{m-1}^{(j)}\in S\setminus U)\\
&= xu(\tilde{a}_{2m-2})(y_{m-1}^{(j)})^{|z_j|_v}v(\tilde{y}_{m-1}')u(\tilde{t}_m)y \text{ (by lemma 3.8.3 as } y_{m-1}^{(j)}\in S\setminus U)\\
&= xv(\tilde{a}_{2m-2})(y_{m-1}^{(j)})^{|z_j|_v}v(\tilde{y}_{m-1}')u(\tilde{t}_m)y\\
&\quad \text{(by Proposition 3.5 as } |z_j|_v\geq q_1 \text{ and } y_{m-1}^{(j)}\in S\setminus U)\\
&= xv(\tilde{a}_{2m-2})v(\tilde{y}_{m-1})u(\tilde{t}_m)y \text{ (by lemma 3.8.3)}\\
&= xv(\tilde{y}_{m-1})v(\tilde{a}_{2m-2})u(\tilde{t}_m)y \text{ (by lemma 3.8.2 as } t_m^{(j)}\in S\setminus U)\\
&= xv(\tilde{y}_{m-1}\tilde{a}_{2m-2})u(\tilde{t}_m)y \text{ (by the dual of lemma 3.8.1)}\\
&= xv(\tilde{y}_m\tilde{a}_{2m-1})u(\tilde{t}_m)y \text{ (from equations (14))}\\
&= xv(\tilde{y}_m)v(\tilde{a}_{2m-1})u(\tilde{t}_m)y \text{ (by the dual of lemma 3.8.1)}\\
&= xv(\tilde{a}_{2m-1})v(\tilde{y}_m)u(\tilde{t}_m)y \text{ (by lemma 3.8.2 as } y_m^{(j)}\in S\setminus U)\\
&= xv(\tilde{a}_{2m-1})(y_m^{(j)})^{|z_j|_v}v(\tilde{y}_m')u(\tilde{t}_m)y \text{ (by lemma 3.8.3 as } y_m^{(j)}\in S\setminus U)
\end{aligned}
$$

$$
\begin{aligned}
&= xu(\tilde{a}_{2m-1})(y_m^{(j)})^{|z_j|_v}v(\tilde{y}'_m)u(\tilde{t}_m)y \\
&\quad \text{(by Proposition 3.5 as } |z_j|_v \geq q_1 \text{ and } y_m^{(j)} \in S \setminus U) \\
&= xu(\tilde{a}_{2m-1})v(\tilde{y}_m)u(\tilde{t}_m)y \text{ (by lemma 3.8.3)} \\
&= xv(\tilde{y}_m)u(\tilde{a}_{2m-1}\tilde{t}_m)y \text{ (by Lemmas 3.8.1 and 3.8.2 as } y_m^{(j)} \in S \setminus U) \\
&= xv(\tilde{y}_m)u(\tilde{a}_{2m})y \text{ (from equations (14))} \\
&= xu(\tilde{a}_{2m})v(\tilde{y}_m)y \text{ (by lemma 3.8.2 as } y_m^{(j)} \in S \setminus U) \\
&= xu(\tilde{a}_{2m})(y_m^{(j)})^{|z_j|_v}v(\tilde{y}'_m)y \text{ (by lemma 3.8.3 as } y_m^{(j)} \in S \setminus U) \\
&= xv(\tilde{a}_{2m})(y_m^{(j)})^{|z_j|_v}v(\tilde{y}'_m)y \text{ (by Proposition 3.5 as } |z_j|_u \geq q_1 \text{ and } y_m^{(j)} \in S \setminus U) \\
&= xv(\tilde{a}_{2m})v(\tilde{y}_m)y \text{ (by lemma 3.8.3)} \\
&= xv(\tilde{y}_m)v(\tilde{a}_{2m})y \text{ (by lemma 3.8.2 as } y_m^{(j)} \in S \setminus U) \\
&= xv(\tilde{y}_m\tilde{a}_{2m})y \text{ (by the dual of lemma 3.8.1)} \\
&= x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}v(\tilde{d})y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}
\end{aligned}
$$

where the last equality follows from equations (14), and as $x = x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}$ and $y = y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}$. This completes the proof of the theorem. □

4. AN IMPROVEMENT IN A SPECIAL CASE

In this section, we find some sufficient condition on semigroup identities, a subclass of the class of semigroup identities of the type considered in Theorem 3.6, to be preserved under epis in conjunction with any seminormal identity (1), and with improved lower bounds for $p_1+\cdots+p_r$ and $q_1+\cdots+q_s$. This modest weaking of lower bounds is achieved by a careful analysis of the interplay of the zigzag equations and the identities in question.

Theorem 4.1: Let (1) be a seminormal identity and let $p_1, p_2, \ldots, p_r,\ q_1, \ldots, q_s$ be any positive integers such that $p_1 \leq p_2 \leq \cdots \leq p_{r-1} \leq p_r$; $q_s \leq q_{s-1} \cdots \leq q_2 \leq q_1 (r, s \geq 1)$ with $p_1 + \ldots + p_r \geq g_0 - 2$ and $q_1 + \ldots + q_s \geq h_0 - 2$ respectively. Then for any integer $p \geq \max\{p_r, q_1\}$, all semigroup identities of the form

$$x_1^{p_1} \cdots x_r^{p_r} z_1^p \cdots z_\ell^p y_1^{q_1} \cdots y_s^{q_s} = x_1^{p_1} \cdots x_r^{p_r} z_{j_1}^p \cdots z_{j_\ell}^p y_1^{q_1} \cdots y_s^{q_s} \tag{15}$$

where $\ell \geq 3$ and j is any permutation of the set $\{1, 2, \ldots, \ell\}$, are preserved under epis in conjunction with (1).

Proof. Take any semigroups U and S with U dense in S. Assume U, and hence S by Result 2.3, satisfy the identity (1). We shall show that if U satisfies (15), then so does S. So let $z_1, z_2, \ldots, z_\ell \in S$. If $z_1, z_2, \ldots, z_\ell \in U$, then the result holds by corollary 3.4.

For ease of writing, we introduce some notation:

$$\begin{aligned} & w_1(x_1, \ldots, x_r, z_{j_1}, \ldots, z_{j_\ell}, y_1, \ldots, y_s) \\ = \ & x_1^{p_1} \cdots x_r^{p_r} z_{j_1}^p \cdots z_{j_\ell}^p y_1^{q_1} \cdots y_s^{q_s} \\ = \ & u_1(x_1, \ldots, x_r, z_1, \ldots, z_\ell, y_1, \ldots, y_s) \end{aligned}$$

and

$$\begin{aligned} & w_2(x_1, \ldots, x_r, z_{j_1}, \ldots, z_{j_\ell}, y_1, \ldots, y_s) \\ = \ & x_1^{p_1} \cdots x_r^{p_r} z_1^p \cdots z_\ell^p y_1^{q_1} \cdots y_s^{q_s} \\ = \ & u_2(x_1, \ldots, x_r, z_1, \ldots, z_\ell, y_1, \ldots, y_s) \end{aligned}$$

Using these definitions, the theorem asserts that

$$w_1(x_1, \ldots, x_r, z_{j_1}, \ldots, z_{j_\ell}, y_1, \ldots, y_s) = w_2(x_1, \ldots, x_r, z_{j_1}, \ldots, z_{j_\ell}, y_1, \ldots, y_s)$$

or, equivalently, that

$$u_1(x_1,\ldots,x_r,z_1,\ldots,z_\ell,y_1,\ldots,y_s)=u_2(x_1,\ldots,x_r,z_1,\ldots,z_\ell,y_1,\ldots,y_s)$$

We will prove the theorem by induction on q, where the elements $x_1,x_2,\ldots,x_r,y_1,$ $y_2,\ldots,y_s,z_{j_1},\ldots,z_{j_{q-1}}$ lie in S $(q\geq 2)$ and the remaining elements $z_{j_q},\ldots,z_{j_\ell}$ lie in U.

First for $q=2$, that is, when $x_1,x_2,\ldots,x_r,y_1,y_2,\ldots,y_s,z_{j_1}\in S$ and $z_{j_2},\ldots,z_{j_\ell}\in U$, we wish to show that (15) holds. When $z_{j_1}\in U$, (15) holds by corollary 3.4. If, on the other hand $z_{j_1}\in S\setminus U$, let (2) be a zigzag of minimal length m over U with value z_{j_1}.

Case(i). $j_1=1$. Now

$$\begin{aligned}
&x_1^{p_1}\cdots x_r^{p_r}z_{j_1}^{p}\cdots z_{j_\ell}^{p}y_1^{q_1}\cdots y_s^{q_s}\\
&= x_1^{p_1}\cdots x_r^{p_r}y_m^{p}a_{2m}^{p}z_{j_2}^{p}\cdots z_{j_\ell}^{p}y_1^{q_1}\cdots y_s^{q_s} \text{ (by the zigzag equations and Result 2.9)}\\
&= x_1^{p_1}\cdots x_r^{p_r}y_m^{p-p_r}y_m^{p_r}a_{2m}^{p}z_{j_2}^{p}\cdots z_{j_\ell}^{p}y_1^{q_1}\cdots y_s^{q_s}\\
&= x_1^{p_1}\cdots x_r^{p_r}y_m^{p-p_r}y_m^{(m)^{p_r}}a_1^{(m)^{p_r}}\cdots a_r^{(m)^{p_r}}a_{2m}^{p}z_{j_2}^{p}\cdots z_{j_\ell}^{p}y_1^{q_1}\cdots y_s^{q_s}\\
&\quad\text{(by Results 2.6 and 2.7 for some } a_1^{(m)},\ldots,a_r^{(m)}\in U \text{ and } y_m^{(m)}\in S\setminus U \text{ as}\\
&\quad y_m\in S\setminus U \text{ and } a_{2m}=a_{2m-1}t_m \text{ with } t_m\in S\setminus U)\\
&= x_1^{p_1}\cdots x_r^{p_r}y_m^{p-p_r}y_m^{(m)^{p_r}}w^{(m)}a_1^{(m)^{p_1}}\cdots a_r^{(m)^{p_r}}a_{2m}^{p}z_{j_2}^{p}\cdots z_{j_\ell}^{p}y_1^{q_1}\cdots y_s^{q_s}\text{(by Result}\\
&\quad\text{2.7 as } y_m^{(m)},t_m\in S\setminus U \text{ and where } w^{(m)}=a_1^{(m)^{p_r-p_1}}\cdots a_{r-1}^{(m)^{p_r-p_{r-1}}})\\
&= x_1^{p_1}\cdots x_r^{p_r}y_m^{p-p_r}y_m^{(m)^{p_r}}w^{(m)}a_1^{(m)^{p_1}}\cdots a_r^{(m)^{p_r}}a_{2m}^{p}z_2^{p}\cdots z_\ell^{p}y_1^{q_1}\cdots y_s^{q_s}\\
&\quad\text{(by corollary 3.4)}\\
&= x_1^{p_1}\cdots x_r^{p_r}y_m^{p-p_r}y_m^{(m)^{p_r}}a_1^{(m)^{p_r}}\cdots a_r^{(m)^{p_r}}a_{2m}^{p}z_2^{p}\cdots z_\ell^{p}y_1^{q_1}\cdots y_s^{q_s}\\
&\quad\text{(by definition of } w^{(m)})
\end{aligned}$$

$$= \quad x_1^{p_1} \cdots x_r^{p_r} y_m^p a_{2m}^p z_2^p \cdots z_\ell^p y_1^{q_1} \cdots y_s^{q_s}$$
(by Results 2.6 and 2.7 as $y_m^{p_r} = y_m^{(m)^{p_r}} a_1^{(m)^{p_r}} \cdots a_r^{(m)^{p_r}}$)

$$= \quad x_1^{p_1} \cdots x_r^{p_r} z_1^p \cdots z_\ell^p y_1^{q_1} \cdots y_s^{q_s}$$ (by the zigzag equations and Result 2.9)

as required.

Case(ii). $1 < j_1 < \ell$. Now letting $k = j_1$, we have

$$x_1^{p_1} \cdots x_r^{p_r} z_{j_1}^p \cdots z_{j_\ell}^p y_1^{q_1} \cdots y_s^{q_s}$$

$$= \quad x_1^{p_1} \cdots x_r^{p_r} y_m^p a_{2m}^p \cdots z_{j_\ell}^p y_1^{q_1} \cdots y_s^{q_s}$$ (by the zigzag equations and Result 2.9)

$$= \quad x_1^{p_1} \cdots x_r^{p_r} y_m^{p-p_r} y_m^{p_r} a_{2m}^p \cdots z_{j_\ell}^p y_1^{q_1} \cdots y_s^{q_s}$$

$$= \quad x_1^{p_1} \cdots x_r^{p_r} y_m^{p-p_r} y_m^{(m)^{p_r}} a_1^{(m)^{p_r}} \cdots a_r^{(m)^{p_r}} a_{2m}^p \cdots z_{j_\ell}^p y_1^{q_1} \cdots y_s^{q_s}$$
(by Results 2.6 and 2.7 for some $a_1^{(m)}, \ldots, a_r^{(m)} \in U$ and $y_m^{(m)} \in S \backslash U$ as $y_m \in S \backslash U$ and $a_{2m} = a_{2m-1} t_m$ with $t_m \in S \backslash U$)

$$= \quad x_1^{p_1} \cdots x_r^{p_r} y_m^{p-p_r} y_m^{(m)^{p_r}} w^{(m)} a_1^{(m)^{p_1}} \cdots a_r^{(m)^{p_r}} a_{2m}^p \cdots z_{j_\ell}^p y_1^{q_1} \cdots y_s^{q_s}$$ (by Result 2.7 as $y_m^{(m)},\ t_m \ \in\ S \backslash U$ and where $w^{(m)} = a_1^{(m)^{p_r - p_1}} \cdots a_{r-1}^{(m)^{p_r - p_{r-1}}}$)

$$= \quad x_1^{p_1} \cdots x_r^{p_r} y_m^{p-p_r} y_m^{(m)^{p_r}} w^{(m)} w_1(a_1^{(m)}, \ldots, a_r^{(m)}, a_{2m}, \ldots, z_{j_\ell}, y_1, \ldots, y_s)$$

$$= \quad x_1^{p_1} \cdots x_r^{p_r} y_m^{p-p_r} y_m^{(m)^{p_r}} w^{(m)} w_2(a_1^{(m)}, \ldots, a_r^{(m)}, a_{2m}, \ldots, z_{j_\ell}, y_1, \ldots, y_s)$$
(by corollary 3.4)

$$= \quad x_1^{p_1} \cdots x_r^{p_r} y_m^{p-p_r} y_m^{(m)^{p_r}} w^{(m)} a_1^{(m)^{p_1}} \cdots a_r^{(m)^{p_r}} z_1^p \cdots z_{k-1}^p a_{2m}^p z_{k+1}^p \cdots z_\ell^p y_1^{q_1} \cdots y_s^{q_s}$$

$$= \quad x_1^{p_1} \cdots x_r^{p_r} y_m^{p-p_r} y_m^{(m)^{p_r}} w^{(m)} a_1^{(m)^{p_1}} \cdots a_r^{(m)^{p_r}} z_1^p \cdots z_{k-1}^p a_{2m-1}^p t_m^p z_{k+1}^p \cdots z_\ell^p y_1^{q_1} \cdots y_s^{q_s},$$

where the last equality holds by the zigzag equations and Result 2.7 as $y_m,\ t_m$ in $S \setminus U$. Now, setting $u^{(m)} = x_1^{p_1} \cdots x_r^{p_r} y_m^{p-p_r} {y_m^{(m)}}^{p_r} w^{(m)}$, we have

$$
\begin{aligned}
& x_1^{p_1} \cdots x_r^{p_r} z_{j_1}^p \cdots z_{j_\ell}^p y_1^{q_1} \cdots y_s^{q_s} \\
= \;& u^{(m)} {a_1^{(m)}}^{p_1} \cdots {a_r^{(m)}}^{p_r} z_1^p \cdots z_{k-1}^p a_{2m-1}^p t_m^p z_{k+1}^p \cdots z_\ell^p y_1^{q_1} \cdots y_s^{q_s} \\
= \;& u^{(m)} {a_1^{(m)}}^{p_1} \cdots {a_r^{(m)}}^{p_r} z_1^p \cdots z_{k-1}^p a_{2m-1}^p b_{k+1}^{(m)p} \cdots b_\ell^{(m)p} t_m^{(m)^p} z_{k+1}^p \cdots z_\ell^p y_1^{q_1} \cdots y_s^{q_s} \\
& \text{(by Results 2.6 and 2.7 for some } b_{k+1}^{(m)}, \ldots, b_l^{(m)} \in U \text{ and } t_m^{(m)} \in S \backslash U \\
& \text{as } y_m^{(m)} \text{ and } t_m \in S \backslash U) \\
= \;& u^{(m)} {a_1^{(m)}}^{p_1} \cdots {a_r^{(m)}}^{p_r} z_1^p \cdots z_{k-1}^p a_{2m-1}^p b_{k+1}^{(m)p} \cdots b_\ell^{(m)p} c_1^{(m)^p} \cdots c_s^{(m)^p} {t_m^{(m)}}'^p d \\
& \text{(by Results 2.6 and 2.7 for some } c_1^{(m)}, \ldots, c_s^{(m)} \in U \text{ and } {t_m^{(m)}}' \in S \backslash U \\
& \text{as } y_m^{(m)} \text{ and } t_m^{(m)} \in S \backslash U; \text{ where } d = z_{k+1}^p \cdots z_\ell^p y_1^{q_1} \cdots y_s^{q_s}) \\
= \;& u^{(m)} {a_1^{(m)}}^{p_1} \cdots {a_r^{(m)}}^{p_r} z_1^p \cdots z_{k-1}^p a_{2m-1}^p b_{k+1}^{(m)p} \cdots b_\ell^{(m)p} {c_1^{(m)}}^{q_1} \cdots {c_s^{(m)}}^{q_s} e^{(m)} {t_m^{(m)}}'^p d \\
& \text{(by Result 2.7 as } y_m^{(m)}, {t_m^{(m)}}' \in \ S \backslash U \text{ and where } e^{(m)} = {c_1^{(m)}}^{p-q_1} \cdots {c_s^{(m)}}^{p-q_s}) \\
= \;& u^{(m)} {a_1^{(m)}}^{p_1} \cdots {a_r^{(m)}}^{p_r} z_1^p \cdots z_{k-1}^p a_{2m-1}^p b_{k+1}^{(m)p} \cdots b_\ell^{(m)p} {c_1^{(m)}}^{q_1} \cdots {c_s^{(m)}}^{q_s} v^{(m)} \\
& \text{(where } v^{(m)} = e^{(m)} {t_m^{(m)}}'^p d) \\
= \;& u^{(m)} u_2(a_1^{(m)}, \ldots, a_r^{(m)}, z_1, \ldots, z_{k-1}, a_{2m-1}, b_{k+1}^{(m)}, \ldots, b_\ell^{(m)}, c_1^{(m)}, \ldots, c_s^{(m)}) v^{(m)} \\
= \;& u^{(m)} u_1(a_1^{(m)}, \ldots, a_r^{(m)}, z_1, \ldots, z_{k-1}, a_{2m-1}, b_{k+1}^{(m)}, \ldots, b_\ell^{(m)}, c_1^{(m)}, \ldots, c_s^{(m)}) v^{(m)} \\
& \text{(by corollary 3.4)}
\end{aligned}
$$

Now the word $u_1(\xi_1, \ldots, \xi_r, z_1, z_2, \ldots, z_\ell, {\xi'}_1, \ldots, {\xi'}_s)$ begins with $\xi_1^{p_1} \cdots \xi_r^{p_r} z_{j_1}^p$ which equals $\xi_1^{p_1} \cdots \xi_r^{p_r} z_k^p$. So, the above product in S contains $y_m^{p-p_r} y_m^{p_r} a_{2m-1}^p$. Thus, using Result 2.7 and the fact that $y_m a_{2m-1} = y_{m-1} a_{2m-2}$ from the zigzag equations, we have

$$
\begin{aligned}
& x_1^{p_1}\cdots x_r^{p_r}z_{j_1}^p\cdots z_{j_\ell}^p y_1^{q_1}\cdots y_s^{q_s}\\
= \;& u^{(m)}u_1(a_1^{(m)},\ldots,a_r^{(m)},z_1,\ldots,z_{k-1},a_{2m-1},b_{k+1}^{(m)},\ldots,b_\ell^{(m)},c_1^{(m)},\ldots,c_s^{(m)})v^{(m)}\\
= \;& u^{(m-1)}u_1(a_1^{(m-1)},\ldots,a_r^{(m-1)},z_1,\ldots,z_{k-1},a_{2m-2},b_{k+1}^{(m)},\ldots,b_\ell^{(m)},c_1^{(m)},\ldots,c_s^{(m)})v^{(m)}\\
& \text{(where } u^{(m-1)}=x_1^{p_1}\cdots x_r^{p_r}y_{m-1}^{p-p_r}y_{m-1}^{(m-1)p_r}a_1^{(m-1)p_r-p_1}\cdots a_{r-1}^{(m-1)p_r-p_{r-1}} \text{ and by}\\
& \text{Results 2.6 and 2.7 for some a}_1^{(m-1)},\ldots,a_r^{(m-1)}\in U \text{ and } y_{m-1}^{(m-1)}\in S\backslash U\\
& \text{as } y_{m-1} \text{ and } t_m^{(m)'}\in S\backslash U)\\
= \;& u^{(m-1)}u_2(a_1^{(m)},\ldots,a_r^{(m)},z_1,\ldots,z_{k-1},a_{2m-2},b_{k+1}^{(m)},\ldots,b_\ell^{(m)},c_1^{(m)},\ldots,c_s^{(m)})v^{(m)}\\
& \text{(by corollary 3.4)}\\
= \;& x_1^{p_1}\cdots x_r^{p_r}y_{m-1}^{p-p_r}y_{m-1}^{p_r}z_1^p\cdots z_{k-1}^p a_{2m-2}^p b_{k+1}^{(m)p}\cdots b_\ell^{(m)p}c_1^{(m)^{q_1}}\cdots c_s^{(m)^{q_s}}e^{(m)}t_m^{(m)'^p}d\\
& \text{(by definition of } u^{(m-1)} \text{ and } v^{(m)})\\
= \;& x_1^{p_1}\cdots x_r^{p_r}y_{m-1}^{p-p_r}y_{m-1}^{p_r}z_1^p\cdots z_{k-1}^p a_{2m-2}^p b_{k+1}^{(m)p}\cdots b_\ell^{(m)p}c_1^{(m)^p}\cdots c_s^{(m)^p}t_m^{(m)'^p}d\\
& \text{(by definition of } e^{(m)})\\
= \;& x_1^{p_1}\cdots x_r^{p_r}y_{m-1}^{p-p_r}y_{m-1}^{p_r}z_1^p\cdots z_{k-1}^p a_{2m-2}^p b_{k+1}^{(m)p}\cdots b_\ell^{(m)p}t_m^{(m)^p}d\\
& \text{(by Result 2.7 as } t_m^{(m)^p}=c_1^{(m)^p}\cdots c_s^{(m)^p}t_m^{(m)'^p})\\
= \;& x_1^{p_1}\cdots x_r^{p_r}y_{m-1}^{p-p_r}y_{m-1}^{p_r}z_1^p\cdots z_{k-1}^p a_{2m-2}^p t_m^p d\\
& \text{(by Result 2.7 as } t_m^p=b_{k+1}^{(m)p}\cdots b_\ell^{(m)p}t_m^{(m)^p})\\
= \;& x_1^{p_1}\cdots x_r^{p_r}y_{m-1}^{p-p_r}y_{m-1}^{p_r}z_1^p z_2^p\cdots z_{k-1}^p a_{2m-3}^p t_{m-1}^p d,\\
& \text{(by the zigzag equations and Result 2.7 as } y_{m-1},t_{m-1}\in S\backslash U)
\end{aligned}
$$

Continuing this way, we obtain

$$x_1^{p_1} \cdots x_r^{p_r} z_{j_1}^p z_{j_2}^p \cdots z_{j_\ell}^p y_1^{q_1} \cdots y_s^{q_s}$$

$$= x_1^{p_1} \cdots x_r^{p_r} y_{m-1}^{p-p_r} y_{m-1}^{p_r} z_1^p z_2^p \cdots z_{k-1}^p a_{2m-3}^p t_{m-1}^p d$$

$$\vdots$$

$$= x_1^{p_1} \cdots x_r^{p_r} y_1^{p-p_r} y_1^{p_r} z_1^p z_2^p \cdots z_{k-1}^p a_1^p t_1^p d$$

$= x_1^{p_1} \cdots x_r^{p_r} y_1^{p-p_r} {y_1^{(1)}}^{p_r} {a_1^{(1)}}^{p_r} \cdots {a_r^{(1)}}^{p_r} z_1^p \cdots z_{k-1}^p a_1^p t_1^p d$ (by Results 2.6 and 2.7 for some $a_1^{(1)}, \ldots, a_r^{(1)} \in U$ and $y_1^{(1)} \in S \backslash U$ as $y_1, t_1 \in S \backslash U$)

$= x_1^{p_1} \cdots x_r^{p_r} y_1^{p-p_r} {y_1^{(1)}}^{p_r} w^{(1)} {a_1^{(1)}}^{p_1} \cdots {a_r^{(1)}}^{p_r} z_1^p \cdots z_{k-1}^p a_1^p t_1^p d$
(by Result 2.7 as $y_1^{(1)},\ t_1 \in S \backslash U$ and where $w^{(1)} = {a_1^{(1)}}^{p_r - p_1} \cdots {a_{r-1}^{(1)}}^{p_r - p_{r-1}}$)

$= u^{(1)} {a_1^{(1)}}^{p_1} \cdots {a_r^{(1)}}^{p_r} z_1^p \cdots z_{k-1}^p a_1^p t_1^p d$
(where $u^{(1)} = x_1^{p_1} \cdots x_r^{p_r} y_1^{p-p_r} {y_1^{(1)}}^{p_r} w^{(1)}$)

$= u^{(1)} {a_1^{(1)}}^{p_1} \cdots {a_r^{(1)}}^{p_r} z_1^p \cdots z_{k-1}^p a_1^p {b_{k+1}^{(1)}}^p \cdots {b_\ell^{(1)}}^p {t_1^{(1)}}^p d$ (by Results 2.6 and 2.7 for some $b_{k+1}^{(1)}, \ldots, b_l^{(1)} \in U$ and $t_1^{(1)} \in S \backslash U$ as $y_1^{(1)}$ and $t_1 \in S \backslash U$)

$= u^{(1)} {a_1^{(1)}}^{p_1} \cdots {a_r^{(1)}}^{p_r} z_1^p \cdots z_{k-1}^p a_1^p {b_{k+1}^{(1)}}^p \cdots {b_\ell^{(1)}}^p {c_1^{(1)}}^p \cdots {c_s^{(1)}}^p {t_1^{(1)}}'^p d$
(by Results 2.6 and 2.7 for some $c_1^{(1)}, \ldots, c_s^{(1)} \in U$ and ${t_1^{(1)}}' \in S \backslash U$ as $y_1^{(1)}$ and $t_1^{(1)} \in S \backslash U$)

$= u^{(1)} {a_1^{(1)}}^{p_1} \cdots {a_r^{(1)}}^{p_r} z_1^p \cdots z_{k-1}^p a_1^p {b_{k+1}^{(1)}}^p \cdots {b_\ell^{(1)}}^p {c_1^{(1)}}^{q_1} \cdots {c_s^{(1)}}^{q_s} e^{(1)} {t_1^{(1)}}'^p d$
(by Result 2.7 as $y_1^{(1)},\ {t_1^{(1)}}' \in S \backslash U$ and where $e^{(1)} = {c_1^{(1)}}^{p-q_1} \cdots {c_s^{(1)}}^{p-q_s}$)

$= u^{(1)} {a_1^{(1)}}^{p_1} \cdots {a_r^{(1)}}^{p_r} z_1^p \cdots z_{k-1}^p a_1^p {b_{k+1}^{(1)}}^p \cdots {b_\ell^{(1)}}^p {c_1^{(1)}}^{q_1} \cdots {c_s^{(1)}}^{q_s} v^{(1)}$
(where $v^{(1)} = e^{(1)} {t_1^{(1)}}'^p d$)

$$= u^{(1)} u_2(a_1^{(1)}, \ldots, a_r^{(1)}, z_1, \ldots, z_{k-1}, a_1, b_{k+1}^{(1)}, \ldots, b_\ell^{(1)}, c_1^{(1)}, \ldots, c_s^{(1)}) v^{(1)}$$

$$= u^{(1)}u_1(a_1^{(1)},\dots,a_r^{(1)},z_1,\dots,z_{k-1},a_1,b_{k+1}^{(1)},\dots,b_\ell^{(1)},c_1^{(1)},\dots,c_s^{(1)})v^{(1)}$$
(by corollary 3.4)

As before, the above product contains the subproduct $y_1^{p-p_r}y_1^{p_r}a_1^p$. Thus, using Result 2.7 and the fact that $y_1a_1 = a_0$ from the zigzag equations, we have

$$x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}z_{j_1}^pz_{j_2}^p\cdots z_{j_\ell}^py_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}$$

$$= u^{(1)}u_1(a_1^{(1)},\dots,a_r^{(1)},z_1,\dots,z_{k-1},a_1,b_{k+1}^{(1)},\dots,b_\ell^{(1)},c_1^{(1)},\dots,c_s^{(1)})v^{(1)}$$

$$= u_1(x_1,\dots,x_r,z_1,\dots,z_{k-1},a_0,b_{k+1}^{(1)},\dots,b_\ell^{(1)},c_1^{(1)},\dots,c_s^{(1)})v^{(1)}$$

$$= u_2(x_1,\dots,x_r,z_1,\dots,z_{k-1},a_0,b_{k+1}^{(1)},\dots,b_\ell^{(1)},c_1^{(1)},\dots,c_s^{(1)})v^{(1)} \text{ (by corollary 3.4)}$$

$$= x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}z_1^pz_2^p\cdots z_{k-1}^pa_0^pb_{k+1}^{(1)p}\cdots b_\ell^{(1)p}c_1^{(1)^{q_1}}\cdots c_s^{(1)^{q_s}}e^{(1)}t_1^{(1)'^p}d$$
(as $v^{(1)} = e^{(1)}t_1^{(1)'^p}d$)

$$= x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}z_1^pz_2^p\cdots z_{k-1}^pa_0^pb_{k+1}^{(1)p}\cdots b_\ell^{(1)p}c_1^{(1)^p}\cdots c_s^{(1)^p}t_1^{(1)'^p}d$$
(by definition of $e^{(1)}$)

$$= x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}z_1^pz_2^p\cdots z_{k-1}^pa_0^pb_{k+1}^{(1)p}\cdots b_\ell^{(1)p}t_1^{(1)^p}d$$
(by Results 2.6 and 2.7 as $c_1^{(1)^p}\cdots c_s^{(1)^p}t_1^{(1)'^p} = t_1^{(1)^p}$)

$$= x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}z_1^pz_2^p\cdots z_{k-1}^pa_0^pt_1^pd$$
(by Results 2.6 and 2.7 as $b_{k+1}^{(1)p}\cdots b_\ell^{(1)p}t_1^{(1)^p} = t_1^p$)

$$= x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}z_1^pz_2^p\cdots z_{k-1}^pz_k^pz_{k+1}^p\cdots z_\ell^py_1^{q_1}y_2^{q_2}\cdots y_s^{q_s},$$

where the last equality follows by the zigzag equations and Result 2.9, and as $d = z_{k+1}^p\cdots z_\ell^py_1^{q_1}\cdots y_s^{q_s}$. This completes the proof in Case(ii).

Case(iii). $j_1 = \ell$. Now

$$
\begin{aligned}
& x_1^{p_1}\cdots x_r^{p_r} z_{j_1}^p z_{j_2}^p \cdots z_{j_\ell}^p y_1^{q_1}\cdots y_s^{q_s} \\
= \;& x_1^{p_1}\cdots x_r^{p_r} y_m^p a_{2m}^p z_{j_2}^p \cdots z_{j_\ell}^p y_1^{q_1}\cdots y_s^{q_s} \\
& \text{(by the zigzag equations and Result 2.9)} \\
= \;& x_1^{p_1}\cdots x_r^{p_r} y_m^{p-p_r} y_m^{p_r} a_{2m}^p z_{j_2}^p \cdots z_{j_\ell}^p y_1^{q_1}\cdots y_s^{q_s} \\
= \;& x_1^{p_1}\cdots x_r^{p_r} y_m^{p-p_r} {y_m^{(m)}}^{p_r} {a_1^{(m)}}^{p_r} \cdots {a_r^{(m)}}^{p_r} a_{2m}^p z_{j_2}^p \cdots z_{j_\ell}^p y_1^{q_1}\cdots y_s^{q_s} \\
& \text{(by Results 2.6 and 2.7 for some } a_1^{(m)},\ldots,a_r^{(m)} \in U \text{ and } y_m^{(m)} \in S\backslash U \text{ as} \\
& y_m \in S\backslash U \text{ and } a_{2m} = a_{2m-1}t_m \text{ with } t_m \in S\backslash U) \\
= \;& x_1^{p_1}\cdots x_r^{p_r} y_m^{p-p_r} {y_m^{(m)}}^{p_r} w^{(m)} {a_1^{(m)}}^{p_1} \cdots {a_r^{(m)}}^{p_r} a_{2m}^p z_{j_2}^p \cdots z_{j_\ell}^p y_1^{q_1}\cdots y_s^{q_s} \text{ (by Result 2.7} \\
& \text{as } y_m^{(m)},\ t_m \in S\backslash U \text{ and where } w^{(m)} = {a_1^{(m)}}^{p_r-p_1} \cdots {a_{r-1}^{(m)}}^{p_r-p_{r-1}}) \\
= \;& u^{(m)} w_1(a_1^{(m)},\ldots,a_r^{(m)}, a_{2m}, z_{j_2},\ldots,z_{j_\ell}, y_1,\ldots,y_s) \\
& \text{(where } u^{(m)} = x_1^{p_1}\cdots x_r^{p_r} y_m^{p-p_r} {y_m^{(m)}}^{p_r} w^{(m)}) \\
= \;& u^{(m)} w_2(a_1^{(m)},\ldots,a_r^{(m)}, a_{2m}, z_{j_2},\ldots,z_{j_\ell}, y_1,\ldots,y_s) \text{ (by corollary 3.4)} \\
= \;& u^{(m)} {a_1^{(m)}}^{p_1} \cdots {a_r^{(m)}}^{p_r} z_1^p z_2^p \cdots z_{\ell-1}^p a_{2m}^p y_1^{q_1}\cdots y_s^{q_s} \\
= \;& u^{(m)} {a_1^{(m)}}^{p_1} \cdots {a_r^{(m)}}^{p_r} z_1^p \cdots z_{\ell-1}^p a_{2m-1}^p t_m^p y_1^{q_1}\cdots y_s^{q_s} \\
& \text{(by the zigzag equations and Result 2.7)} \\
= \;& u^{(m)} {a_1^{(m)}}^{p_1} \cdots {a_r^{(m)}}^{p_r} z_1^p \cdots z_{\ell-1}^p a_{2m-1}^p {c_1^{(m)}}^p \cdots {c_s^{(m)}}^p {t_m^{(m)}}^p y_1^{q_1}\cdots y_s^{q_s} \\
& \text{(by Results 2.6 and 2.7 for some } c_1^{(m)},\ldots,c_s^{(m)} \in U \text{ and } t_m^{(m)} \in S\backslash U \\
& \text{as } y_m \text{ and } t_m \in S\backslash U)
\end{aligned}
$$

$$= u^{(m)} a_1^{(m)^{p_1}} \cdots a_r^{(m)^{p_r}} z_1^p \cdots z_{\ell-1}^p a_{2m-1}^p c_1^{(m)^{q_1}} \cdots c_s^{(m)^{q_s}} e^{(m)} t_m^{(m)^p} y_1^{q_1} \cdots y_s^{q_s}$$

(by Result 2.7 as $y_m,\ t_m^{(m)} \in S\backslash U$ and where $e^{(m)} = c_1^{(m)^{p-q_1}} \cdots c_s^{(m)^{p-q_s}}$)

$$= u^{(m)} a_1^{(m)^{p_1}} \cdots a_r^{(m)^{p_r}} z_1^p \cdots z_{\ell-1}^p a_{2m-1}^p c_1^{(m)^{q_1}} \cdots c_s^{(m)^{q_s}} v^{(m)}$$

(where $v^{(m)} = e^{(m)} t_m^{(m)^p} y_1^{q_1} \cdots y_s^{q_s}$)

$$= u^{(m)} u_2(a_1^{(m)}, \ldots, a_r^{(m)}, z_1, \ldots, z_{l-1}, a_{2m-1}, c_1^{(m)}, \ldots, c_s^{(m)}) v^{(m)}$$

$$= u^{(m)} u_1(a_1^{(m)}, \ldots, a_r^{(m)}, z_1, \ldots, z_{l-1}, a_{2m-1}, c_1^{(m)}, \ldots, c_s^{(m)}) v^{(m)} \text{(by corollary 3.4).}$$

Now the word $u_1(\xi_1, \cdots, \xi_r, z_1, z_2, \ldots, z_\ell, {\xi'}_1, \cdots, {\xi'}_s)$ begins with $\xi_1^{p_1} \cdots \xi_r^{p_r} z_{j_1}^p$ which equals $\xi_1^{p_1} \cdots \xi_r^{p_r} z_l^p$. So, the above product in S contains $y_m^{p-p_r} y_m^{p_r} a_{2m-1}^p$. Thus, using Result 2.7 and the fact that $y_m a_{2m-1} = y_{m-1} a_{2m-2}$ from the zigzag equations, we have

$$x_1^{p_1} \cdots x_r^{p_r} z_{j_1}^p \cdots z_{j_\ell}^p y_1^{q_1} \cdots y_s^{q_s}$$

$$= u^{(m)} u_1(a_1^{(m)}, \ldots, a_r^{(m)}, z_1, \ldots, z_{l-1}, a_{2m-1}, c_1^{(m)}, \ldots, c_s^{(m)}) v^{(m)}$$

$$= u^{(m-1)} u_1(a_1^{(m-1)}, \ldots, a_r^{(m-1)}, z_1, \ldots, z_{l-1}, a_{2m-2}, c_1^{(m)}, \ldots, c_s^{(m)}) v^{(m)}$$

(where $u^{(m-1)} = x_1^{p_1} \cdots x_r^{p_r} y_{m-1}^{p-p_r} y_{m-1}^{(m-1)^{p_r}} a_1^{(m-1)^{p_r-p_1}} \cdots a_{r-1}^{(m-1)^{p_r-p_{r-1}}}$
and by Results 2.6 and 2.7 for some $a_1^{(m-1)}, \ldots, a_r^{(m-1)} \in U$ and
$y_{m-1}^{(m-1)} \in S\backslash U$ as y_{m-1} and $t_m^{(m)} \in S\backslash U$)

$$= x_1^{p_1} \cdots x_r^{p_r} y_{m-1}^{p-p_r} y_{m-1}^{p_r} z_1^p z_2^p \cdots z_{\ell-1}^p a_{2m-2}^p c_1^{(m)^{q_1}} \cdots c_s^{(m)^{q_s}} e^{(m)} t_m^{(m)^p} y_1^{q_1} \cdots y_s^{q_s}$$

(by definitions of $u^{(m-1)}$ and $v^{(m)}$)

$$= x_1^{p_1} \cdots x_r^{p_r} y_{m-1}^{p-p_r} y_{m-1}^{p_r} z_1^p z_2^p \cdots z_{\ell-1}^p a_{2m-2}^p c_1^{(m)^p} \cdots c_s^{(m)^p} t_m^{(m)^p} y_1^{q_1} \cdots y_s^{q_s}$$

(by definition of $e^{(m)}$)

$$\begin{aligned}
&= x_1^{p_1}\cdots x_r^{p_r} y_{m-1}^{p-p_r} y_{m-1}^{p_r} z_1^p z_2^p \cdots z_{\ell-1}^p a_{2m-2}^p t_m^p y_1^{q_1}\cdots y_s^{q_s} \\
&\quad \text{(by Result 2.7 as } t_m^p = c_1^{(m)^p}\cdots c_s^{(m)^p} t_m^{(m)^p}) \\
&= x_1^{p_1}\cdots x_r^{p_r} y_{m-1}^{p-p_r} y_{m-1}^{p_r} z_1^p z_2^p \cdots z_{\ell-1}^p a_{2m-3}^p t_{m-1}^p y_1^{q_1}\cdots y_s^{q_s},
\end{aligned}$$

where the last equality follows by Result 2.7 and the zigzag equations.

Continuing this way and letting $y = y_1^{q_1} y_2^{q_2}\cdots y_s^{q_s}$, we obtain

$$\begin{aligned}
&x_1^{p_1}\cdots x_r^{p_r} z_{j_1}^p z_{j_2}^p \cdots z_{j_\ell}^p y_1^{q_1} y_2^{q_2}\cdots y_s^{q_s} \\
&= x_1^{p_1}\cdots x_r^{p_r} y_{m-1}^{p-p_r} y_{m-1}^{p_r} z_1^p \cdots z_{\ell-1}^p a_{2m-3}^p t_{m-1}^p y \\
&\vdots \\
&= x_1^{p_1}\cdots x_r^{p_r} y_1^{p-p_r} y_1^{p_r} z_1^p \cdots z_{\ell-1}^p a_1^p t_1^p y \\
&= x_1^{p_1}\cdots x_r^{p_r} y_1^{p-p_r} y_1^{(1)^{p_r}} a_1^{(1)^{p_r}} \cdots a_r^{(1)^{p_r}} z_1^p \cdots z_{l-1}^p a_1^p t_1^p y \text{ (by Results 2.6 and 2.7} \\
&\quad \text{for some } a_1^{(1)},\ldots,a_r^{(1)} \in U \text{ and } y_1^{(1)} \in S\backslash U \text{ as } y_1, t_1 \in S\backslash U) \\
&= x_1^{p_1}\cdots x_r^{p_r} y_1^{p-p_r} y_1^{(1)^{p_r}} w^{(1)} a_1^{(1)^{p_1}} \cdots a_r^{(1)^{p_r}} z_1^p \cdots z_{l-1}^p a_1^p t_1^p y \\
&\quad \text{(by Result 2.7 as } y_1^{(1)},\ t_1 \in S\backslash U \text{ and where } w^{(1)} = a_1^{(1)^{p_r-p_1}} \cdots a_{r-1}^{(1)\ p_r-p_{r-1}}) \\
&= u^{(1)} a_1^{(1)^{p_1}} \cdots a_r^{(1)^{p_r}} z_1^p \cdots z_{l-1}^p a_1^p t_1^p y \\
&\quad \text{(where } u^{(1)} = x_1^{p_1}\cdots x_r^{p_r} y_1^{p-p_r} y_1^{(1)^{p_r}} w^{(1)}) \\
&= u^{(1)} a_1^{(1)^{p_1}} \cdots a_r^{(1)^{p_r}} z_1^p \cdots z_{l-1}^p a_1^p c_1^{(1)^p} \cdots c_s^{(1)^p} t_1^{(1)^p} y \text{ (by Results 2.6 and 2.7} \\
&\quad \text{for some } c_1^{(1)},\ldots,c_s^{(1)} \in U \text{ and } t_1^{(1)} \in S\backslash U \text{ as } y_1^{(1)} \text{ and } t_1 \in S\backslash U) \\
&= u^{(1)} a_1^{(1)^{p_1}} \cdots a_r^{(1)^{p_r}} z_1^p \cdots z_{l-1}^p a_1^p c_1^{(1)^{q_1}} \cdots c_s^{(1)^{q_s}} e^{(1)} t_1^{(1)^p} y \\
&\quad \text{(by Result 2.7 as } y_1^{(1)}, t_1^{(1)} \in S\backslash U \text{ and where } e^{(1)} = c_1^{(1)^{p-q_1}} \cdots c_s^{(1)^{p-q_s}})
\end{aligned}$$

$$= \quad u^{(1)}u_2(a_1^{(1)},\ldots,a_r^{(1)},z_1,\ldots,z_{l-1},a_1,c_1^{(1)},\ldots,c_s^{(1)})v^{(1)}$$
$$\text{(where } v^{(1)} = e^{(1)}{t_1^{(1)}}^p y\text{)}$$

$$= \quad u^{(1)}u_1(a_1^{(1)},\ldots,a_r^{(1)},z_1,\ldots,z_{l-1},a_1,c_1^{(1)},\ldots,c_s^{(1)})v^{(1)} \text{ (by corollary 3.4)}$$

As before, the above product in S contains the subproduct $y_1^{p-p_r}y_1^{p_r}a_1^p$. Thus, using Result 2.7 and the fact that $y_1a_1 = a_0$ from the zigzag equations, we have

$$x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}z_{j_1}^pz_{j_2}^p\cdots z_{j_\ell}^py_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}$$

$$= \quad u^{(1)}u_1(a_1^{(1)},\ldots,a_r^{(1)},z_1,\ldots,z_{l-1},a_1,c_1^{(1)},\ldots,c_s^{(1)})v^{(1)}$$

$$= \quad u_1(x_1,\ldots,x_r,z_1,\ldots,z_{\ell-1},a_0,c_1^{(1)},\ldots,c_s^{(1)})v^{(1)}$$

$$= \quad u_2(x_1,\ldots,x_r,z_1,\ldots,z_{\ell-1},a_0,c_1^{(1)},\ldots,c_s^{(1)})v^{(1)}\text{(by corollary 3.4)}$$

$$= \quad x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}z_1^pz_2^p\cdots z_{l-1}^pa_0^p{c_1^{(1)}}^{q_1}\ldots {c_s^{(1)}}^{q_s}e^{(1)}{t_1^{(1)}}^p y \text{ (by definition of } v^{(1)}\text{)}$$

$$= \quad x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}z_1^pz_2^p\cdots z_{l-1}^pa_0^p{c_1^{(1)}}^{p}\cdots {c_s^{(1)}}^{p}{t_1^{(1)}}^p y \text{ (by definition of } e^{(1)}\text{)}$$

$$= \quad x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}z_1^pz_2^p\cdots z_{l-1}^pa_0^pt_1^py$$
$$\text{(by Results 2.6 and 2.7 as } {c_1^{(1)}}^{p}\cdots {c_s^{(1)}}^{p}{t_1^{(1)}}^p = t_1^p\text{)}$$

$$= \quad x_1^{p_1}\cdots x_r^{p_r}z_1^pz_2^p\cdots z_{\ell-1}^pz_\ell^py_1^{q_1}\cdots y_s^{q_s}$$

where the last equality follows by zigzag equations, Result 2.9; and as $j_1 = \ell$ and $y = y_1^{q_1}\cdots y_s^{q_s}$.

This is the end of the proof in case(iii) and, thus, of the base $q = 2$ of the induction.

Next, assume inductively that the result holds when $x_1, \ldots, x_r, y_1, \ldots, y_s, z_{j_1}, \ldots, z_{j_{q-1}}$ are in S ($q > 2$) and $z_{j_q}, \ldots, z_{j_\ell} \in U$. From this assumption, we shall prove that the result also holds when $x_1, \ldots, x_r, y_1, \ldots, y_s, z_{j_1}, \ldots, z_{j_{q-1}}, z_{j_q} \in S$ and $z_{j_{q+1}}, \ldots, z_{j_\ell}$ in U. So take any $x_1, \ldots, x_r, y_1, \ldots, y_s, z_{j_1}, z_{j_2}, \ldots, z_{j_{q-1}}, z_{j_q} \in S$ and $z_{j_{q+1}}, \ldots, z_{j_\ell} \in U$. Assume that $z_{j_q} \in S \setminus U$. Let (2) be a zigzag of minimal length m over U with value z_{j_q}. Put $k = j_q$ and $t = j_{q-1}$. As equalities (16) and (17) below hold by Proposition 3.2 as $y_m \in S \setminus U$ and $p_1 + \cdots + p_r + 1 \geq g_0 - 1$, we have

Case (1). $t = k - 1$. Now

$$
\begin{aligned}
& x_1^{p_1} \cdots x_r^{p_r} z_{j_1}^p \cdots z_{j_\ell}^p y_1^{q_1} \cdots y_s^{q_s} \\
= \;& x_1^{p_1} \cdots x_r^{p_r} z_{j_1}^p \cdots z_{j_{q-1}}^p y_m^p a_{2m}^p z_{j_{q+1}}^p \cdots z_{j_\ell}^p y_1^{q_1} \cdots y_s^{q_s} \\
& \text{(by the zigzag equations and Result 2.9)} \\
= \;& x_1^{p_1} \cdots x_r^{p_r} z_{j_1}^p \cdots (z_{j_{q-1}} y_m)^p a_{2m}^p z_{j_{q+1}}^p \cdots z_{j_\ell}^p y_1^{q_1} \cdots y_s^{q_s} \qquad (16) \\
= \;& w_1(x_1, \ldots, x_r, z_{j_1}, \ldots, z_{j_{q-2}}, z_{j_{q-1}} y_m, a_{2m}, z_{j_{q+1}}, \ldots, z_{j_\ell}, y_1, \ldots, y_s) \\
= \;& w_2(x_1, \ldots, x_r, z_{j_1}, \ldots, z_{j_{q-2}}, z_{j_{q-1}} y_m, a_{2m}, z_{j_{q+1}}, \ldots, z_{j_\ell}, y_1, \ldots, y_s) \\
& \text{(by the inductive hypothesis)} \\
= \;& x_1^{p_1} \cdots x_r^{p_r} z_1^p \cdots z_{k-2}^p (z_{j_{q-1}} y_m)^p a_{2m}^p z_{k+1}^p \cdots z_\ell^p y_1^{q_1} \cdots y_s^{q_s} \text{ (as } t = k-1) \\
= \;& x_1^{p_1} \cdots x_r^{p_r} z_1^p \cdots z_{k-2}^p z_{j_{q-1}}^p y_m^p a_{2m}^p z_{k+1}^p \cdots z_\ell^p y_1^{q_1} \cdots y_s^{q_s} \qquad (17) \\
= \;& x_1^{p_1} \cdots x_r^{p_r} z_1^p \cdots z_{k-2}^p z_{j_{q-1}}^p z_k^p z_{k+1}^p \cdots z_\ell^p y_1^{q_1} \cdots y_s^{q_s} \\
& \text{(by the zigzag equations and Result 2.9)} \\
= \;& x_1^{p_1} \cdots x_r^{p_r} z_1^p \cdots z_{k-2}^p z_{k-1}^p z_k^p z_{k+1}^p \cdots z_\ell^p y_1^{q_1} \cdots y_s^{q_s} \text{ (as } j_{q-1} = k-1)
\end{aligned}
$$

as required.

Case (2). $t < k-1$ and $k \leq \ell$. Now, as above, we have

$$
\begin{aligned}
& x_1^{p_1}\cdots x_r^{p_r} z_{j_1}^{p}\cdots z_{j_\ell}^{p} y_1^{q_1}\cdots y_s^{q_s} \\
= \;& x_1^{p_1}\cdots x_r^{p_r} z_{j_1}^{p}\cdots z_{j_{q-1}}^{p} y_m^{p} a_{2m}^{p} z_{j_{q+1}}^{p}\cdots z_{j_\ell}^{p} y_1^{q_1}\cdots y_s^{q_s} \\
& \text{(by the zigzag equations and Result 2.9)} \\
= \;& x_1^{p_1}\cdots x_r^{p_r} z_{j_1}^{p}\cdots (z_{j_{q-1}} y_m)^{p} a_{2m}^{p} z_{j_{q+1}}^{p}\cdots z_{j_\ell}^{p} y_1^{q_1}\cdots y_s^{q_s} \\
& \text{(by Proposition 3.2 as } y_m \in S\setminus U \text{ and } p_1+\ldots+p_r+1 \geq g_0-1) \\
= \;& w_1(x_1,\ldots,x_r, z_{j_1},\ldots, z_{j_{q-2}}, z_{j_{q-1}} y_m, a_{2m}, z_{j_{q+1}},\ldots, z_{j_\ell}, y_1,\ldots,y_s) \\
= \;& w_2(x_1,\ldots,x_r, z_{j_1},\ldots, z_{j_{q-2}}, z_{j_{q-1}} y_m, a_{2m}, z_{j_{q+1}},\ldots, z_{j_\ell}, y_1,\ldots,y_s) \\
& \text{(by the inductive hypothesis)} \\
= \;& x_1^{p_1}\cdots x_r^{p_r} z_1^{p}\cdots z_{t-1}^{p} (z_{j_{q-1}} y_m)^{p} z_{t+1}^{p}\cdots z_{k-1}^{p} a_{2m}^{p} z_{k+1}^{p}\cdots z_\ell^{p} y_1^{q_1}\cdots y_s^{q_s} \\
= \;& x_1^{p_1}\cdots x_r^{p_r} z_1^{p}\cdots z_{t-1}^{p} z_{j_{q-1}}^{p} z_{t+1}^{p}\cdots z_{k-1}^{p} y_m^{p} a_{2m}^{p} z_{k+1}^{p}\cdots z_\ell^{p} y_1^{q_1}\cdots y_s^{q_s} \qquad (18) \\
= \;& x_1^{p_1}\cdots x_r^{p_r} z_1^{p}\cdots z_{t-1}^{p} z_t^{p} z_{t+1}^{p}\cdots z_{k-1}^{p} z_k^{p} z_{k+1}^{p}\cdots z_\ell^{p} y_1^{q_1}\cdots y_s^{q_s},
\end{aligned}
$$

where the last equality holds by the zigzag equations and Result 2.9, and the equality (18) above follows by Proposition 3.2 as $p_1+\cdots+p_r+1 \geq g_0-1$ and $a_{2m} = a_{2m-1}t_m$ with $t_m \in S\setminus U$, as required.

Case (3). $k+1 = t$. As the equality (19) holds by the inductive hypothesis, and equalities (20) and (21) follow by the zigzag equations and by Proposition 3.2 as y_m in $S\setminus U$ and $p_1+\cdots+p_r+1 \geq g_0-1$ respectively, we have

$$x_1^{p_1}\cdots x_r^{p_r}z_{j_1}^p\cdots z_{j_\ell}^p y_1^{q_1}\cdots y_s^{q_s}$$

$$= x_1^{p_1}\cdots x_r^{p_r}z_{j_1}^p\cdots z_{j_{q-1}}^p y_m^p a_{2m}^p z_{j_{q+1}}^p\cdots z_{j_\ell}^p y_1^{q_1}\cdots y_s^{q_s}$$
(by the zigzag equations and Result 2.9)

$$= x_1^{p_1}\cdots x_r^{p_r}z_{j_1}^p\cdots (z_{j_{q-1}}y_m)^p a_{2m}^p\cdots z_{j_\ell}^p y_1^{q_1}\cdots y_s^{q_s}$$
(by Proposition 3.2 as $y_m\in S\setminus U$ and $p_1+\cdots+p_r+1\geq g_0-1$)

$$= w_1(x_1,\ldots,x_r,z_{j_1},\ldots,z_{j_{q-2}},z_{j_{q-1}}y_m,a_{2m},z_{j_{q+1}},\ldots,z_{j_\ell},y_1,\ldots,y_s)$$

$$= w_2(x_1,\ldots,x_r,z_{j_1},\ldots,z_{j_{q-2}},z_{j_{q-1}}y_m,a_{2m},z_{j_{q+1}},\ldots,z_{j_\ell},y_1,\ldots,y_s) \tag{19}$$

$$= w_2(x_1,\ldots,x_r,z_{j_1},\ldots,z_{j_{q-2}},z_{j_{q-1}}y_m,a_{2m-1}t_m,z_{j_{q+1}},\ldots,z_{j_\ell},y_1,\ldots,y_s) \tag{20}$$

$$= x_1^{p_1}\cdots x_r^{p_r}z_1^p\cdots z_{k-1}^p(a_{2m-1}t_m)^p(z_{j_{q-1}}y_m)^p z_{t+1}^p\cdots z_\ell^p y_1^{q_1}\cdots y_s^{q_s}$$

$$= x_1^{p_1}\cdots x_r^{p_r}z_1^p\cdots z_{k-1}^p a_{2m-1}^p(t_m z_{j_{q-1}}y_m)^p z_{t+1}^p\cdots z_\ell^p y_1^{q_1}\cdots y_s^{q_s} \tag{21}$$

$$= u_2(x_1,\ldots,x_r,z_1,\ldots,z_{k-1},a_{2m-1},t_m z_{j_{q-1}}y_m,z_{t+1},\ldots,z_\ell,y_1,\ldots,y_s)$$

$$= u_1(x_1,\ldots,x_r,z_1,\ldots,z_{k-1},a_{2m-1},t_m z_{j_{q-1}}y_m,z_{t+1},\ldots,z_\ell,y_1,\ldots,y_s),$$

where the last equality holds by the inductive hypothesis. Since $z_{j_{q-1}}z_{j_q}$ is a subword of the word $u_1(\xi_1,\ldots,\xi_r,z_1,\ldots,z_\ell,\xi_1',\ldots,\xi_s')$, the above product in S contains $(t_m z_{j_{q-1}}y_m)^p a_{2m-1}^p$. Thus, using dual of Proposition 3.2 as $t_m\in S\setminus U$ and $y_1^{q_1}\cdots y_s^{q_s}\in S^{q_s}$, and the fact that $y_m a_{2m-1}=y_{m-1}a_{2m-2}$ from the zigzag equations, we have

$$
\begin{aligned}
&x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}z_{j_1}^p z_{j_2}^p\cdots z_{j_\ell}^p y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}\\
&= u_1(x_1,\dots,x_r,z_1,z_2,\dots,z_{k-1},a_{2m-1},t_m z_{j_{q-1}}y_m,z_{t+1},\dots,z_\ell,y_1,\dots,y_s)\\
&= u_1(x_1,\dots,x_r,z_1,z_2,\dots,z_{k-1},a_{2m-2},t_m z_{j_{q-1}}y_{m-1},z_{t+1},\dots,z_\ell,y_1,\dots,y_s)\\
&= u_2(x_1,\dots,x_r,z_1,\dots,z_{k-1},a_{2m-2},t_m z_{j_{q-1}}y_{m-1},z_{t+1},\dots,z_\ell,y_1,\dots,y_s) \qquad (22)\\
&= x_1^{p_1}\cdots x_r^{p_r}z_1^p\cdots z_{k-1}^p a_{2m-2}^p(t_m z_{j_{q-1}}y_{m-1})^p z_{t+1}^p\cdots z_\ell^p y_1^{q_1}\cdots y_s^{q_s}\\
&= x_1^{p_1}\cdots x_r^{p_r}z_1^p\cdots z_{k-1}^p a_{2m-2}^p t_m^p(z_{j_{q-1}}y_{m-1})^p z_{t+1}^p\cdots z_\ell^p y_1^{q_1}\cdots y_s^{q_s},
\end{aligned}
$$

where the last equality and the equality (22) above follow by Proposition 3.2, as y_{m-1} in $S\setminus U$ and $p_1+\cdots+p_r+1\geq g_0-1$, and the inductive hypothesis respectively. As equalities (23) and (24) below follow by zigzag equations and Proposition 3.2, as $y_{m-1}\in S\setminus U$ and $p_1+\cdots+p_r+1\geq g_0-1$, we have

$$
\begin{aligned}
&x_1^{p_1}\cdots x_r^{p_r}z_{j_1}^p\cdots z_{j_\ell}^p y_1^{q_1}\cdots y_s^{q_s}\\
&= x_1^{p_1}\cdots x_r^{p_r}z_1^p\cdots z_{k-1}^p a_{2m-2}^p t_m^p(z_{j_{q-1}}y_{m-1})^p z_{t+1}^p\cdots z_\ell^p y_1^{q_1}\cdots y_s^{q_s}\\
&= x_1^{p_1}\cdots x_r^{p_r}z_1^p\cdots z_{k-1}^p a_{2m-3}^p t_{m-1}^p(z_{j_{q-1}}y_{m-1})^p z_{t+1}^p\cdots z_\ell^p y_1^{q_1}\cdots y_s^{q_s} \qquad (23)\\
&= x_1^{p_1}\cdots x_r^{p_r}z_1^p z_2^p\cdots z_{k-1}^p a_{2m-3}^p(t_{m-1}z_{j_{q-1}}y_{m-1})^p z_{t+1}^p\cdots z_\ell^p y_1^{q_1}\cdots y_s^{q_s} \qquad (24)
\end{aligned}
$$

Thus, continuing this way, we obtain

$$
\begin{aligned}
&x_1^{p_1}\cdots x_r^{p_r}z_{j_1}^p\cdots z_{j_\ell}^p y_1^{q_1}\cdots y_s^{q_s}\\
&= x_1^{p_1}\cdots x_r^{p_r}z_1^p\cdots z_{k-1}^p a_{2m-3}^p(t_{m-1}z_{j_{q-1}}y_{m-1})^p z_{t+1}^p\cdots z_\ell^p y_1^{q_1}\cdots y_s^{q_s}\\
&\ \vdots
\end{aligned}
$$

$$= \quad x_1^{p_1}\cdots x_r^{p_r} z_1^p z_2^p \cdots z_{k-1}^p a_1^p (t_1 z_{j_{q-1}} y_1)^p z_{t+1}^p \cdots z_\ell^p y_1^{q_1}\cdots y_s^{q_s}$$

$$= \quad u_2(x_1,\ldots,x_r,z_1,\ldots,z_{k-1},a_1,t_1 z_{j_{q-1}} y_1, z_{t+1},\ldots,z_\ell,y_1,\ldots,y_s)$$

$$= \quad u_1(x_1,\ldots,x_r,z_1,\ldots,z_{k-1},a_1,t_1 z_{j_{q-1}} y_1, z_{t+1},\ldots,z_\ell,y_1,\ldots,y_s)$$

(by the inductive hypothesis)

As before, the word $u_1(\xi_1,\ldots,\xi_r,z_1,\ldots,z_\ell,\xi'_1,\ldots,\xi'_s)$ contains $z_{j_{q-1}} z_{j_q}$ as a subword, so the above product in S contains $(t_1 z_{j_{q-1}} y_1)^p a_1^p$. Thus, using Proposition 3.2 and the fact that $y_1 a_1 = a_0$ from the zigzag equations, we have

$$x_1^{p_1}\cdots x_r^{p_r} z_{j_1}^p \cdots z_{j_\ell}^p y_1^{q_1}\cdots y_s^{q_s}$$

$$= \quad u_1(x_1,\ldots,x_r,z_1,\ldots,z_{k-1},a_1,t_1 z_{j_{q-1}} y_1, z_{t+1},\ldots,z_\ell,y_1,\ldots,y_s)$$

$$= \quad u_1(x_1,\ldots,x_r,z_1,z_2,\ldots,z_{k-1},a_0,t_1 z_{j_{q-1}}, z_{t+1},\ldots,z_\ell,y_1,\ldots,y_s)$$

$$= \quad u_2(x_1,\ldots,x_r,z_1,z_2,\ldots,z_{k-1},a_0,t_1 z_{j_{q-1}}, z_{t+1},\ldots,z_\ell,y_1,\ldots,y_s)$$

(by the inductive hypothesis)

$$= \quad x_1^{p_1}\cdots x_r^{p_r} z_1^p \cdots z_{k-1}^p a_0^p (t_1 z_{j_{q-1}})^p z_{t+1}^p \cdots z_\ell^p y_1^{q_1}\cdots y_s^{q_s}$$

$$= \quad x_1^{p_1}\cdots x_r^{p_r} z_1^p \cdots z_{k-1}^p a_0^p t_1^p z_{j_{q-1}}^p \cdots z_\ell^p y_1^{q_1}\cdots y_s^{q_s}$$

(by Proposition 3.2 as $t_1 \in S\setminus U$ and $p_1+\ldots+p_r+1 \geq g_0-1$)

$$= \quad x_1^{p_1}\cdots x_r^{p_r} z_1^p \cdots z_{k-1}^p z_k^p z_{j_{q-1}}^p z_{t+1}^p \cdots z_\ell^p y_1^{q_1}\cdots y_s^{q_s}$$

(by Result 2.9 and the zigzag equations)

$$= \quad x_1^{p_1}\cdots x_r^{p_r} z_1^p \cdots z_{k-1}^p z_k^p z_{k+1}^p z_{t+1}^p \cdots z_\ell^p y_1^{q_1}\cdots y_s^{q_s} \text{ (as } j_{q-1}=t)$$

as required.

Case (4). $k+1<t$ and $t<\ell$. As the equality (25) below follows by Proposition 3.2 because $y_m \in S\setminus U$ and $p_1+\cdots+p_r+1\geq g_0-1$, we have

$$x_1^{p_1}\cdots x_r^{p_r}z_{j_1}^p\cdots z_{j_\ell}^p y_1^{q_1}\cdots y_s^{q_s}$$

$$= x_1^{p_1}\cdots x_r^{p_r}z_{j_1}^p\cdots z_{j_{q-1}}^p y_m^p a_{2m}^p z_{j_{q+1}}^p\cdots z_{j_\ell}^p y_1^{q_1}\cdots y_s^{q_s}$$
(by the zigzag equations and Result 2.9)

$$= x_1^{p_1}\cdots x_r^{p_r}z_{j_1}^p\cdots z_{j_{q-2}}^p (z_{j_{q-1}}y_m)^p a_{2m}^p z_{j_{q+1}}^p\cdots z_{j_\ell}^p y_1^{q_1}\cdots y_s^{q_s} \tag{25}$$

$$= w_1(x_1,\ldots,x_r,z_{j_1},\ldots,z_{j_{q-2}},z_{j_{q-1}}y_m,a_{2m},z_{j_{q+1}},\ldots,z_{j_\ell},y_1,\ldots,y_s)$$

$$= w_2(x_1,\ldots,x_r,z_{j_1},\ldots,z_{j_{q-2}},z_{j_{q-1}}y_m,a_{2m},z_{j_{q+1}},\ldots,z_{j_\ell},y_1,\ldots,y_s)$$
(by the inductive hypothesis)

$$= x_1^{p_1}\cdots x_r^{p_r}z_1^p\cdots z_{k-1}^p a_{2m}^p z_{k+1}^p\cdots z_{t-1}^p (z_{j_{q-1}}y_m)^p z_{t+1}^p\cdots z_\ell^p y_1^{q_1}\cdots y_s^{q_s}$$

$$= x_1^{p_1}\cdots x_r^{p_r}z_1^p\cdots z_{k-1}^p a_{2m-1}^p t_m^p z_{k+1}^p\cdots z_{t-1}^p (z_{j_{q-1}}y_m)^p z_{t+1}^p\cdots z_\ell^p y_1^{q_1}\cdots y_s^{q_s},$$

where the last equality holds by the zigzag equations and Proposition 3.2 as t_m is in $S\setminus U$ and $p_1+\cdots+p_r+1\geq g_0-1$. As the equality (26) below holds by Results 2.6 and Proposition 3.2 for some $b_{k+1}^{(m)},\ldots,b_{k+(t-k-1)}^{(m)}\in U$ and $t_m^{(m)}\in S\setminus U$, as $t_m\in S\setminus U$ and letting $z=z_{t+1}^p\cdots z_\ell^p y_1^{q_1}\cdots y_s^{q_s}$ and $w_i^{(k)}=t_i^{(i)}z_{k+1}\cdots z_{t-1}(z_{j_{q-1}}y_k)$ where $i,k\in\{1,2,\ldots,m\}$, we have

$$x_1^{p_1}\cdots x_r^{p_r}z_{j_1}^p\cdots z_{j_\ell}^p y_1^{q_1}\cdots y_s^{q_s}$$

$$= x_1^{p_1}\cdots x_r^{p_r}z_1^p\cdots z_{k-1}^p a_{2m-1}^p t_m^p z_{k+1}^p\cdots z_{t-1}^p (z_{j_{q-1}}y_m)^p z$$

$$= x_1^{p_1}\cdots x_r^{p_r}z_1^p\cdots a_{2m-1}^p b_{k+1}^{(m)p}\cdots b_{k+(t-k-1)}^{(m)p} t_m^{(m)p} z_{k+1}^p\cdots z_{t-1}^p (z_{j_{q-1}}y_m)^p z \tag{26}$$

$$\begin{aligned}
&= x_1^{p_1}\cdots x_r^{p_r}z_1^p\cdots a_{2m-1}^p b_{k+1}^{(m)p}\cdots b_{k+(t-k-1)}^{(m)p}(w_m^{(m)})^p z_{t+1}^p\cdots z_\ell^p y_1^{q_1}\cdots y_s^{q_s} \qquad (27)\\
&\quad\text{(by the definition of } z)\\
&= u_2(x_1,\ldots,x_r,z_1,\ldots,a_{2m-1},b_{k+1}^{(m)},\ldots,b_{k+(t-k-1)}^{(m)},w_m^{(m)},\ldots,z_\ell,y_1,\ldots,y_s)\\
&= u_1(x_1,\ldots,x_r,z_1,\ldots,a_{2m-1},b_{k+1}^{(m)},\ldots,b_{k+(t-k-1)}^{(m)},w_m^{(m)},\ldots,z_\ell,y_1,\ldots,y_s),
\end{aligned}$$

where the last equality holds by the inductive hypothesis and equality (27) above holds by Proposition 3.2, as $y_m \in S\setminus U$ and $p_1+\cdots+p_r+1 \geq g_0-1$. Also equalities (28), (29) and (30) follow respectively by the inductive hypothesis, by Proposition 3.2 as $y_{m-1} \in S\setminus U$ and $p_1+\cdots+p_r+1 \geq g_0-1$, and by Result 2.6 and Proposition 3.2 as $b_{k+1}^{(m)p}\cdots b_{k+(t-k-1)}^{(m)p}t_m^{(m)p} = t_m^p$. The above product in S contains the subproduct $(t_m^{(m)}z_{k+1}\cdots z_{t-1}(z_{j_{q-1}}y_m))^p a_{2m-1}^p$ because the word $u_1(\xi_1,\ldots,\xi_r,z_1,\ldots,z_\ell,\xi_1',\ldots,\xi_s')$ contains $z_{j_{q-1}}z_{j_q}$ as a subword. Thus, using Proposition 3.2 and the fact that $y_m a_{2m-1} = y_{m-1}a_{2m-2}$ from the zigzag equations, we have

$$\begin{aligned}
&x_1^{p_1}\cdots x_r^{p_r}z_{j_1}^p\cdots z_{j_\ell}^p y_1^{q_1}\cdots y_s^{q_s}\\
&= u_1(x_1,\ldots,x_r,z_1,\ldots,a_{2m-1},b_{k+1}^{(m)},\ldots,b_{k+(t-k-1)}^{(m)},w_m^{(m)},\ldots,z_\ell,y_1,\ldots,y_s)\\
&= u_1(x_1,\ldots,x_r,z_1,\ldots,a_{2m-2},b_{k+1}^{(m)},\ldots,b_{k+(t-k-1)}^{(m)},w_m^{(m-1)},\ldots,z_\ell,y_1,\ldots,y_s)\\
&= u_2(x_1,\ldots,x_r,z_1,\ldots,a_{2m-2},b_{k+1}^{(m)},\ldots,b_{k+(t-k-1)}^{(m)},w_m^{(m-1)},\ldots,z_\ell,y_1,\ldots,y_s) \qquad (28)\\
&= x_1^{p_1}\cdots x_r^{p_r}z_1^p\cdots a_{2m-2}^p b_{k+1}^{(m)p}\cdots b_{k+(t-k-1)}^{(m)p}(w_m^{(m-1)})^p\cdots z_\ell^p y\\
&\quad\text{(where } y = y_1^{q_1}\cdots y_s^{q_s})\\
&= x_1^{p_1}\cdots x_r^{p_r}z_1^p\cdots a_{2m-2}^p b_{k+1}^{(m)p}\cdots b_{k+(t-k-1)}^{(m)p}t_m^{(m)p}z_{k+1}^p\cdots z_{t-1}^p(z_{j_{q-1}}y_{m-1})^p\cdots z_\ell^p y \qquad (29)\\
&\quad\text{(by definition of } w_m^{(m-1)})
\end{aligned}$$

$$= \quad x_1^{p_1} \cdots x_r^{p_r} z_1^p \cdots z_{k-1}^p a_{2m-2}^p t_m^p z_{k+1}^p \cdots z_{t-1}^p (z_{j_{q-1}} y_{m-1})^p z_{t+1}^p \cdots z_\ell^p y \qquad (30)$$

$$= \quad x_1^{p_1} \cdots x_r^{p_r} z_1^p \cdots z_{k-1}^p a_{2m-3}^p t_{m-1}^p z_{k+1}^p \cdots z_{t-1}^p (z_{j_{q-1}} y_{m-1})^p z_{t+1}^p \cdots z_\ell^p y,$$

where the last equality holds by the zigzag equations and Proposition 3.2. Continuing this way, we obtain

$$\begin{aligned}
& x_1^{p_1} \cdots x_r^{p_r} z_{j_1}^p z_{j_2}^p \cdots z_{j_\ell}^p y_1^{q_1} \cdots y_s^{q_s} \\
&= \quad x_1^{p_1} \cdots x_r^{p_r} z_1^p \cdots z_{k-1}^p a_{2m-3}^p t_{m-1}^p z_{k+1}^p \cdots z_{t-1}^p (z_{j_{q-1}} y_{m-1})^p z_{t+1}^p \cdots z_\ell^p y \\
&\ \vdots \\
&= \quad x_1^{p_1} \cdots x_r^{p_r} z_1^p \cdots z_{k-1}^p a_1^p t_1^p z_{k+1}^p \cdots z_{t-1}^p (z_{j_{q-1}} y_1)^p z_{t+1}^p \cdots z_\ell^p y \\
&= \quad x_1^{p_1} \cdots x_r^{p_r} z_1^p \cdots z_{k-1}^p a_1^p b_{k+1}^{(1)p} \cdots b_{k+(t-k-1)}^{(1)p} t_1^{(1)p} z_{k+1}^p \cdots z_{t-1}^p (z_{j_{q-1}} y_1)^p \cdots z_\ell^p y \qquad (31) \\
&= \quad x_1^{p_1} \cdots x_r^{p_r} z_1^p \cdots z_{k-1}^p a_1^p b_{k+1}^{(1)p} \cdots b_{k+(t-k-1)}^{(1)p} (w_1^{(1)})^p \cdots z_\ell^p y,
\end{aligned}$$

where the last equality follows by Proposition 3.2 as $p_1 + \cdots + p_r + 1 \geq g_0 - 1$ and y_1 is in $S \setminus U$; and the equality (31) above follows by Result 2.6 and Proposition 3.2 for some $b_{k+1}^{(1)}, \ldots, b_{k+(t-k-1)}^{(1)}$ in U and $t_1^{(1)} \in S \setminus U$, as $p_1 + \cdots + p_r + 1 \geq g_0 - 1$ and $t_1 \in S \setminus U$. Therefore,

$$\begin{aligned}
& x_1^{p_1} \cdots x_r^{p_r} z_{j_1}^p z_{j_2}^p \cdots z_{j_\ell}^p y_1^{q_1} \cdots y_s^{q_s} \\
&= \quad x_1^{p_1} \cdots x_r^{p_r} z_1^p \cdots z_{k-1}^p a_1^p b_{k+1}^{(1)p} \cdots b_{k+(t-k-1)}^{(1)p} (w_1^{(1)})^p z_{t+1}^p \cdots z_\ell^p y \\
&= \quad u_2(x_1, \ldots, x_r, z_1, \ldots, z_{k-1}, a_1, b_{k+1}^{(1)}, \ldots, b_{k+(t-k-1)}^{(1)}, w_1^{(1)}, \ldots, z_\ell, y_1, \ldots, y_s) \\
&\qquad \text{(by the definition of } y) \\
&= \quad u_1(x_1, \ldots, x_r, z_1, \ldots, z_{k-1}, a_1, b_{k+1}^{(1)}, \ldots, b_{k+(t-k-1)}^{(1)}, w_1^{(1)}, \ldots, z_\ell, y_1, \ldots, y_s),
\end{aligned}$$

where the last equality holds by the inductive hypothesis, and the equality (32) below holds by the inductive hypothesis and Proposition 3.2 respectively as, $t_1^{(1)} \in S \setminus U$ and $q_1 + \cdots + q_s + 1 \geq h_0 - 1$. Also equalities (33), (34) and (35) below follow respectively by the inductive hypothesis, Result 2.8 as, $q_1 + \cdots + q_s + 1 \geq h_0 - 1$ and $t_1^{(1)} \in S \setminus U$; and as $t_1^{(1)} \in S \setminus U$ and $b_{k+1}^{(1)} = y_{k+1}^{(1)} c_{k+1}^{'}$ with $y_{k+1}^{(1)} \in S \setminus U$ and $c_{k+1}^{'} \in U$, and by the zigzag equations and Result 2.9 respectively. As before, the word $u_1(\xi_1, \ldots, \xi_r, z_1, \ldots, z_\ell, \xi_1', \ldots, \xi_s')$ contains $z_{j_{q-1}} z_{j_q}$ as a subword, the above product in S contains $(z_{j_{q-1}} y_1)^p a_1^p$. Thus, using Proposition 3.2 as $t_1^{(1)} \in S \setminus U$ and $q_1 + \cdots + q_s + 1 \geq h_0 - 1$ and the fact that $y_1 a_1 = a_0$ from the zigzag equations, we have

$$
\begin{aligned}
& x_1^{p_1} x_2^{p_2} \cdots x_r^{p_r} z_{j_1}^p z_{j_2}^p \cdots z_{j_\ell}^p y_1^{q_1} y_2^{q_2} \cdots y_s^{q_s} \\
= \; & u_1(x_1, \ldots, x_r, z_1, \ldots, z_{k-1}, a_1, b_{k+1}^{(1)}, \ldots, b_{k+(t-k-1)}^{(1)}, w_1^{(1)}, \ldots, z_\ell, y_1, \ldots, y_s) \\
= \; & u_1(x_1, \ldots, x_r, z_1, \ldots, z_{k-1}, a_0, b_{k+1}^{(1)}, \ldots, b_{k+(t-k-1)}^{(1)}, f, \ldots, z_\ell, y_1, \ldots, y_s) \\
& \text{(by definition of } w_1^{(1)} \text{ and where } f = t_1^{(1)} z_{k+1} \cdots z_{t-1} z_{j_{q-1}}) \\
= \; & u_2(x_1, \ldots, x_r, z_1, \ldots, z_{k-1}, a_0, b_{k+1}^{(1)}, \ldots, b_{k+(t-k-1)}^{(1)}, f, \ldots, z_\ell, y_1, \ldots, y_s) \qquad (32) \\
= \; & x_1^{p_1} \cdots x_r^{p_r} z_1^p \cdots z_{k-1}^p a_0^p b_{k+1}^{(1)p} \cdots b_{k+(t-k-1)}^{(1)p} (t_1^{(1)} z_{k+1} \cdots z_{t-1} z_{j_{q-1}})^p \cdots z_\ell^p y_1^{q_1} \cdots y_s^{q_s} \\
& \text{(by definition of } f) \\
= \; & x_1^{p_1} \cdots x_r^{p_r} z_1^p \cdots z_{k-1}^p a_0^p b_{k+1}^{(1)p} \cdots b_{k+(t-k-1)}^{(1)p} t_1^{(1)p} z_{k+1}^p \cdots z_{t-1}^p z_{j_{q-1}}^p \cdots z_\ell^p y_1^{q_1} \cdots y_s^{q_s} \qquad (33) \\
= \; & x_1^{p_1} \cdots x_r^{p_r} z_1^p \cdots z_{k-1}^p a_0^p t_1^p z_{k+1}^p \cdots z_{t-1}^p z_{j_{q-1}}^p z_{t+1}^p \cdots z_\ell^p y_1^{q_1} \cdots y_s^{q_s} \qquad (34) \\
= \; & x_1^{p_1} \cdots x_r^{p_r} z_1^p \cdots z_{k-1}^p z_k^p z_{k+1}^p \ldots . z_{t-1}^p z_{j_{q-1}}^p \cdots z_\ell^p y_1^{q_1} \cdots y_s^{q_s} \qquad (35) \\
= \; & x_1^{p_1} \cdots x_r^{p_r} z_1^p \cdots z_{k-1}^p z_k^p z_{k+1}^p \cdots z_{t-1}^p z_t^p z_{t+1}^p \cdots z_\ell^p y_1^{q_1} \cdots y_s^{q_s} \text{ (as } j_{q-1} = t)
\end{aligned}
$$

as required.

Case (5). $k+1<t$ and $t=\ell$.

The proof in this case may be obtained by modifying the proof of Case (4) in the following way:

Letting $\tilde{x}=(x_1,\ldots,x_r)$ and $\tilde{y}=(y_1,\ldots,y_s)$, define

$$w_1(\tilde{x},z_{j_1},\ldots,z_{j_\ell},\tilde{y})=x_1^{p_1}\cdots x_r^{p_r}z_{j_1}^{p}\cdots z_{j_\ell}^{p}y_1^{q_1}\cdots y_s^{q_s}=u_1(\tilde{x},z_1,\ldots,z_\ell,\tilde{y})$$

and

$$w_2(\tilde{x},z_{j_1},\ldots,z_{j_\ell},\tilde{y})=x_1^{p_1}\cdots x_r^{p_r}z_1^{p}\cdots z_\ell^{p}y_1^{q_1}\cdots y_s^{q_s}=u_2(\tilde{x},z_1,\ldots,z_\ell,\tilde{y})$$

(a) Replace the word $z_{t+1}^p\cdots z_\ell^p$ by 1;

(b) Replace

$$(\tilde{x},z_1,\ldots,z_{k-1},a_{2c-1},b_{k+1}^{(c)},\ldots,b_{k+(t-k-1)}^{(c)},t_c^{(c)}z_{k+1}\cdots z_{t-1}(z_{j_{q-1}}y_c),z_{t+1},\ldots,z_\ell,\tilde{y})$$

by

$$(\tilde{x},z_1,z_2,\ldots,z_{k-1},a_{2c-1},b_{k+1}^{(c)},\ldots,b_{\ell-1}^{(c)},t_c^{(c)}z_{k+1}\cdots z_{\ell-1}(z_{j_{q-1}}y_c),\tilde{y})$$

for all $c=1,2,\ldots,m$;

(c) Replace

$$(\tilde{x},z_1,\ldots,z_{k-1},a_{2c-2},b_{k+1}^{(c)},\ldots,b_{k+(t-k-1)}^{(c)},t_c^{(c)}z_{k+1}\cdots z_{t-1}(z_{j_{q-1}}y_{c-1}),\ldots,z_\ell,\tilde{y})$$

by

$$(\tilde{x},z_1,\ldots,z_{k-1},a_{2c-2},b_{k+1}^{(c)},\ldots,b_{\ell-1}^{(c)},t_c^{(c)}z_{k+1}\cdots z_{\ell-1}(z_{j_{q-1}}y_{c-1}),\tilde{y})$$

for all $c=1,2,\ldots,m$; and y_0 by 1 when c is 1.

Thus, the proof of the theorem is completed. $\square$

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