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# A NEW APPROACH TO SEMIGROUP THEORY I: SOFT UNION SEMIGROUPS, IDEALS AND BI-IDEALS

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**Abstract.** In this paper, soft union semigroups, soft union left (right, two-sided) ideals and bi-ideals of semigroups are defined, their properties and interrelations are given and regular, intra-regular, completely regular, weakly regular and quasi-regular semigroups are characterized in terms of these ideals. This paper is a new approach to classical semigroup theory via soft set theory.

**Keywords:** soft set; soft union left (right, two-sided) ideal; soft union bi-ideal; soft union semiprime; regular semigroups

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## 1. Introduction

Soft set theory was introduced by Molodtsov [1] in 1999 as a new mathematical tool for dealing with uncertainties. It has seen a many applications in algebraic structures such as groups [2,3], semirings [4], rings [5], BCK/BCI-algebras [6,7,8], BL-algebras [9], near-rings [10] and soft substructures and union soft substructures [11,12] since its inception.

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Soft set operations has been studied by [13], [14], [15], [16] as well and soft set theory has also a wide-ranging applications as in the following studies: [17,18,19,20,21,22,23].

In this paper, with the concept of soft union semigroup, a new approach to semigroup theory via soft set theory is made. The paper reads as follows: In Section 2, we remind some basic definitions about soft sets and semigroups. In Section 3, we define soft union product and obtain its basic properties. In Section 4, soft union semigroup, Section 5, soft union left (right, two-sided) ideals, Section 6, soft union bi-ideals and soft union semiprime ideals are defined and studied with respect to soft set operations and soft union product. In the following five sections, regular, intra-regular, completely regular, weakly regular and quasi-regular semigroups are characterized by the properties of these ideals, respectively.

## 2. Preliminaries

In this section, we recall some basic notions relevant to semigroups and soft sets. A *semigroup* $S$ is a nonempty set with an associative binary operation. Note that throughout this paper, $S$ denotes a semigroup.

A nonempty subset $A$ of $S$ is called a *subsemigroup* of $S$ if $AA \subseteq A$ and is called a *right ideal* of $S$ if $AS \subseteq A$ and is called a *left ideal* of $S$ if $SA \subseteq A$. By *two-sided ideal* (or simply *ideal*), we mean a subset of $S$, which is both a left and right ideal of $S$. A subsemigroup $X$ of $S$ is called a *bi-ideal* of $S$ if $XSX \subseteq X$. A subset $P$ of a semigroup $S$ is called *semiprime* if $\forall a \in S$, $a^2 \in P$ implies that $a \in P$. We denote by $L[a](R[a], J[a], B[a])$, the principal left ideal (right ideal, two-sided ideal, bi-ideal) of a semigroup $S$ generated by $a \in S$, that is,

$$\begin{aligned} L[a] &= \{a\} \cup Sa, \\ R[a] &= \{a\} \cup aS, \\ J[a] &= \{a\} \cup Sa \cup aS \cup SaS \\ B[a] &= \{a\} \cup \{a^2\} \cup aSa \end{aligned}$$

A *semilattice* is a structure $S = (S, .)$, where "." is an infix binary operation, called the *semilattice operation*, such that "." is associative, commutative and idempotent. For all undefined concepts and notions about semigroups, we refer to [24,25,26]. Note that, throughout this paper the product of ordered pairs will be considered componentwise. From now on, $U$ refers to an initial universe, $E$ is a set of parameters, $P(U)$ is the power set of $U$ and $A,B,C \subseteq E$.

**Definition 2.1.** ([1,18]) *A soft set $f_A$ over $U$ is a set defined by*

$$f_A : E \to P(U) \text{ such that} f_A(x) = \emptyset \text{ if } x \notin A.$$

*Here $f_A$ is also called an approximate function. A soft set over $U$ can be represented by the set of ordered pairs*

$$f_A = \{(x, f_A(x)) : x \in E, f_A(x) \in P(U)\}.$$

It is clear to see that a soft set is a parametrized family of subsets of the set $U$. Note that the set of all soft sets over $U$ will be denoted by $S(U)$.

**Definition 2.2.** [18] *Let $f_A, f_B \in S(U)$. Then, $f_A$ is called a soft subset of $f_B$ and denoted by $f_A \tilde{\subseteq} f_B$, if $f_A(x) \subseteq f_B(x)$ for all $x \in E$.*

**Definition 2.3.** [18] *Let $f_A$, $f_B \in S(U)$. Then, union of $f_A$ and $f_B$, denoted by $f_A \tilde{\cup} f_B$, is defined as $f_A \tilde{\cup} f_B = f_{A\tilde{\cup}B}$, where $f_{A\tilde{\cup}B}(x) = f_A(x) \cup f_B(x)$ for all $x \in E$.*

**Definition 2.4.** [18] *Let $f_A$, $f_B \in S(U)$. Then, intersection of $f_A$ and $f_B$, denoted by $f_A \tilde{\cap} f_B$, is defined as $f_A \tilde{\cap} f_B = f_{A\tilde{\cap}B}$, where $f_{A\tilde{\cap}B}(x) = f_A(x) \cap f_B(x)$ for all $x \in E$.*

**Definition 2.5.** [18] *Let $f_A$, $f_B \in S(U)$. Then, $\wedge$-product of $f_A$ and $f_B$, denoted by $f_A \wedge f_B$, is defined as $f_A \wedge f_B = f_{A\wedge B}$, where $f_{A\wedge B}(x,y) = f_A(x) \cap f_B(y)$ for all $(x,y) \in E \times E$.*

**Definition 2.6.** [27] *Let $f_A$ and $f_B$ be soft sets over the common universe $U$ and $\Psi$ be a function from $A$ to $B$. Then, soft anti image of $f_A$ under $\Psi$, denoted by $\Psi^\star(f_A)$, is a soft set over $U$ by*

$$(\Psi^\star(f_A))(b) = \begin{cases} \bigcap\{f_A(a) \mid a \in A \text{ and } \Psi(a) = b\}, & \text{if } \Psi^{-1}(b) \neq \emptyset, \\ \emptyset, & otherwise \end{cases}$$

*for all $b \in B$. And soft pre-image (or soft inverse image) of $f_B$ under $\Psi$, denoted by $\Psi^{-1}(f_B)$, is a soft set over $U$ by $(\Psi^{-1}(f_B))(a) = f_B(\Psi(a))$ for all $a \in A$.*

**Definition 2.7.** [28] *Let $f_A$ be a soft set over $U$ and $\alpha \subseteq U$. Then, lower $\alpha$-inclusion of $f_A$, denoted by $\mathscr{L}(f_A;\alpha)$, is defined as*

$$\mathscr{L}(f_A : \alpha) = \{x \in A \mid f_A(x) \subseteq \alpha\}.$$

## 3. Soft union product and soft anti characteristic function

In this section, we define soft union product and soft anti characteristic function and study their properties.

**Definition 3.1.** *Let $f_S$ and $g_S$ be soft sets over the common universe $U$. Then, soft union product $f_S * g_S$ is defined by*

$$(f_S * g_S)(x) = \begin{cases} \bigcap_{x=yz}\{f_S(y) \cup g_S(z)\}, & \text{if } \exists y,z \in S \ \ such \ \ that \ \ x = yz, \\ U, & otherwise \end{cases}$$

*for all $x \in S$.*

Note that soft union product is abbreviated by soft uni-product in what follows.

**Example 3.2.** *Consider the semigroup $S = \{a,b,c,d\}$ defined by the following table:*

| . | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | $a$ | $a$ | $a$ | $a$ |
| $b$ | $a$ | $a$ | $a$ | $a$ |
| $c$ | $a$ | $a$ | $b$ | $a$ |
| $d$ | $a$ | $a$ | $b$ | $b$ |

*Let $U = D_2 = \{< x,y >: x^2 = y^2 = e, xy = yx\} = \{e,x,y,yx\}$ be the universal set. Let $f_S$ and $g_S$ be soft sets over $U$ such that $f_S(a) = \{e,y,yx\},\ f_S(b) = \{e,x\},\ f_S(c) = \{y,yx\},\ f_S(d) = \{e,x,y\}$ and $g_S(a) = \{x,y\},\ g_S(b) = \{e,yx\},\ g_S(c) = \{yx\},\ g_S(d) = \{e,y\}$. Since $b = cc$, $b = dc$ and $b = dd$, then*

$$(f_S * g_S)(b) = \{f_S(c) \cup g_S(c)\} \cap \{f_S(d) \cup g_S(c)\} \cap \{f_S(d) \cup g_S(d)\} = \{y\}$$

*Similarly, $(f_S * g_S)(a) = \emptyset$, $(f_S * g_S)(c) = (f_S * g_S)(d) = U$.*

**Theorem 3.3.** *Let $f_S, g_S, h_S \in S(U)$. Then,*

*i)* $(f_S * g_S) * h_S = f_S * (g_S * h_S)$.

*ii)* $f_S * g_S \neq g_S * f_S$, *generally.*

*iii)* $f_S * (g_S \widetilde{\cup} h_S) = (f_S * g_S) \widetilde{\cup} (f_S * h_S)$ *and* $(f_S \widetilde{\cup} g_S) * h_S = (f_S * h_S) \widetilde{\cup} (g_S * h_S)$.

*iv)* $f_S * (g_S \widetilde{\cap} h_S) = (f_S * g_S) \widetilde{\cap} (f_S * h_S)$ *and* $(f_S \widetilde{\cap} g_S) * h_S = (f_S * h_S) \widetilde{\cap} (g_S * h_S)$.

*v) If $f_S \tilde{\subseteq} g_S$, then $f_S * h_S \tilde{\subseteq} g_S * h_S$ and $h_S * f_S \tilde{\subseteq} h_S * g_S$.*

*vi) If $t_S, l_S \in S(U)$ such that $t_S \tilde{\subseteq} f_S$ and $l_S \tilde{\subseteq} g_S$, then $t_S * l_S \tilde{\subseteq} f_S * g_S$.*

**Proof.** $i)$ and $ii)$ follows from Definition 3.1. and Example 3.2.

$iii)$ Let $a \in S$. If $a$ is not expressible as $a = xy$, then $(f_S * (g_S \tilde{\cup} h_S))(a) = U$. Similarly,

$$((f_S * g_S) \tilde{\cup} (f_S * h_S))(a) = (f_S * g_S)(a) \cup (f_S * h_S)(a) = U \cup U = U$$

Now, let there exist $x, y \in S$ such that $a = xy$. Then,

$$\begin{aligned}
(f_S * (g_S \tilde{\cup} h_S))(a) &= \bigcap_{a=xy} (f_S(x) \cup (g_S \tilde{\cup} h_S)(y)) \\
&= \bigcap_{a=xy} (f_S(x) \cup (g_S(y) \cup h_S(y))) \\
&= \bigcap_{a=xy} [(f_S(x) \cup g_S(y)) \cup (f_S(x) \cup h_S(y))] \\
&= [\bigcap_{a=xy} (f_S(x) \cup g_S(y))] \cup [\bigcap_{a=xy} (f_S(x) \cup h_S(y))] \\
&= (f_S * g_S)(a) \cup (f_S * h_S)(a) \\
&= [(f_S * g_S) \tilde{\cup} (f_S * h_S)](a)
\end{aligned}$$

Thus, $(f_S \tilde{\cup} g_S) * h_S = (f_S * h_S) \tilde{\cup} (g_S * h_S)$ and $(iv)$ can be proved similarly.

$v)$ Let $x \in S$. If $x$ is not expressible as $x = yz$, then $(f_S * h_S)(x) = (g_S * h_S)(x) = U$. Otherwise,

$$\begin{aligned}
(f_S * h_S)(x) &= \bigcap_{x=yz} (f_S(y) \cup h_S(z)) \\
&\subseteq \bigcap_{x=yz} (g_S(y) \cup h_S(z)) \ (since\ f_S(y) \subseteq g_S(y)) \\
&= (g_S * h_S)(x)
\end{aligned}$$

Similarly, one can show that $h_S * f_S \tilde{\subseteq} h_S * g_S$.

$(vi)$ can be proved similar to $(v)$.

**Definition 3.4.** *Let X be a subset of S. We denote by $\mathscr{S}_{X^c}$ the soft characteristic function of the complement X and define as*

$$\mathscr{S}_{X^c}(x) = \begin{cases} \emptyset, & \text{if } x \in X, \\ U, & \text{if } x \in S \setminus X \end{cases}$$

**Theorem 3.5.** *Let X and Y be nonempty subsets of a semigroup S. Then, the following properties hold:*

*i) If $Y \subseteq X$, then $\mathscr{S}_{X^c} \tilde{\subseteq} \mathscr{S}_{Y^c}$.*

*ii) $\mathscr{S}_{X^c} \widetilde{\cap} \mathscr{S}_{Y^c} = \mathscr{S}_{X^c \cap Y^c}$, $\mathscr{S}_{X^c} \widetilde{\cup} \mathscr{S}_{Y^c} = \mathscr{S}_{X^c \cup Y^c}$.*

**Proof.** $i$) is straightforward by Definition 3.4.

$ii$) Let $s$ be any element of $S$. Suppose $s \in X^c \cap Y^c$. Then, $s \in X^c$ and $s \in Y^c$. Thus, we have

$$(\mathscr{S}_{X^c} \widetilde{\cap} \mathscr{S}_{Y^c})(s) = \mathscr{S}_{X^c}(s) \cap \mathscr{S}_{Y^c}(s) = U \cap U = U = \mathscr{S}_{X^c \cap Y^c}(s)$$

Suppose $s \notin X^c \cap Y^c$. Then, $s \notin X^c$ or $s \notin Y^c$. Hence, we have

$$(\mathscr{S}_{X^c} \widetilde{\cap} \mathscr{S}_{Y^c})(s) = \mathscr{S}_{X^c}(s) \cap \mathscr{S}_{Y^c}(s) = \emptyset = \mathscr{S}_{X^c \cap Y^c}(s)$$

Let $s$ be any element of $S$. Suppose $s \in X^c \cup Y^c$. Then, $s \in X^c$ or $s \in Y^c$. Thus, we have

$$(\mathscr{S}_{X^c} \widetilde{\cup} \mathscr{S}_{Y^c})(s) = \mathscr{S}_{X^c}(s) \cup \mathscr{S}_{Y^c}(s) = U = \mathscr{S}_{X^c \cup Y^c}(s)$$

Suppose $s \notin X^c \cup Y^c$. Then, $s \in S$ and $s \in Y$. Hence, we have

$$(\mathscr{S}_{X^c} \widetilde{\cup} \mathscr{S}_{Y^c})(s) = \mathscr{S}_{X^c}(s) \cup \mathscr{S}_{Y^c}(s) = \emptyset = \mathscr{S}_{X^c \cup Y^c}(s)$$

## 4. Soft union semigroup

In this section, we define soft union semigroups, study their basic properties with respect to soft operations and soft uni-product.

**Definition 4.1.** *Let $S$ be a semigroup and $f_S$ be a soft set over $U$. Then, $f_S$ is called a soft union semigroup of $S$, if*

$$f_S(xy) \subseteq f_S(x) \cup f_S(y)$$

*for all $x, y \in S$.*

For the sake of brevity, soft union semigroup is abbreviated by $SU$-semigroup in what follows.

**Example 4.2.** *Let $S = \{a, b, c, d\}$ be the semigroup in Example 2.1. and $f_S$ be a soft set over $U = S_3$, symmetric group. If we construct a soft set such that $f_S(a) = \{(1)\}$, $f_S(b) = \{(1), (123)\}$, $f_S(c) = \{(1), (12), (123)\}, f_S(d) = \{(1), (123)\}$ then, one can easily show that $f_S$ is an $SU$-semigroup over $U$.*

*Now, let $U = \left\{ \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} \mid x, y \in \mathbb{Z}_4 \right\}$, $2 \times 2$ matrices with $\mathbb{Z}_3$ terms, be the universal set. We construct a soft set $g_S$ over $U$ by*

$$g_S(a) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\},$$

$$g_S(b) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \right\},$$

$$g_S(c) = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \right\}$$

$$g_S(d) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix} \right\}.$$

*Then, since*

$$g_S(dd) = g_S(b) \nsubseteq g_S(d) \cup g_S(d),$$

*$g_S$ is not an $SU$-semigroup over $U$.*

**Note 4.3.** *It is easy to see that if $f_S(x) = \emptyset$ for all $x \in S$, then $f_S$ is an $SU$-semigroup over $U$. We denote such a kind of $SU$-semigroup by $\widetilde{\theta}$. It is obvious that $\widetilde{\theta} = \mathscr{S}_{S^c}$, i.e. $\widetilde{\theta}(x) = \emptyset$ for all $x \in S$.*

**Lemma 4.4.** *Let $f_S$ be any $SU$-semigroup over $U$. Then, we have the followings:*

*i)* $\widetilde{\theta} * \widetilde{\theta} \widetilde{\supseteq} \widetilde{\theta}$.

*ii)* $f_S * \widetilde{\theta} \widetilde{\supseteq} \widetilde{\theta}$ *and* $\widetilde{\theta} * f_S \widetilde{\supseteq} \widetilde{\theta}$.

*iii)* $f_S \widetilde{\cap} \widetilde{\theta} = \widetilde{\theta}$ *and* $f_S \widetilde{\cup} \widetilde{\theta} = f_S$.

**Theorem 4.5.** *Let $f_S$ be a soft set over $U$. Then, $f_S$ is an $SU$-semigroup over $U$ if and only if*

$$f_S * f_S \widetilde{\supseteq} f_S$$

**Proof.** Assume that $f_S$ is an $SU$-semigroup over $U$. Let $a \in S$. If $(f_S * f_S)(a) = U$, then it is obvious that

$$(f_S * f_S)(a) \supseteq f_S(a), \ thus \ f_S * f_S \widetilde{\supseteq} f_S.$$

Otherwise, there exist elements $x, y \in S$ such that $a = xy$. Then, since $f_S$ is an $SU$-semigroup over $U$, we have:

$$\begin{aligned} (f_S * f_S)(a) &= \bigcap_{a=xy} (f_S(x) \cup f_S(y)) \\ &\supseteq \bigcap_{a=xy} f_S(xy) \\ &= \bigcap_{a=xy} f_S(a) \\ &= f_S(a) \end{aligned}$$

Thus, $f_S * f_S \widetilde{\supseteq} f_S$.

Conversely, assume that $f_S * f_S \widetilde{\supseteq} f_S$. Let $x, y \in S$ and $a = xy$. Then, we have:

$$\begin{aligned} f_S(xy) &= f_S(a) \\ &\subseteq (f_S * f_S)(a) \\ &= \bigcap_{a=xy} (f_S(x) \cup f_S(y)) \\ &\subseteq f_S(x) \cup f_S(y) \end{aligned}$$

Hence, $f_S$ is an $SU$-semigroup over $U$. This completes the proof.

**Theorem 4.6.** *A non-empty subset $A$ of a semigroup of $S$ is a subsemigroup of $S$ if and only if the soft subset $f_S$ defined by*

$$f_S(x) = \begin{cases} \alpha, & \text{if } x \in S \setminus A, \\ \beta, & \text{if } x \in A \end{cases}$$

*is an SU-semigroup, where $\alpha, \beta \subseteq U$ such that $\alpha \supseteq \beta$.*

**Proof.** Suppose $A$ is a subsemigroup of $S$ and $x, y \in S$. If $x, y \in A$, then $xy \in A$. Hence, $f_S(xy) = f_S(x) = f_S(y) = \beta$ and so, $f_S(xy) \subseteq f_S(x) \cup f_S(y)$. If $x, y \notin A$, then $xy \in A$ or $xy \notin A$. In any case, $f_S(xy) \subseteq f_S(x) \cup f_S(y) = \alpha$. Thus, $f_S$ is an $SU$-semigroup.

Conversely assume that $f_S$ is an $SU$-semigroup of $S$. Let $x, y \in A$. Then, $f_S(xy) \subseteq f_S(x) \cup f_S(y) = \beta$. This implies that $f_S(xy) = \beta$. Hence, $xy \in A$ and so $A$ is a subsemigroup of $S$.

**Theorem 4.7.** *Let $X$ be a nonempty subset of a semigroup $S$. Then, $X$ is a subsemigroup of $S$ if and only if $\mathscr{S}_{X^c}$ is an SU-semigroup of $S$.*

**Proof.** Since

$$\mathscr{S}_{X^c}(x) = \begin{cases} U, & \text{if } x \in S \setminus X, \\ \emptyset, & \text{if } x \in X \end{cases}$$

and $U \supseteq \emptyset$, the rest of the proof follows from Theorem 4.6.

**Proposition 4.8.** *Let $f_S$ and $f_T$ be SU-semigroup over $U$. Then, $f_S \vee f_T$ is an SU-semigroup over $U$.*

**Proof.** Let $(x_1, y_1), (x_2, y_2) \in S \times T$. Then,

$$\begin{aligned} f_{S \vee T}((x_1, y_1)(x_2, y_2)) &= f_{S \vee T}(x_1 x_2, y_1 y_2) \\ &= f_S(x_1 x_2) \cup f_T(y_1 y_2) \\ &\subseteq (f_S(x_1) \cup f_S(x_2)) \cup (f_T(y_1) \cup f_T(y_2)) \\ &= (f_S(x_1) \cup f_T(y_1)) \cup (f_S(x_2) \cup f_T(y_2)) \\ &= f_{S \vee T}(x_1, y_1) \cup f_{S \vee T}(x_2, y_2) \end{aligned}$$

Therefore, $f_S \vee f_T$ is an $SU$-semigroup over $U$.

**Proposition 4.9.** *If $f_S$ and $h_S$ are SU-semigroups over $U$, then so is $f_S \widetilde{\cup} h_S$ over $U$.*

**Proof.** Let $x,y \in S$, then

$$\begin{aligned}(f_S\widetilde{\cup}h_S)(xy) &= f_S(xy)\cup h_S(xy)\\ &\subseteq (f_S(x)\cup f_S(y))\cup(h_S(x)\cup h_S(y))\\ &= (f_S(x)\cup h_S(x))\cup(f_S(y)\cup h_S(y))\\ &= (f_S\widetilde{\cup}h_S)(x)\cup(f_S\widetilde{\cup}h_S)(y)\end{aligned}$$

Therefore, $f_S\widetilde{\cup}h_S$ is an $SU$-semigroup over $U$.

**Proposition 4.10.** *Let $f_S$ be a soft set over $U$ and $\alpha$ be a subset of $U$ such that $\alpha \in Im(f_S)$, where $Im(f_S) = \{\alpha \subseteq U : f_S(x) = \alpha,\ for\ x \in S\}$. If $f_S$ is an $SU$-semigroup over $U$, then $\mathscr{L}(f_S;\alpha)$ is a subsemigroup of $S$.*

**Proof.** Since $f_S(x) = \alpha$ for some $x \in S$, then $\emptyset \neq \mathscr{L}(f_S;\alpha) \subseteq S$. Let $x,y \in \mathscr{L}(f_S;\alpha)$, then $f_S(x) \subseteq \alpha$ and $f_S(y) \subseteq \alpha$. We need to show that $xy \in \mathscr{U}(f_S;\alpha)$ for all $x,y \in \mathscr{L}(f_S;\alpha)$. Since $f_S$ is an $SU$-semigroup over $U$, it follows that $f_S(xy) \subseteq f_S(x)\cup f_S(y) \subseteq \alpha\cup\alpha = \alpha$ implying that $xy \in \mathscr{L}(f_S;\alpha)$. Thus, the proof is completed.

**Definition 4.11.** *Let $f_S$ be an $SU$-semigroup over $U$. Then, the subsemigroups $\mathscr{L}(f_S;\alpha)$ are called lower $\alpha$-subsemigroups of $f_S$.*

**Proposition 4.12.** *Let $f_S$ be a soft set over $U$, $\mathscr{L}(f_S;\alpha)$ be lower $\alpha$-subsemigroups of $f_S$ for each $\alpha \subseteq U$ and $Im(f_S)$ be an ordered set by inclusion. Then, $f_S$ is an $SU$-semigroup over $U$.*

**Proof.** Let $x,y \in S$ and $f_S(x) = \alpha_1$ and $f_S(y) = \alpha_2$. Suppose that $\alpha_1 \subseteq \alpha_2$. It is obvious that $x \in \mathscr{L}(f_S;\alpha_1)$ and $y \in \mathscr{L}(f_S;\alpha_2)$. Since $\alpha_1 \subseteq \alpha_2$, $x,y \in \mathscr{L}(f_S;\alpha_2)$ and since $\mathscr{L}(f_S;\alpha)$ is a subsemigroup of $S$ for all $\alpha \subseteq U$, it follows that $xy \in \mathscr{U}(f_S;\alpha_2)$. Hence, $f_S(xy) \subseteq \alpha_2 = \alpha_1\cup\alpha_2 = f_S(x)\cup f_S(y)$. Thus, $f_S$ is an $SU$-semigroup over $U$.

**Proposition 4.13.** *Let $f_S$ and $f_T$ be soft sets over $U$ and $\Psi$ be a semigroup isomorphism from $S$ to $T$. If $f_S$ is an $SU$-semigroup over $U$, then so is $\Psi^\star(f_S)$.*

**Proof.** Let $t_1,t_2 \in T$. Since $\Psi$ is surjective, then there exist $s_1,s_2 \in S$ such that $\Psi(s_1) = t_1$ and $\Psi(s_2) = t_2$. Then,

$$
\begin{aligned}
&(\Psi^\star(f_S))(t_1t_2)\\
&=\bigcap\{f_S(s): s\in S, \Psi(s)=t_1t_2\}\\
&=\bigcap\{f_S(s): s\in S, s=\Psi^{-1}(t_1t_2)\}\\
&=\bigcap\{f_S(s): s\in S, s=\Psi^{-1}(\Psi(s_1s_2))=s_1s_2\}\\
&=\bigcap\{f_S(s_1s_2): s_i\in S, \Psi(s_i)=t_i, i=1,2\}\\
&\subseteq\bigcap\{f_S(s_1)\cup f_S(s_2): s_i\in S, \Psi(s_i)=t_i, i=1,2\}\\
&=(\bigcap\{f_S(s_1): s_1\in S, \Psi(s_1)=t_1\})\cup(\bigcap\{f_S(s_2): s_2\in S, \Psi(s_2)=t_2\})\\
&=(\Psi^\star(f_S))(t_1)\cup(\Psi(f_S))(t_2)
\end{aligned}
$$

Hence, $\Psi(f_S)$ is an $SU$-semigroup over $U$.

**Proposition 4.14.** *Let $f_S$ and $f_T$ be soft sets over $U$ and $\Psi$ be a semigroup homomorphism from $S$ to $T$. If $f_T$ is an $SU$-semigroup over $U$, then so is $\Psi^{-1}(f_T)$.*

**Proof.** Let $s_1, s_2 \in S$. Then,

$$
\begin{aligned}
(\Psi^{-1}(f_T))(s_1s_2) &= f_T(\Psi(s_1s_2))\\
&= f_T(\Psi(s_1)\Psi(s_2))\\
&\subseteq f_T(\Psi(s_1))\cup f_T(\Psi(s_2))\\
&= (\Psi^{-1}(f_T))(s_1)\cup(\Psi^{-1}(f_T))(s_2)
\end{aligned}
$$

Hence, $\Psi^{-1}(f_T)$ is an $SU$-semigroup over $U$.

## 5. Soft union left (right, two-sided) ideals of semigroups

In this section, we define soft union left (right, two-sided) ideal of semigroups and obtain their basic properties related with soft set operations and soft uni-product.

**Definition 5.1.** *A soft set over $U$ is called a soft union left (right) ideal of $S$ over $U$ if*

$$f_S(ab) \subseteq f_S(b) \; (f_S(ab) \subseteq f_S(a))$$

*for all $a, b \in S$. A soft set over $U$ is called a soft union two-sided ideal (soft union ideal) of $S$ if it is both soft union left and soft union right ideal of $S$ over $U$.*

For the sake of brevity, soft union left (right) ideal is abbreviated by $SU$-left (right) ideal in what follows.

**Example 5.2.** *Consider the semigroup $S=\{0,x,1\}$ defined by the following table:*

| . | 0 | x | 1 |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| x | 0 | x | x |
| 1 | 0 | x | 1 |

*Let $f_S$ be a soft set over $S$ such that $f_S(0)=\{0\},\ f_S(1)=\{0,1,x\},\ f_S(x)=\{0,x\}$. Then, one can easily show that $f_S$ is an $SU$-ideal of $S$ over $U$. However if we define a soft set $h_S$ over $S$ such that $h_S(0)=\{0,1\},\ h_S(1)=\{1\},\ h_S(x)=\{0,x,1\}$, then, $h_S(x1)=h_S(x)\not\supseteq h_S(1)$ Thus, $h_S$ is not an $SU$-left ideal of $S$ and moreover since $h_S(1x)=h_S(x)\not\supseteq h_S(1)$, $h_S$ is not an $SU$-right ideal of $S$ over $U$.*

**Theorem 5.3.** *Let $f_S$ be a soft set over $U$. Then, $f_S$ is an $SU$-left ideal of $S$ over $U$ if and only if*

$$\widetilde{\theta}*f_S\widetilde{\supseteq}f_S.$$

**Proof.** First assume that $f_S$ is an $SU$-left ideal of $S$ over $U$. Let $s\in S$. If

$$(\widetilde{\theta}*f_S)(s)=U,$$

then it is clear that $\widetilde{\theta}*f_S\widetilde{\supseteq}f_S$. Otherwise, there exist elements $x,y\in S$ such that $s=xy$. Then, since $f_S$ is an $SU$-left ideal of $S$ over $U$, we have:

$$\begin{aligned}(\widetilde{\theta}*f_S)(s) &= \bigcap_{s=xy}(\widetilde{\theta}(x)\cup f_S(y))\\ &\supseteq \bigcap_{s=xy}(\emptyset\cup f_S(xy))\\ &= \bigcap_{s=xy}(f_S(xy))\\ &= f_S(s)\end{aligned}$$

Thus, we have $\widetilde{\theta}*f_S\widetilde{\supseteq}f_S$.

Conversely, assume that $\widetilde{\theta} * f_S \widetilde{\supseteq} f_S$. Let $x, y \in S$ and $s = xy$. Then, we have:

$$\begin{aligned} f_S(xy) &= f_S(s) \\ &\subseteq (\widetilde{\theta} * f_S)(s) \\ &= \bigcap_{s=mn} (\widetilde{\theta}(m) \cup f_S(n)) \\ &\subseteq \widetilde{\theta}(x) \cup f_S(y) \\ &= \emptyset \cup f_S(y) \\ &= f_S(y) \end{aligned}$$

Hence, $f_S$ is an $SU$-left ideal over $U$. This completes the proof.

**Theorem 5.4.** *Let $f_S$ be a soft set over $U$. Then, $f_S$ is an $SU$-right ideal of $S$ over $U$ if and only if*

$$f_S * \widetilde{\theta} \widetilde{\supseteq} f_S$$

**Proof.** Similar to the proof of Theorem 5.3.

**Theorem 5.5.** *Let $f_S$ be a soft set over $U$. Then, $f_S$ is an $SU$-ideal of $S$ over $U$ if and only if*

$$f_S * \widetilde{\theta} \widetilde{\supseteq} f_S \text{ and } \widetilde{\theta} * f_S \widetilde{\supseteq} f_S$$

**Corollary 5.6.** *$\widetilde{\theta}$ is both $SU$-right and $SU$-left ideal of $S$.*

**Theorem 5.7.** *A non-empty subset $L$ of a semigroup of $S$ is a left (right) ideal of $S$ if and only if the soft subset $f_S$ defined by*

$$f_S(x) = \begin{cases} \alpha, & \text{if } x \in S \setminus L, \\ \beta, & \text{if } x \in L \end{cases}$$

*is an $SU$-left (right) ideal of $S$, where $\alpha, \beta \subseteq U$ such that $\alpha \supseteq \beta$.*

**Proof.** Suppose $L$ is a left ideal of $S$ and $x, y \in S$. If $y \in L$, then $xy \in L$. Hence, $f_S(xy) = f_S(y) = \beta$. If $y \notin L$, then $xy \in L$ or $xy \notin L$. In any case, $f_S(xy) \subseteq f_S(y) = \alpha$. Thus, $f_S$ is an $SU$-left ideal of $S$.

Conversely assume that $f_S$ is an $SU$-left ideal of $S$. Let $y \in L$ and $x \in S$. Then, $f_S(xy) \subseteq f_S(y) = \beta$. This implies that $f_S(xy) = \beta$. Hence, $xy \in L$ and so $L$ is a left ideal of $S$.

**Theorem 5.8.** *Let $X$ be a nonempty subset of a semigroup $S$. Then, $X$ is a left (right, two-sided) ideal of $S$ if and only if $\mathscr{S}_{X^c}$ is an $SU$-left (right, two-sided) ideal of $S$ over $U$.*

**Proof.** It follows from Theorem 5.7.

**Proposition 5.9.** *Let $f_S$ be a soft set over $U$. Then, $f_S$ is an $SU$-ideal of $S$ over $U$ if and only if*

$$f_S(xy) \subseteq f_S(x) \cap f_S(y)$$

*for all $x, y \in S$.*

**Proof.** Let $f_S$ be an $SU$-ideal of $S$ over $U$. Then,

$$f_S(xy) \subseteq f_S(x) \text{ and } f_S(xy) \subseteq f_S(y)$$

for all $x, y \in S$. Thus, $f_S(xy) \subseteq f_S(x) \cap f_S(y)$ Conversely, suppose that $f_S(xy) \subseteq f_S(x) \cap f_S(y)$ for all $x, y \in S$. It follows that

$$f_S(xy) \subseteq f_S(x) \cap f_S(y) \subseteq f_S(x) \text{ and } f_S(xy) \subseteq f_S(x) \cap f_S(y) \subseteq f_S(y)$$

so $f_S$ is an $SU$-ideal of $S$ over $U$.

It is obvious that every left (right, two-sided) ideal of $S$ is a subsemigroup of $S$. Moreover, we have the following:

**Theorem 5.10.** *Let $f_S$ be a soft set over $U$. Then, if $f_S$ is an $SU$-left (right, two-sided) ideal of $S$ over $U$, $f_S$ is an $SU$-semigroup over $U$.*

**Proof.** *We give the proof for $SU$-left ideals. Let $f_S$ be an $SU$-left ideal of $S$ over $U$. Then, $f_S(xy) \subseteq f_S(y)$ for all $x, y \in S$. Thus, $f_S(xy) \subseteq f_S(y) \subseteq f_S(x) \cup f_S(y)$, so $f_S$ is an $SU$-semigroup over $U$.*

**Proposition 5.11.** *If $f_S$ is an $SU$-right (left) ideal of $S$ over $U$, then*

$$f_S \widetilde{\cap} (\widetilde{\theta} * f_S) \ (f_S \widetilde{\cap} (f_S * \widetilde{\theta})$$

*is an $SU$-ideal of $S$ over $U$.*

**Proof.** *Assume that $f_S$ is an SU-right ideal of S. Then,*

$$\begin{aligned}
\widetilde{\theta} * (f_S \widetilde{\cap} (\widetilde{\theta} * f_S)) &= (\widetilde{\theta} * f_S) \widetilde{\cap} (\widetilde{\theta} * (\widetilde{\theta} * f_S)) \ (by\ Theorem 3.3.(iii)) \\
&= (\widetilde{\theta} * f_S) \widetilde{\cap} ((\widetilde{\theta} * \widetilde{\theta}) * f_S) \ (by\ Theorem 3.3(i)) \\
&\widetilde{\supseteq} (\widetilde{\theta} * f_S) \widetilde{\cap} (\widetilde{\theta} * f_S) \ (by\ Lemma 4.4.(i)) \\
&= \widetilde{\theta} * f_S \\
&\widetilde{\supseteq} f_S \widetilde{\cap} (\widetilde{\theta} * f_S)
\end{aligned}$$

*Thus, $f_S \widetilde{\cap} (\widetilde{\theta} * f_S)$ is an SU-left ideal of S over U. Also,*

$$\begin{aligned}
(f_S \widetilde{\cap} (\widetilde{\theta} * f_S)) * \widetilde{\theta} &= (f_S * \widetilde{\theta}) \widetilde{\cap} ((\widetilde{\theta} * f_S) * \widetilde{\theta}) \\
&= (f_S * \widetilde{\theta}) \widetilde{\cap} (\widetilde{\theta} * (f_S * \widetilde{\theta})) \\
&\widetilde{\supseteq} (f_S * \widetilde{\theta}) \widetilde{\cap} (\widetilde{\theta} * f_S) \ (since f_S * \widetilde{\theta} \widetilde{\supseteq} f_S) \\
&\widetilde{\supseteq} f_S \widetilde{\cap} (\widetilde{\theta} * f_S)
\end{aligned}$$

*Hence, $f_S \widetilde{\cap} (\widetilde{\theta} * f_S)$ is an SU-right ideal of S over U. This completes the proof.*

**Theorem 5.12.** *Let $f_S$ be an SU-right ideal of S over U and $g_S$ be an SU-left ideal of S over U. Then*

$$f_S * g_S \widetilde{\supseteq} f_S \widetilde{\cup} g_S$$

**Proof.** *Let $f_S$ and $g_S$ be SU-right and SU-left ideal of S over U, respectively. Then, since $f_S, g_S \widetilde{\supseteq} \widetilde{\theta}$ always holds, we have:*

$$f_S * g_S \widetilde{\supseteq} f_S * \widetilde{\theta} \widetilde{\supseteq} f_S \text{ and } f_S * g_S \widetilde{\supseteq} \widetilde{\theta} * g_S \widetilde{\supseteq} g_S$$

*It follows that $f_S * g_S \widetilde{\supseteq} f_S \widetilde{\cup} g_S$.*

*Now, we show that if $f_S$ is an SU-right ideal of S over U and $g_S$ is an SU-left ideal of S over U, then*

$$f_S * g_S \widetilde{\not\subseteq} f_S \widetilde{\cap} g_S$$

*with the following example:*

**Example 5.13.** *Consider the semigroup S and SU-ideal $f_S$ in Example 5.2. Let $g_S$ be a soft set over S such that $g_S(0)=\{x,1\},\ g_S(x)=\{x\},\ g_S(1)=\{x\}$, One can easily show that $g_S$ is an SU-ideal of S over U. However,*

$$(f_S * g_S)(x) = \bigcap_{x=ab}(f_S(a)\cup g_S(b)) = \{0,1,x\} \nsubseteq (f_S \widetilde{\cap} g_S)(x) = \{x\}.$$

**Proposition 5.14.** *Let $f_S$ and $h_S$ be SU-left (right) ideals of S over U. Then, $f_S \circ h_S$ is an SU-left (right) ideal of S over U.*

**Proof.** *Let $f_S$ and $h_S$ be SU-left ideal of S and $x,y\in S$. Then,*

$$(f_S * h_S)(y) = \bigcap_{y=pq}(f_S(p)\cup h_S(q))$$

*If $y=pq$, then $xy=x(pq)=(xp)q$. Since $f_S$ is an SU-left ideal of S, $f_S(xp)\subseteq f_S(p)$. Thus,*

$$\begin{aligned}(f_S * h_S)(y) &= \bigcap_{y=pq}(f_S(p)\cup h_S(q))\\ &\supseteq \bigcap_{xy=xpq}(f_S(xp)\cup h_S(q))\\ &= (f_S * h_S)(xy)\end{aligned}$$

*So,*

$$(f_S * h_S)(xy) \subseteq (f_S * h_S)(y)$$

*If y is not expressible as $y=pq$, then $(f_S*h_S)(y)=U\supseteq(f_S*h_S)(xy)$. Thus, $f_S*h_S$ is an SU-left ideal of S.*

*We give the following propositions without proof. The proofs are similar to those in Section 4.*

**Proposition 5.16.** *Let $f_S$ and $f_T$ be SU-left (right) ideals of S over U. Then, $f_S\vee f_T$ is an SU-left (right) ideal of $S\times T$ over U.*

**Proposition 5.17.** *If $f_S$ and $h_S$ are two SU-left (right) ideals of S over U, then so is $f_S\widetilde{\cup}h_S$ of S over U.*

**Proposition 5.18.** *Let $f_S$ be a soft set over U and $\alpha$ be a subset of U such that $\alpha\in Im(f_S)$. If $f_S$ is an SU-left (right) ideal of S over U, then $\mathscr{L}(f_S;\alpha)$ is a left (right) ideal of S.*

**Definition 5.19.** *Let $f_S$ be an SU-left (right) ideal of S over U. Then, the left (right) ideals $\mathscr{L}(f_S;\alpha)$ are called lower $\alpha$-left (right) ideals of $f_S$.*

**Proposition 5.20.** *Let $f_S$ be a soft set over U, $\mathscr{L}(f_S;\alpha)$ be lower $\alpha$-ideals of $f_S$ for each $\alpha \subseteq U$ and $Im(f_S)$ be an ordered set by inclusion. Then, $f_S$ is an SU-left (right) ideal of S over U.*

*In order to show Proposition 5.18., we have the following example:*

**Example 5.21.** *Consider the semigroup in Example 3.2. Define a soft set $f_S$ over $U = D_2 = \{e,x,y,yx\}$ such that $f_S(a) = \{x\}$, $f_S(b) = \{e,x\}$, $f_S(c) = \{e,x,y\}$, $f_S(d) = \{e,x,yx\}$. Then, one can easily show that $f_S$ is an SU-ideal of S over U. By taking into account $Im(f_S)$, we have: $\mathscr{L}(f_S;\{x\}) = \{a\}, \mathscr{L}(f_S;\{e,x\}) = \{a,b\}, \mathscr{L}(f_S;\{e,x,y\}) = \{a,b,c\}, \mathscr{L}(f_S;\{e,x,yx\}) = \{a,b,d\}$ One can easily show that $\{a\},\{a,b\},\{a,b,c\}$ and $\{a,b,d\}$ are two-sided ideals of S.*

*In order to show Proposition 5.20., we have the following example:*

**Example 5.22.** *Consider the semigroup in Example 3.2. Define a soft set $f_S$ over $U = D_2 = \{e,x,y,yx\}$ such that $f_S(a) = \{e\}$, $f_S(b) = \{e,y\}$, $f_S(c) = \{e,y,yx\}$, $f_S(d) = \{e,x,y,yx\}$, By taking into account*

$$Im(f_S) = \{\{e\},\{e,y\},\{e,y,yx\},\{e,x,y,yx\}\}$$

*and considering that $Im(f_S)$ is ordered by inclusion, we have:*

$$\mathscr{L}(f_S;\alpha) = \begin{cases} \{a\}, & \text{if } \alpha = \{e\} \\ \{a,b\}, & \text{if } \alpha = \{e,y\} \\ \{a,b,c\}, & \text{if } \alpha = \{e,y,yx\} \\ \{a,b,c,d\}, & \text{if } \alpha = \{e,x,y,yx\} \end{cases}$$

*Since $\{a\},\{a,b\},\{a,b,c\}$ and $\{a,b,c,d\}$ are two-sided ideals of S, $f_S$ is an SU-ideal of S over U.*

*Now we define a soft set $h_S$ over $U = D_2$ such that $h_S(a) = \{x\}, h_S(b) = \{e,x,y,yx\}$, $h_S(c) = \{e,x\}$, $h_S(d) = \{e,x,yx\}$. By taking into account $Im(f_S) = \{\{e,x,y,yx\},\{e,x,yx\},\{e,x\},\{x\}\}$*

*and considering that $Im(f_S)$ is ordered by inclusion, we have:*

$$\mathscr{L}(f_S;\alpha)=\begin{cases}\{a\}, & \text{if } \alpha=\{x\}\\ \{a,c\}, & \text{if } \alpha=\{e,x\}\\ \{a,b,d\}, & \text{if } \alpha=\{e,x,yx\}\\ \{a,b,c,d\}, & \text{if } \alpha=\{e,x,y,yx\}\end{cases}$$

*Since $\{a,c\}S \nsubseteq \{a,c\}$ and $S\{a,c\} \nsubseteq \{a,c\}$ $\{a,c\}$ is not a two-sided ideal of S. Moreover, since; $h_S(cc)=h_S(b) \nsubseteq h_S(c)$ $h_S$ is not an SU-ideal of S over U.*

**Proposition 5.23.** *Let $f_S$ and $f_T$ be soft sets over $U$ and $\Psi$ be a semigroup isomorphism from S to $T$. If $f_S$ is an SU-left (right) ideal of S over U, then so is $\Psi^\star(f_S)$ of $T$ over U.*

**Proposition 5.24.** *Let $f_S$ and $f_T$ be soft sets over $U$ and $\Psi$ be a semigroup homomorphism from S to $T$. If $f_T$ is an SU-left (right) ideal of $T$ over U, then so is $\Psi^{-1}(f_T)$ of S over U.*

## 6. Soft union bi-ideals of semigroups

*In this section, we define soft union bi-ideals and study their properties as regards soft set operations and soft uni-product.*

**Definition 6.1.** *An SU-semigroup $f_S$ over $U$ is called a soft union bi-ideal of S over U if*

$$f_S(xyz)\subseteq f_S(x)\cup f_S(z)$$

*for all $x,y,z\in S$.*

*For the sake of brevity, soft union bi-ideal is abbreviated by SU-bi-ideal in what follows.* **Example 3.1.** *Let $S=\{0,a,b,c\}$ be the semigroup with the operation table given below.*

| + | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| a | 0 | a | b | 0 |
| b | 0 | 0 | 0 | 0 |
| c | 0 | c | 0 | 0 |

*Define the soft set $f_S$ over $U = \mathbb{Z}_5$ such that $f_S(0) = \{\overline{0}\}$, $f_S(a) = \{\overline{0}, \overline{1}\}$, $f_S(b) = \{\overline{0}, \overline{3}\}$, $f_S(c) = \{\overline{0}, \overline{2}\}$. Then, one can easily show that $f_S$ is an SU-bi-ideal of S over U.*

**Theorem 6.2.** *Let $f_S$ be a soft set over $U$. Then, $f_S$ is an SU-bi-ideal of S over U if and only if*

$$f_S * f_S \widetilde{\supseteq} f_S \text{ and } f_S * \widetilde{\theta} * f_S \widetilde{\supseteq} f_S$$

**Proof.** *First assume that $f_S$ is an SU-bi-ideal of S over U. Since $f_S$ is an SU-semigroup over U, by Theorem 4.5., we have*

$$f_S * f_S \widetilde{\supseteq} f_S.$$

*Let $s \in S$. In the case, when $(f_S * \widetilde{\theta} * f_S)(s) = U$, then it is clear that $f_S * \widetilde{\theta} * f_S \widetilde{\supseteq} f_S$, Otherwise, there exist elements $x, y, p, q \in S$ such that*

$$s = xy \text{ and } x = pq$$

*Then, since $f_S$ is an SU-bi-ideal of S over U, we have:*

$$f_S(s) = f_S(xy) = f_S((pq)y) \subseteq f_S(p) \cup f_S(y)$$

*Thus, we have*

$$\begin{aligned}
(f_S * \widetilde{\theta} * f_S)(s) &= [(f_S * \widetilde{\theta}) * f_S](s) \\
&= \bigcap_{s=xy} [(f_S * \widetilde{\theta})(x) \cup f_S(y)] \\
&= \bigcap_{s=xy} [(\bigcap_{x=pq} (f_S(p) \cup \widetilde{\theta}(q)) \cup f_S(y)] \\
&= \bigcap_{s=xy} [(\bigcap_{x=pq} (f_S(p) \cup \emptyset) \cup f_S(y)] \\
&= \bigcap_{s=pqy} (f_S(p) \cup f_S(y)) \\
&\supseteq \bigcap_{s=pqy} f_S(pqy) \\
&= f_S(xy) \\
&= f_S(s)
\end{aligned}$$

*Hence, $f_S * \widetilde{\theta} * f_S \widetilde{\supseteq} f_S$. Here, note that if $x \neq pq$, then $(f_S * \widetilde{\theta})(x) = U$, and so, $(f_S * \widetilde{\theta} * f_S)(s) = U \supseteq f_S(s)$.*

*Conversely, assume that $f_S * f_S \widetilde{\supseteq} f_S$. By Theorem 4.5., $f_S$ is an SU-semigroup of S. Let $x,y,z \in S$ and $s = xyz$. Then, since $f_S * \widetilde{\theta} * f_S \widetilde{\supseteq} f_S$, we have*

$$
\begin{aligned}
f_S(xyz) &= f_S(s) \\
&\subseteq (f_S * \widetilde{\theta} * f_S)(s) \\
&= [(f_S * \widetilde{\theta}) * f_S](s) \\
&= \bigcap_{s=mn} [(f_S * \widetilde{\theta})(m) \cup f_S(n)] \\
&\subseteq (f_S * \widetilde{\theta})(xy) \cup f_S(z) \\
&= [\bigcap_{xy=pq} (f_S(p) \cup \widetilde{\theta}(q)] \cup f_S(z) \\
&\subseteq ((f_S(x) \cup \widetilde{\theta}(y)) \cup f_S(z) \\
&= f_S(x) \cup f_S(z)
\end{aligned}
$$

*Thus, $f_S$ is an SU-bi-ideal of S over U. This completes the proof.*

**Theorem 6.3.** *A non-empty subset B of a semigroup of S is a bi-ideal of S if and only if the soft subset $f_S$ defined by*

$$
f_S(x) = \begin{cases} \alpha, & \text{if } x \in S \setminus B, \\ \beta, & \text{if } x \in B \end{cases}
$$

*is an SU-bi-ideal of S, where $\alpha, \beta \subseteq U$ such that $\alpha \supseteq \beta$.*

**Theorem 6.4.** *Let X be a nonempty subset of a semigroup S. Then, X is a bi-ideal of S if and only if $\mathscr{S}_{X^c}$ is an SU-bi-ideal of S over U.*

**Proof.** *It follows from Theorem 6.3.*

*It is known that every left (right, two sided) ideal of a semigroup S is a bi-ideal of S. Moreover, we have the following:*

**Theorem 6.5.** *Every SU-left (right, two sided) ideal of a semigroup S over U is an SU-bi-ideal of S over U.*

**Proof.** *Let $f_S$ be an SU-left (right, two sided) ideal of S over U and $x,y,z \in S$. Then, $f_S$ is as SU-semigroup by Theorem 6.5. Moreover,*

$$f_S(xyz) = f_S((xy)z) \subseteq f_S(z) \subseteq f_S(x) \cup f_S(z)$$

*Thus, $f_S$ is an SU-bi-ideal of S.*

**Theorem 6.6.** *Let $f_S$ be any soft subset of a semigroup S and $g_S$ be any SU-bi-ideal of S over U. Then, the soft uni-products $f_S * g_S$ and $g_S * f_S$ are SU-bi-ideals of S over U.*

**Proof.** *We show the proof for $f_S * g_S$. To see that $f_S * g_S$ is an SU-bi-ideal of S over U, first we need to show that $f_S * g_S$ is an SU-semigroup over U. Thus,*

$$\begin{aligned}
(f_S * g_S) * (f_S * g_S) &= f_S * (g_S * (f_S * g_S)) \\
&\widetilde{\supseteq} f_S * (g_S * (\widetilde{\theta} * g_S)) \ (since\ f_S \widetilde{\supseteq} \widetilde{\theta}) \\
&= f_S * (g_S * \widetilde{\theta} * g_S) \\
&\widetilde{\supseteq} f_S * g_S \ (since\ g_S * \widetilde{\theta} * g_S \widetilde{\supseteq} g_S))
\end{aligned}$$

*Hence, by Theorem 4.5., $f_S * g_S$ is an SU-semigroup over U. Moreover we have:*

$$\begin{aligned}
(f_S * g_S) * \widetilde{\theta} * (f_S * g_S) &= f_S * (g_S * (\widetilde{\theta} * f_S) * g_S) \\
&\widetilde{\supseteq} f_S * (g_S * \widetilde{\theta} * g_S) \ (since\ \widetilde{\theta} * f_S \widetilde{\supseteq} \widetilde{\theta}) \\
&\widetilde{\supseteq} f_S * g_S
\end{aligned}$$

*Thus, it follows that $f_S * g_S$ is an SU-bi-ideal of S over U. It can be seen in a similar way that $g_S * f_S$ is an SU-bi-ideal of S over U. This completes the proof.*

**Proposition 6.7.** *Let $f_S$ and $f_T$ be SU-bi-ideals over U. Then, $f_S \vee f_T$ is an SU-bi-ideal of $S \times T$ over U.*

**Proposition 6.8.** *If $f_S$ and $h_S$ are two SU-bi-ideals of S over U, then so is $f_S \widetilde{\cup} h_S$ of S over U.*

**Proposition 6.9.** *Let $f_S$ be a soft set over U and $\alpha$ be a subset of U such that $\alpha \in Im(f_S)$. If $f_S$ is an SU-bi-ideal of S over U, then $\mathscr{L}(f_S;\alpha)$ is a bi-ideal of S.*

**Definition 6.10.** *If $f_S$ is an SU-bi-ideal of S over U, then bi-ideals $\mathscr{L}(f_S;\alpha)$ are called lower $\alpha$ bi-ideals of $f_S$.*

**Proposition 6.11.** *Let $f_S$ be a soft set over U, $\mathscr{L}(f_S;\alpha)$ be lower $\alpha$ bi-ideals of $f_S$ for each $\alpha \subseteq U$ and $Im(f_S)$ be an ordered set by inclusion. Then, $f_S$ is an SU-bi-ideal of S over U.*

**Proposition 6.12.** *Let $f_S$ and $f_T$ be soft sets over U and $\Psi$ be a semigroup isomorphism from S to T. If $f_S$ is an SU-bi-ideal of S over U, then so is $\Psi^{\star}(f_S)$ of T over U.*

**Proposition 6.13.** *Let $f_S$ and $f_T$ be soft sets over U and $\Psi$ be a semigroup homomorphism from S to T. If $f_T$ is an SU-bi-ideal of T over U, then so is $\Psi^{-1}(f_T)$ of S over U.*

## 7. Regular semigroups

*In this section, we characterize a regular semigroup in terms of SU-ideals.*

*A semigroup S is called regular if for every element a of S there exists an element x in S such that*

$$a = axa$$

*or equivalently $a \in aSa$. There is a characterization of a regular semigroup in [31]iseki as follows:*

**Proposition 7.1.** *[31] For a semigroup S, the following conditions are equivalent:*

*1) S is regular.*

*2) $RL = R \cap L$ for every right ideal R and left ideal L of S.*

**Theorem 7.2.** *For a semigroup S, the following conditions are equivalent:*

*1) S is regular.*

*2) $f_S * g_S = f_S \widetilde{\cup} g_S$ for every SU-right ideal $f_S$ of S over U and SU-left ideal $g_S$ of S over U.*

**Proof.** *Let S be a regular semigroup and $f_S$ be an SU-right ideal of S and $g_S$ be an SU-left ideal of S over U. In Theorem 5.12., we show that*

$$f_S * g_S \widetilde{\supseteq} f_S \widetilde{\cup} g_S$$

*for every SU-right ideal $f_S$ of S and SU-left ideal $g_S$ of S over U. Therefore, it suffices to show that $f_S \widetilde{\cup} g_S \widetilde{\supseteq} f_S * g_S$. Let s be any element of S. Then, since S is regular, there exists an element x in S such that $s = sxs$. Thus, we have*

$$\begin{aligned}(f_S * g_S)(s) &= \bigcap_{s=ab} (f_S(a) \cup g_S(b)) \\ &\subseteq f_S(sx) \cup g_S(s) \\ &\subseteq f_S(s) \cup g_S(s) \\ &= (f_S \widetilde{\cup} g_S)(s)\end{aligned}$$

*Thus, $f_S * g_S = f_S \widetilde{\cap} g_S$.*

*Conversely, assume that (2) holds. In order to show that S is regular, we need to illustrate that $RL = R \cap L$ for every for every right ideal R of S and left ideal L of S over U. Let R and L be any right ideal and left ideal of S, respectively. It is known that $RL \subseteq R \cap L$ always holds. So it is enough to show that $R \cap L \subseteq RL$. On the contrary, let there exists $a \in R \cap L$ such that $a \notin RL$. By Theorem 5.8., the soft characteristic functions $\mathscr{S}_{R^c}$ and $\mathscr{S}_{L^c}$ are SU-right ideal and SU-left ideal of S, respectively. Since $a \in R \cap L$, $a \in R$ and $a \in L$. Thus,*

$$\mathscr{S}_{R^c}(a) = \mathscr{S}_{L^c}(a) = \emptyset$$

*On the other hand, since $a \notin RL$, this implies that there do not exist $x \in R$ and $y \in L$ such that $a = xy$. Thus,*

$$(\mathscr{S}_{R^c} * \mathscr{S}_{L^c})(a) = \bigcap_{a=bc} (\mathscr{S}_{R^c}(b) \cup \mathscr{S}_{L^c}(c)) = \bigcap_{a=bc} (U \cup U) = U$$

*But this contradicts our hypothesis. Hence, $R \cap L \subseteq RL$. It follows by Proposition 7.1. that S is regular. Hence (2) implies (1).*

**Corollary 7.3.** *For a semigroup S, the following conditions are equivalent:*

*1) S is regular.*

*2) $f_S * g_S = f_S \widetilde{\cup} g_S$ for every SU-ideals $f_S$ and $g_S$ of S over U.*

**Proposition 7.4.** *A semigroup S is regular if and only if every SU-ideal of S is idempotent.*

**Proof.** *Let $S$ be a regular semigroup and $h_S$ be an SU-ideal of $S$. Since $h_S$ is an SU-right ideal of $S$, we have*

$$h_S * h_S \widetilde{\supseteq} h_S * \widetilde{\theta} \widetilde{\supseteq} h_S.$$

*Now, we show that $h_S \widetilde{\supseteq} h_S * h_S$. Since $S$ is regular, there exists an element $x \in S$ such that $a = axa$ for all $a \in S$. So, we have;*

$$\begin{aligned} (h_S * h_S)(a) &= \bigcap_{a=axa} (h_S(ax) \cup h_S(a)) \\ &\subseteq h_S(a) \cup h_S(a) \\ &= h_S(a) \end{aligned}$$

*Hence, $h_S \widetilde{\supseteq} h_S * h_S$ and so $(h_S)^2 = h_S * h_S = h_S$.*

*Now, let $k_S$ be any SU-ideal of $S$. Since it is an SI-left ideal of $S$, we have*

$$k_S * k_S \widetilde{\supseteq} \widetilde{\theta} * k_S \widetilde{\supseteq} k_S.$$

*Thus, we show that $k_S \widetilde{\supseteq} k_S * k_S$. Since $S$ is regular, there exists an element $x \in S$ such that $a = axa$ for all $a \in S$. Thus, we have;*

$$\begin{aligned} (k_S * k_S)(a) &= \bigcap_{a=axa} (k_S(a) \cup k_S(xa)) \\ &\subseteq (k_S(a) \cup k_S(a)) \\ &= k_S(a) \end{aligned}$$

*Hence, $k_S \widetilde{\supseteq} k_S * k_S$ and so $(k_S)^2 = k_S * k_S = k_S$.*

*For the converse, let $f_S$ and $k_S$ be an SU-ideals of $S$. In view of Corollary 7.3., it is sufficient to show that $h_S * k_S = f_S \widetilde{\cup} k_S$. It is obvious that $f_S * k_S \widetilde{\supseteq} f_S \widetilde{\cup} k_S$. For the inverse inclusion, we argue as follows:*

$$(f_S \widetilde{\cup} k_S)(x) = (f_S \widetilde{\cup} k_S)^2(x)$$

*by idempotency of $f_S \widetilde{\cup} k_S$ and $x \in R$. Thus,*

$$\begin{aligned}(f_S \widetilde{\cup} k_S)(x) &= (f_S \widetilde{\cup} k_S)^2(x) \\ &= \bigcap_{x=\sum_{i=1}^{m} a_i b_i} (f_S \cup k_S)(a_i) \cup (f_S \widetilde{\cup} k_S)(b_i) \\ &\supseteq \bigcap_{x=\sum_{i=1}^{m} a_i b_i} f_S(a_i) \cup k_S(b_i) \\ &= (f_S * k_S)(x)\end{aligned}$$

*Hence, $f_S \widetilde{\cup} k_S \widetilde{\supseteq} f_S * k_S$, whence $f_S \widetilde{\cup} k_S = f_S * k_S$.*

**Corollary 7.5.** *Every SU-left (right) of a regular semigroup is idempotent.*

**Corollary 7.6.** *The set of all SU-ideals of a regular semigroup S forms a semilattice under the soft uni-product.*

**Proposition 7.7.** *Let the set of all SU-ideals of S be a regular semigroup of S under the soft uni-product. Then, every SU-ideal of S has the form $f_S = f_S * \widetilde{\theta} * f_S$.*

**Proof.** *Let $f_S$ be an SU-ideal of S. Then, by assumption, there exists an SU-ideal $g_S$ of S such that*

$$f_S = f_S * g_S * f_S.$$

*Thus, we have*

$$f_S = f_S * g_S * f_S \widetilde{\supseteq} f_S * \widetilde{\theta} * f_S \widetilde{\supseteq} (f_S * \widetilde{\theta}) \widetilde{\cap} (\widetilde{\theta} * f_S) \widetilde{\supseteq} f_S \widetilde{\cap} f_S = f_S,$$

*since*

$$f_S * \widetilde{\theta} * f_S \widetilde{\supseteq} f_S * \widetilde{\theta} * \widetilde{\theta} \widetilde{\supseteq} f_S * \widetilde{\theta}$$

*and*

$$f_S * \widetilde{\theta} * f_S \widetilde{\supseteq} \widetilde{\theta} * \widetilde{\theta} * f_S \widetilde{\supseteq} \widetilde{\theta} * f_S.$$

*Hence, $f_S = f_S * \widetilde{\theta} * f_S$.*

**Definition 7.8.** *An SU-ideal $f_S$ of a semigroup S is said to be soft strongly irreducible if and only if for every SU-ideals $g_S$ and $h_S$ of S, $g_S \widetilde{\cup} h_S \widetilde{\supseteq} f_S$ implies that $g_S \widetilde{\supseteq} f_S$ or $h_S \widetilde{\supseteq} f_S$.*

**Definition 7.9.** *An SU-ideal $h_S$ of a semigroup S is said to be soft prime ideal if for any SU-ideals $f_S$ and $g_S$ of S, $f_S * g_S \widetilde{\supseteq} h_S$ implies that $f_S \widetilde{\supseteq} h_S$ or $g_S \widetilde{\supseteq} h_S$.*

**Definition 7.10.** *The set of SU-ideals of a semigroup is called totally ordered under inclusion if for any SU-ideals $f_S$ and $g_S$ of S, either $f_S \widetilde{\supseteq} g_S$ or $g_S \widetilde{\supseteq} f_S$.*

**Proposition 7.11.** *In a regular semigroup S, an SU-ideal is soft strongly irreducible if and only if it is soft prime.*

**Proof.** *It follows from Corollary 7.3., Definition 7.8. and Definition 7.9.*

**Proposition 7.12.** *Every SU-ideal of a regular semigroup S is soft prime if and only if the set of SU-ideals of S is totally ordered under inclusion.*

**Proof.** *It follows from Corollary 7.3., Definition 7.9. and Definition 7.10.*

*As is known a semigroup S is regular if and only if $B = BSB$ for all bi-ideals B of S. Now, we shall give a characterization of a regular semigroup by SU-bi-ideals.*

**Theorem 7.13.** *For a semigroup S, the following conditions are equivalent:*

*1) S is regular.*

*2) $f_S = f_S * \widetilde{\theta} * f_S$ for every SU-bi-ideal $f_S$ of S over U.*

**Proof.** *First assume that (1) holds. Let $f_S$ be any SU-bi-ideal $f_S$ of S over U and s be any element of S. Then, since S is regular, there exists an element $x \in S$ such that $s = sxs$. Thus, we have;*

$$\begin{aligned}
(f_S * \widetilde{\theta} * f_S)(s) &= [(f_S * \widetilde{\theta}) * f_S](s) \\
&= \bigcap_{s=ab} [(f_S * \widetilde{\theta})(a) \cup f_S(b)] \\
&\subseteq (f_S * \widetilde{\theta})(sx) \cup f_S(s) \\
&= \bigcap_{sx=mn} \{(f_S(m) \cup \widetilde{\theta}(n)\} \cup f_S(s) \\
&\subseteq (f_S(s) \cup \widetilde{\theta}(x)) \cup f_S(s) \\
&= (f_S(s) \cup \emptyset) \cup f_S(s) \\
&= f_S(s)
\end{aligned}$$

*and so, we have* $f_S * \widetilde{\theta} * f_S \widetilde{\subseteq} f_S$. *Since* $f_S$ *is an SU-bi-ideal of S,* $f_S * \widetilde{\theta} * f_S \widetilde{\supseteq} f_S$. *Thus,* $f_S * \widetilde{\theta} * f_S = f_S$ *which means that (1) implies (2).*

*Conversely assume that (2) holds. In order to show that S is regular, we need to illustrate that* $B = BSB$ *for every bi-ideal B of S. It is obvious that* $BSB \subseteq B$. *Therefore, it is enough to show that* $B \subseteq BSB$. *On the contrary, let there exists* $a \in B$ *such that* $a \notin BSB$. *By Theorem 6.4., the soft characteristic function* $\mathscr{S}_{B^c}$ *is an SU-bi-ideal of S. Since* $a \in B$, *thus,*

$$\mathscr{S}_{B^c}(a) = \emptyset$$

*On the other hand, since* $a \notin BSB$, *this implies that there do not exist* $x, z \in B$ *and* $y \in S$ *such that* $a = xyz$. *Thus,*

$$(\mathscr{S}_{B^c} * \mathscr{S}_{S^c} * \mathscr{S}_{B^c})(a) = (\mathscr{S}_{B^c} * \widetilde{\theta} * \mathscr{S}_{B^c})(a) = U$$

*But this contradicts our hypothesis. Thus,* $B \subseteq BSB$ *and so* $B = BSB$. *It follows that S is regular, so (2) implies (1).*

**Theorem 7.14.** *Let* $f_S$ *be a soft set of a regular semigroup S. Then, the following conditions are equivalent:*

1) $f_S$ *is an SU-bi-ideal of S.*
2) $f_S$ *may be presented in the form* $f_S = g_S * h_S$, *where* $g_S$ *is an SU-right ideal and* $h_S$ *is an SU-left ideal of S over U.*

**Proof.** *First assume that (1) holds. Since S is regular, it follows from Theorem 7.13. that* $f_S = f_S * \widetilde{\theta} * f_S$. *Thus, we have*

$$\begin{aligned} f_S &= f_S * \widetilde{\theta} * f_S \\ &= f_S * \widetilde{\theta} * (f_S * \widetilde{\theta} * f_S) \\ &= [f_S * (\widetilde{\theta} * f_S)] * (\widetilde{\theta} * f_S) \\ &\widetilde{\supseteq} (f_S * \widetilde{\theta}) * (\widetilde{\theta} * f_S) \ (since\ \widetilde{\theta} * f_S \widetilde{\supseteq} \widetilde{\theta}) \end{aligned}$$

*Similarly,*

$$\begin{aligned}(f_S*\widetilde{\theta})*(\widetilde{\theta}*f_S) &= f_S*(\widetilde{\theta}*\widetilde{\theta})*f_S)\\ &\widetilde{\supseteq} f_S*\widetilde{\theta}*f_S \ (since\ \widetilde{\theta}*\widetilde{\theta}\widetilde{\supseteq}\widetilde{\theta})\\ &= f_S\end{aligned}$$

*Namely,* $f_S=(f_S*\widetilde{\theta})*(\widetilde{\theta}*f_S)$. *Here, we can easily show that* $f_S*\widetilde{\theta}$ *is an SU-right ideal of S and* $\widetilde{\theta}*f_S$ *is an SU-left ideal of S. In fact*

$$(f_S*\widetilde{\theta})*\widetilde{\theta}=f_S*(\widetilde{\theta}*\widetilde{\theta})\widetilde{\supseteq}f_S*\widetilde{\theta}$$

*Similarly*

$$\widetilde{\theta}*(\widetilde{\theta}*f_S)=(\widetilde{\theta}*\widetilde{\theta})*f_S\widetilde{\supseteq}\widetilde{\theta}*f_S$$

*implying that* $\widetilde{\theta}*f_S$ *is an SU-left ideal of S.*

*Conversely assume that (2) holds. It means that there exists an SU-right ideal* $g_S$ *and SU-left ideal* $h_S$ *of S such that* $f_S=g_S*h_S$. *By Theorem 6.5., every SU-left (right) ideal of S is an SU-bi-ideal of S. Thus,* $g_S$ *and* $h_S$ *are SU-bi-ideals of S. Moreover,* $g_S*h_S=f_S$ *is an SU-bi-ideal of S by Theorem 6.6. Therefore, we obtain that (2) implies (1). This completes the proof.*

**Theorem 7.15.** *For a semigroup S, the following conditions are equivalent:*

*1) S is regular.*

*2)* $f_S\widetilde{\cup}g_S=f_S*g_S*f_S$ *for every SU-bi-ideal* $f_S$ *of S and SU-ideal* $g_S$ *of S over U.*

**Proof.** *First assume that (1) holds. Let* $f_S$ *be any SU-bi-ideal and* $g_S$ *be SU-ideal of S over U. Then,*

$$f_S*g_S*f_S\widetilde{\supseteq}f_S*\widetilde{\theta}*f_S\widetilde{\supseteq}f_S$$

*and*

$$f_S*g_S*f_S\widetilde{\supseteq}\widetilde{\theta}*(g_S*\widetilde{\theta})\widetilde{\supseteq}\widetilde{\theta}*g_S\widetilde{\supseteq}g_S$$

*so* $f_S*g_S*f_S\widetilde{\supseteq}f_S\widetilde{\cup}g_S$. *To show that* $f_S\widetilde{\cup}g_S\widetilde{\supseteq}f_S*g_S*f_S$ *holds, let s be any element of S. Since S is regular, there exists an element x in S such that*

$$s=sxs\ (s=sx(sxs))$$

*Since $g_S$ is an SU-ideal of S, we have*

$$g_s(xsx) = g_S(x(sx)) \subseteq g_S(sx) \subseteq g_S(s)$$

*Therefore, we have*

$$\begin{aligned}
(f_S * g_S * f_S)(s) &= [f_S * (g_S * f_S)](s) \\
&= \bigcap_{s=mn} [f_S(m) \cup (g_S * f_S)(n)] \\
&\subseteq f_S(s) \cup (g_S * f_S)(xsxs) \\
&= f_S(s) \cup \{ \bigcap_{xsxs=yz} [g_S(y) \cup f_S(z)]\} \\
&= f_S(s) \cup (g_S(xsx) \cup f_S(s)) \\
&\subseteq (f_S(s) \cup g_S(s) \cup f_S(s) \\
&\subseteq f_S(s) \cup g_S(s) \\
&= (f_S \widetilde{\cup} g_S)(s)
\end{aligned}$$

*so we have $f_S \widetilde{\cup} g_S \widetilde{\supseteq} f_S * g_S * f_S$. Thus we obtain that $f_S \widetilde{\cup} g_S = f_S * g_S * f_S$, hence (1) implies (2).*

*Conversely assume that (2) holds. In order to show that S is regular, it is enough to show that $f_S = f_S * \widetilde{\theta} * f_S$ for all SU-bi-ideals of S over U by Theorem 7.13. Since $\widetilde{\theta}$ is an SU-ideal of S, we have $f_S = f_S \widetilde{\cup} \widetilde{\theta} = f_S * \widetilde{\theta} * f_S$ Thus, (2) implies (1). This completes the proof.*

**Theorem 7.16.** *For a semigroup S, the following conditions are equivalent:*

1) *S is regular.*
2) *$h_S \widetilde{\cup} f_S \widetilde{\cup} g_S \widetilde{\supseteq} h_S * f_S * g_S$ for every SU-right ideal $h_S$, every SU-bi-ideal $f_S$ and every SU-left ideal $g_S$ of S.*

**Proof.** *Assume that (1) holds. Let $h_S$, $f_S$ and $g_S$ be SU-right, SU-bi-ideal and SU-left ideal of S, respectively. Let a be any element of S. Since S is regular, there exists an element x in S such* *that $a = axa$. Hence, we have:*

$$\begin{aligned}(h_S * f_S * g_S)(a) &= [h_S * (f_S * g_S)](a)\\ &= \bigcap_{a=yz}[h_S(y) \cup (f_S * g_S)(z)]\\ &\subseteq h_S(ax) \cup (f_S * g_S)(a)\\ &= h_S(ax) \cap \{\bigcap_{a=pq}[f_S(p) \cup g_S(q)]\}\\ &\subseteq h_S(a) \cup (f_S(a) \cup g_S(xa))\\ &\subseteq h_S(a) \cup (f_S(a) \cup g_S(a))\\ &= (h_S \tilde{\cup} f_S \tilde{\cup} g_S)(a)\end{aligned}$$

*so we have $h_S * f_S * g_S \tilde{\supseteq} h_S \cup f_S \cup g_S$. Thus, (1) implies (2).*

*Conversely assume that (2) holds. Let $h_S$ and $g_S$ be any SU-right ideal and SU-left ideal of S, respectively. It is obvious that*

$$h_S * g_S \tilde{\supseteq} h_S \tilde{\cap} g_S.$$

*Since $\widetilde{\theta}$ itself is an SU-bi-ideal of S by Theorem 6.2., by assumption we have:*

$$h_S \tilde{\cup} g_S = h_S \tilde{\cup} \widetilde{\theta} \tilde{\cup} g_S \tilde{\supseteq} h_S * \widetilde{\theta} * g_S = h_S * (\widetilde{\theta} * g_S) \tilde{\supseteq} h_S * g_S$$

*It follows that $h_S \tilde{\cup} g_S \tilde{\supseteq} h_S * g_S$ for every SU-right ideal $h_S$ and SU-left ideal $g_S$ of S. It follows by Theorem 7.2. that S is regular. Hence, (2) implies (1). This completes the proof.*

**Theorem 7.17.** *For a regular semigroup S, the following conditions are equivalent:*

*1) Every bi-ideal of S is a right (left, two-sided) ideal of S.*

*2) Every SU-bi-ideal of S is an SU-right (left, two-sided) ideal of S.*

**Proof.** *We give the proof for the SU-right ideals. First assume that (1) holds. Let $f_S$ any SU-bi-ideal of S and $a, b$ any elements in S. One easily show that aSa is a bi-ideal of S. By assumption, aSa is a right ideal of S. Since S is regular,*

$$ab \in (aSa)S = a((Sa)S) \subseteq aSa$$

*This implies that there exists an element $x \in S$ such that*

$$ab = axa.$$

*Then, since $f_S$ is an SI bi-ideal of S, we have*

$$f_S(ab) = f_S(axa) \subseteq f_S(a) \cup f_S(a) = f_S(a).$$

*This means that $f_S$ is an SU-right ideal of S and that (1) implies (2).*

*Conversely, assume that (2) holds. Let B be any bi-ideal of S. Then, by Theorem 6.4., the soft characteristic function $\mathscr{S}_{B^c}$ is an SU-bi-ideal of S. Thus, by assumption, $\mathscr{S}_{B^c}$ is an SU-right ideal of S. Again, by Theorem 6.4., B is a right ideal of S. Therefore, (2) implies (1). This completes the proof.*

## 8. Intra-regular semigroups

*In this section, we characterize an intra-regular semigroup in terms of SU-ideals. A semigroup S is called intra-regular if for every element a of S there exist elements x and y in S such that*

$$a = xa^2y$$

**Proposition 8.1.** *[32] For a semigroup S, the following conditions are equivalent:*

*1) S is intra-regular.*

*2) $L \cap R \subseteq LR$ for every left ideal L and every right ideal R of S.*

**Theorem 8.2.** *For a semigroup S, the following conditions are equivalent:*

*1) S is intra-regular.*

*2) $g_S \widetilde{\cup} f_S \widetilde{\supseteq} g_S * f_S$ for every SU-right ideal $f_S$ of S and SU-left ideal $g_S$ of S over U.*

**Proof.** *First assume that (1) holds. Let $f_S$ be any SU-right ideal and $g_S$ be SU-left ideal of S over U and a be any element of S. Then, since S is intra-regular, there exist elements x and y in*

*S such that* $a = xa^2y$*. Thus,*

$$\begin{aligned}(g_S * f_S)(a) &= \bigcap_{a=bc}(g_S(b)\cup f_S(c))\\ &\subseteq (g_S(xa)\cup f_S(ay))\\ &\subseteq (g_S(a)\cup f_S(a))\\ &= (g_S\widetilde{\cup} f_S)(a)\end{aligned}$$

*Thus,* $g_S\widetilde{\cup} f_S \widetilde{\supseteq} g_S * f_S$*, which means that (1) implies (2).*

*Conversely assume that* $g_S\widetilde{\cup} f_S \widetilde{\supseteq} g_S * f_S$ *for every SU-right ideal* $f_S$ *and SU-left ideal* $g_S$ *of S over U. In order to show that S in intra-regular, it suffices to illustrate* $L\cap R\subseteq LR$ *for every left ideal L and for every right ideal R of S. Let L be a left ideal and R be a right ideal of S. On the contrary, let there exists* $a\in L\cap R$ *such that* $a\notin LR$*. Since the soft characteristic functions* $\mathscr{S}_{L^c}$ *and* $\mathscr{S}_{R^c}$ *is an SU-left ideal and SU-right ideal of S, respectively and since* $a\in L\cap R$*, we have*

$$\mathscr{S}_{L^c}(a) = \mathscr{S}_{R^c}(a) = \emptyset$$

*and so* $\mathscr{S}_{L^c}\widetilde{\cup}\mathscr{S}_{L^c} = \emptyset$*. On the other hand, since* $a\notin RL$*, this implies that there do not exist* $x\in L$ *and* $y\in R$ *such that* $a = xy$*. Thus,*

$$(\mathscr{S}_{L^c} * \mathscr{S}_{R^c})(a) = U$$

*But this contradicts our hypothesis. Thus,* $L\cap R\subseteq LR$*. It follows that S is intra-regular, so (2) implies (1).*

*The following characterization of a semigroup is both regular and intra-regular.*

**Proposition 8.3.** *[32] For a semigroup S, the following conditions are equivalent:*

*1) S is both regular and intra-regular.*

*2)* $B^2 = B$ *for every bi-ideal B of S. (That is, every bi-ideal of S is idempotent).*

**Theorem 8.4.** *For a semigroup S, the following conditions are equivalent:*

*1) S is both regular and intra-regular.*

*2)* $f_S * f_S = f_S$ *for every SU-bi-ideal* $f_S$ *of S. (That is, every SU-bi-ideal of S is idempotent).*

*3)* $f_S\widetilde{\cup} g_S \widetilde{\supseteq} (f_S * g_S)\widetilde{\cup}(g_S * f_S)$ *for every SU-bi-ideals* $f_S$ *and* $g_S$ *of S.*

*4)* $f_S \widetilde{\cup} g_S \widetilde{\supseteq} (f_S * g_S) \widetilde{\cup} (g_S * f_S)$ *for every SI bi-ideal* $f_S$ *and for every SU-left ideal* $g_S$ *of S.*

*5)* $f_S \widetilde{\cup} g_S \widetilde{\supseteq} (f_S * g_S) \widetilde{\cup} (g_S * f_S)$ *for every SI bi-ideal* $f_S$ *and for every SU-right ideal* $g_S$ *of S.*

*6)* $f_S \widetilde{\cup} g_S \widetilde{\supseteq} (f_S * g_S) \widetilde{\cup} (g_S * f_S)$ *for every SU-right ideal* $f_S$ *and for every SU-left ideal* $g_S$ *of S.*

**Proof.** *First assume that (1) holds. In order to show that (3) holds, let* $f_S$ *and* $g_S$ *be SU-bi-ideals of S and* $a \in S$. *Since S is intra-regular, there exist elements y and z in S such that* $a = ya^2z$ *for every element a of S. Thus,*

$$a = axa = (axa)xa = ax(ya^2z)xa = (axya)(azxa)$$

*Since* $f_S$ *and* $g_S$ *be SU-bi-ideals of S, we have;*

$$f_S(a(xy)a) \subseteq f_S(a) \cup f_S(a) = f_S(a)$$

$$g_S(a(zx)a) \subseteq g_S(a) \cup g_S(a) = g_S(a)$$

*Then, we have:*

$$\begin{aligned}(f_S * g_S)(a) &= \bigcap_{a=bc} (f_S(b) \cup g_S(c)) \\ &\subseteq (f_S(axya) \cup g_S(azxa)) \\ &\subseteq f_S(a) \cup g_S(a) \\ &= (f_S \widetilde{\cup} g_S)(a)\end{aligned}$$

*and so we have* $f_S * g_S \widetilde{\subseteq} f_S \widetilde{\cup} g_S$. *One can similarly show that* $g_S * f_S \widetilde{\subseteq} g_S \widetilde{\cup} f_S$, *which means that* $f_S \widetilde{\cup} g_S \widetilde{\supseteq} (f_S * g_S) \widetilde{\cup} (g_S * f_S)$. *This shows that (1) implies (3).*

*It is obvious that (3) implies (4), (4) implies (6) , (3) implies (5) and (5) implies (6).*

*Assume that (6) holds. Let* $f_S$ *and* $g_S$ *be any SU-right ideal and SU-left ideal of S, respectively. Then, we have*

$$f_S \widetilde{\cup} g_S = g_S \widetilde{\cap} f_S \widetilde{\supseteq} (f_S * g_S) \widetilde{\cup} (g_S * f_S) \widetilde{\supseteq} g_S * f_S$$

*It follows by Theorem 8.2. that S is intra-regular. On the other hand,*

$$f_S \widetilde{\cup} g_S \widetilde{\supseteq} (f_S * g_S) \widetilde{\cup} (g_S * f_S) \widetilde{\supseteq} f_S * g_S$$

*Since, the inclusion $f_S * g_S \tilde{\supseteq} f_S \tilde{\cup} g_S$ always hold, we have $f_S \tilde{\cup} g_S = f_S * g_S$. It follows that S is regular. Hence, (6) implies (1).*

*It is clear that (3) implies (2). In fact, by taking $g_S$ as $f_S$ in (3), we get*

$$f_S \tilde{\cup} f_S = f_S = (f_S * f_S) \tilde{\cup} (f_S * f_S) = f_S * f_S$$

*Finally assume that (2) holds. In order to show that (1) holds, it is enough to show that $B^2 = B$ for every bi-ideal B of S. Let B be any bi-ideal of S. Then, $BB \subseteq B$ always holds. We show that $B \subseteq BB$. On the contrary, let there exists $b \in B$ such that $b \notin BB$. By Theorem 6.4., the soft characteristic function $\mathscr{S}_{B^c}$ is an SU-bi-ideal of S. Since $b \in B$,*

$$\mathscr{S}_{B^c}(b) = \emptyset$$

*On the other hand, since $b \notin BB$, this implies that there do not exist $x, y \in B$ such that $b = xy$. Thus,*

$$(\mathscr{S}_{B^c} * \mathscr{S}_{B^c})(b) = U$$

*But this contradicts our hypothesis. Thus, $B \subseteq BB$ and so $B = BB = B^2$ . It follows that S is both regular and intra-regular, so (2) implies (1).*

**Theorem 8.5.** *For a semigroup S, the following conditions are equivalent:*

*1) S is both regular and intra-regular.*

*2) $f_S \tilde{\cup} g_S \tilde{\cup} h_S \tilde{\supseteq} f_S * g_S * h_S$ for every SU-bi-ideals $f_S$, $g_S$ and $h_S$ of S.*

*3) $f_S \tilde{\cup} g_S \tilde{\cup} h_S \tilde{\supseteq} f_S * g_S * h_S$ for every SI bi-ideals $f_S$ and $h_S$ of S and for every SU-right ideal $g_S$ of S.*

*4) $f_S \tilde{\cup} g_S \tilde{\cup} h_S \tilde{\supseteq} f_S * g_S * h_S$ for every SU-left ideals $f_S$ and $h_S$ of S and for every SU-right ideal $g_S$ of S.*

**Proof.** *First assume that (1) holds. In order to show that (4) holds, let $f_S$ and $h_S$ be any SU-left ideals of S and $g_S$ be any SU-right ideal of S and a be any element in S. Since S is regular, there exists element x in S such that $a = axa$. Since S is intra-regular, there exist elements $y, z$ in S such that $a = ya^2z$. Thus, we have*

$$a = axa = (axa)x(axa) = (ax(yaaz))x((yaaz)xa) = (axya)(azxya)(azxa)$$

*Therefore, we have*

$$\begin{aligned}(f_S * g_S * h_S)(a) &= [f_S * (g_S * h_S)](a)\\ &= \bigcap_{a=pq}[f_S(p) \cup (g_S * h_S)(q)]\\ &\subseteq f_S(axya) \cup (g_S * h_S)(azxyaazxa)\\ &= f_S(a) \cup \{\bigcap_{azxyaazxa=uv}(g_S(u) \cup h_S(v))\}\\ &\subseteq f_S(a) \cup (g_S(azxya) \cup h_S(azxa))\\ &\subseteq f_S(a) \cup g_S(a) \cup h_S(a)\\ &= (f_S \widetilde{\cap} g_S \widetilde{\cup} h_S)(a)\end{aligned}$$

*so we have* $f_S \widetilde{\cap} g_S \widetilde{\cup} h_S \widetilde{\supseteq} f_S * g_S * h_S$. *Thus, (1) implies (4). Assume that (4) holds. Let* $f_S$ *and* $g_S$ *be SU-left and SU-right ideal of S, respectively. Since* $\widetilde{\theta}$, *itself is an SU-left ideal of S,*

$$g_S \widetilde{\cup} f_S = g_S \widetilde{\cup} \widetilde{\theta} \widetilde{\cup} f_S \widetilde{\supseteq} g_S * \widetilde{\theta} * f_S \widetilde{\supseteq} g_S * f_S$$

*Since the inclusion* $g_S * f_S \widetilde{\supseteq} g_S \widetilde{\cup} f_S$ *always hold,* $g_S \widetilde{\cup} f_S = g_S * f_S$. *Hence, it follows that S is regular. Now, let* $f_S$ *and* $g_S$ *be any SU-left ideal and SU-right ideal of S, respectively. Since* $\widetilde{\theta}$ *itself is an SU-left ideal of S, by assumption we have:*

$$f_S \widetilde{\cup} g_S = f_S \widetilde{\cup} g_S \widetilde{\cap} \widetilde{\theta} \widetilde{\supseteq} f_S * g_S * \widetilde{\theta} = f_S * (g_S * \widetilde{\theta}) \widetilde{\supseteq} f_S * g_S$$

*Thus, it follows by Theorem 8.2. that S is intra-regular. So, (4) implies (1). It is obvious that (2) implies (3) and (3) implies (4). Thus, the proof is completed.*

*Now we give a new characterization for an intra-regular semigroup: First, we have the following definition:*

**Definition 8.6.** *A soft set* $f_S$ *over* $U$ *is called soft union semiprime if for all* $a \in S$,

$$f_S(a) \subseteq f_S(a^2).$$

**Theorem 8.7.** *For a nonempty A of S, the following conditions are equivalent:*

*1) A is semiprime.*

*2) The soft characteristic function* $\mathscr{S}_{A^c}$ *is soft union semiprime.*

**Proof.** *First assume that (1) holds. Let a be any element of S. We need to show that* $\mathscr{S}_{A^c}(a) \subseteq \mathscr{S}_{A^c}(a^2)$ *for all* $a \in S$. *If* $a^2 \in A$, *then since A is semiprime,* $a \in A$. *Thus,*

$$\mathscr{S}_{A^c}(a) = \emptyset = \mathscr{S}_{A^c}(a^2)$$

*If* $a^2 \notin A$, *then*

$$\mathscr{S}_{A^c}(a) \subseteq U = \mathscr{S}_{A^c}(a^2)$$

*In any case,* $\mathscr{S}_{A^c}(a) \subseteq \mathscr{S}_{A^c}(a^2)$ *for all* $a \in S$. *Thus,* $\mathscr{S}_{A^c}$ *is soft union semiprime. Hence (1) implies (2).*

*Conversely assume that (2) holds. Let* $a^2 \in A$ *and* $a \notin A$. *Since* $\mathscr{S}_{A^c}$ *is soft union semiprime, we have*

$$\mathscr{S}_{A^c}(a) = U \subseteq \mathscr{S}_{A^c}(a^2) = \emptyset$$

*But, this is a contradiction. Hence,* $a \in A$ *and so A is semiprime. Thus, (2) implies (1).*

**Theorem 8.8.** *For any SU-semigroup* $f_S$, *the following conditions are equivalent:*

*1)* $f_S$ *is soft union semiprime.*

*2)* $f_S(a) = f_S(a^2)$ *for all* $a \in S$.

**Proof.** *(2) implies (1) is clear. Assume that (1) holds. Let a be any element of S. Since* $f_S$ *is an SU-semigroup, we have;*

$$f_S(a) \subseteq f_S(a^2) = f_S(aa) \subseteq f_S(a) \cup f_S(a) = f_S(a)$$

*So,* $f_S(a^2) = f_S(a)$ *and (1) implies (2). This completes the proof.*

**Theorem 8.9.** *For a semigroup S, the following conditions are equivalent:*

*1) S is intra-regular.*

*2) Every SU-ideal of S is soft union semiprime.*

*3)* $f_S(a) = f_S(a^2)$ *for all SU-ideal of S and for all* $a \in S$.

**Proof.** *First assume that (1) holds. Let* $f_S$ *be any SU-ideal of S and a any element of S. Since S is intra-regular, there exist elements x and y in S such that* $a = xa^2y$. *Thus,*

$$f_S(a) = f_S(xa^2y) \subseteq f_S(xa^2) \subseteq f_S(a^2) = f_S(aa) \subseteq f_S(a)$$

*so, we have $f_S(a) = f_S(a^2)$. Hence, (1) implies (3).*

*Conversely, assume that (3) holds. It is known that $J[a^2]$ is an ideal of S. Thus, the soft characteristic function $\mathscr{S}_{(J[a^2])^c}$ is an SU-ideal of S. Since $a^2 \in J[a^2]$, we have;*

$$\mathscr{S}_{(J[a^2])^c}(a) = \mathscr{S}_{(J[a^2])^c}(a^2) = \emptyset$$

*Thus, $a \in J[a^2] = \{a^2\} \cup Sa^2 \cup a^2S \cup Sa^2S \subseteq Sa^2S$. Here, one can easily show that S is intra-regular. Hence (3) implies (1).*

*It is obvious that (3) implies (2). Now, assume that (2) holds. Let $f_S$ be an SU-ideal of S. Since $f_S$ is a soft union semiprime ideal of S,*

$$f_S(a) \subseteq f_S(a^2) = f_S(aa) \subseteq f_S(a)$$

*Thus, $f_S(a) = f_S(a^2)$. Hence (2) implies (3). This completes the proof.*

**Theorem 8.10.** *Let S be an intra-regular semigroup. Then, for every SU-ideal $f_S$ of S,*

$$f_S(ab) = f_S(ba)$$

*for all $a, b \in S$.*

**Proof.** *Let $f_S$ be an SU-ideal of an intra-regular semigroup S. Then, by Theorem 8.8., we have;*

$$f_S(ab) = f_S((ab)^2) = f_S(a(ba)b) \subseteq f_S(ba) = f_S((ba)^2) = f_S(b(ab)a) \subseteq f_S(ab)$$

*so, we have $f_S(ab) = f_S(ba)$. This completes the proof.*

## 9. Completely regular semigroups

*In this section, we characterize a completely regular semigroups in terms of SU-ideals. An element a of S is called a completely regular if there exists an element $x \in S$ such that*

$$a = axa \text{ and } ax = xa$$

*A semigroup S is called completely regular if every element of S is completely regular. A semigroup is called left (right) regular if for each element a of S, there exists an element $x \in S$ such that $a = xa^2$ ($a = a^2x$).*

**Proposition 9.1.** *[30] For a semigroup S, the following conditions are equivalent:*

*1) S is completely regular.*

*2) S is left and right regular, that is, $a \in Sa^2$ and $a \in a^2S$ for all $a \in S$.*

*3) $a \in a^2Sa^2$ for all $a \in S$.*

**Theorem 9.2.** *For a left regular semigroup S, the following conditions are equivalent:*

*1) Every left ideal of S is a two-sided ideal of S.*

*2) Every SU-left ideal of S is an SU-ideal of S.*

**Proof.** *Assume that (1) holds. Let $f_S$ be any SU-left ideal of S and a and b be any elements of S. Then, since the left ideal Sa is a two-sided ideal by assumption and since S is left regular, we have*

$$ab \in (Sa^2)b \subseteq (Sa)bS \subseteq Sa$$

*This implies that there exists an element $x \in S$ such that $ab = xa$. Thus, since $fS$ is an SU-left ideal of S, we have*

$$f_S(ab) = f_S(xa) \subseteq f_S(a).$$

*Hence, $f_S$ is an SU-right ideal of S and so $f_S$ is an SU-ideal of S. Thus (1) implies (2).*

*Assume that (2) holds. Let A be any left ideal of S. Then, the soft characteristic function $\mathscr{S}_{A^c}$ is an SU-left ideal of S. Then, by assumption, $\mathscr{S}_{A^c}$ is an SU-right ideal of S and so A is a right ideal of S and so A is a two-sided ideal of S. Hence (2) implies (1).*

**Theorem 9.3.** *For a semigroup S, the following conditions are equivalent:*

*1) S is left regular.*

*2) For every SU-left ideal $f_S$ of S, $f_S(a) = f_S(a^2)$ for all $a \in S$.*

**Proof.** *First assume that (1) holds. Let $f_S$ be any SU-left ideal of S and a be any element of S. Since S is left regular, there exists an element x in S such that $a = xa^2$. Thus, we have*

$$f_S(a) = f_S(xa^2) \subseteq f_S(a^2) \subseteq f_S(a)$$

*implying that $f_S(a) = f_S(a^2)$. Hence (1) implies (2).*

*Conversely, assume that (2) holds. Let a be any element of S. Since $L[a^2]$ is a left ideal of S, the soft characteristic function $\mathscr{S}_{(L[a^2])^c}$ is an SU-left ideal of S. Since $a^2 \in L[a^2]$, we have*

$$\mathscr{S}_{(L[a^2])^c}(a) = \mathscr{S}_{(L[a^2])^c}(a^2) = \emptyset$$

*implying that $a \in L[a^2] = \{a^2\} \cup Sa^2$. This obviously means that S is left regular. So (2) implies (1). This completes the proof.*

**Theorem 9.4.** *For a semigroup S, the following conditions are equivalent:*

*1) S is right regular.*

*2) For every SU-right ideal $f_S$ of S, $f_S(a) = f_S(a^2)$ for all $a \in S$.*

**Theorem 9.5.** *For a semigroup S, the following conditions are equivalent:*

*1) S is completely regular.*

*2) Every bi-ideal of S is semiprime.*

*3) Every SU-bi-ideal of S is soft union semiprime.*

*4) $f_S(a) = f_S(a^2)$ for every SU-bi-ideal $f_S$ of S and for all $a \in S$.*

**Proof.** *First assume that (1) holds. Let $f_S$ be any SU-bi-ideal of S. Since S is completely regular, there exists an element $x \in S$ such that $a = a^2xa^2$. Thus, we have*

$$f_S(a) = f_S(a^2xa^2) \subseteq f_S(a^2) \cup f_S(a^2) = f_S(a^2) = f_S(aa) = f_S(a(a^2xa^2) =$$

$$f_S(a(a^2xa)a) \subseteq f_S(a) \cup f_S(a) = f_S(a)$$

*and so, $f_S(a) = f_S(a^2)$. Thus (1) implies (4). (4) implies (3) is clear by Theorem 8.9. Assume that (3) holds. Let B be any bi-ideal of S and $a^2 \in B$ and $a \notin B$ . Since the soft characteristic function $\mathscr{S}_{B^c}$ of B is an SU-bi-ideal of S, it is soft union semiprime by hypothesis. Thus,*

$$\mathscr{S}_{B^c}(a) = U \subseteq \mathscr{S}_B(a^2) = \emptyset.$$

*But this is a contradiction. Hence, $a \in B$ and so B is semiprime. Thus (3) implies (2).*

*Finally assume that (2) holds. Let a be any element of S. Then, since the principal ideal $B[a^2]$ generated by $a^2$ is a bi-ideal and so by assumption semiprime and since $a^2 \in B[a^2]$,*

$$\mathscr{S}_{B[a^2]}(a) = \mathscr{S}_{B[a^2]}(a^2) = U$$

*implying that*

$$a \in B[a^2] = \{a^2\} \cup \{a^4\} \cup a^2Sa^2 \subseteq a^2Sa^2.$$

*This implies that S is completely regular. Thus (2) implies (1). This completes the proof.*

## 10.Weakly Regular Semigroups

*In this section, we characterize a weakly regular semigroup in terms of SU-ideals. A semigroup S is called weakly-regular if for every $x \in S$, $x \in (xS)^2$.*

**Proposition 10.1.** *[30] A monoid is weakly regular if and only if $I \cap J = IJ$ for all right ideal I and all two-sided ideal J of S.*

**Theorem 10.2.** *For a monoid S, the following conditions are equivalent:*

*1) S is weakly regular.*

*2) $f_S \widetilde{\cup} g_S \widetilde{\supseteq} f_S * g_S$ for every SU-right ideal $f_S$ of S and for every SU-ideal $g_S$ of S.*

**Proof.** *First assume that (1) holds. Let $f_S$ be an SU-right ideal of S, $g_S$ be an SU-left ideal of S and $x \in S$. Then, since S is weakly regular, $x \in (xS)^2$. Thus, $x = xsxt$ for some $s,t \in S$. Hence,*

$$\begin{aligned} (f_S * g_S)(x) &= \bigcap_{x=xsxt} (f_S(xs) \cup g_S(xt)) \\ &\subseteq f_S(x) \cup g_S(x) \\ &= (f_S \widetilde{\cup} g_S)(x) \end{aligned}$$

*Since $f_S \widetilde{\cup} g_S \widetilde{\subseteq} f_S * g_S$ always holds for every SU-right ideal $f_S$ and SU-left ideal $g_S$ of S, $f_S \widetilde{\cup} g_S = f_S * g_S$. Thus, (1) implies (2).*

*Conversely assume that (2) holds. In order to show that S is weakly regular, we show that $R \cap L = RL$ for every right ideal R and left ideal L of S. It is obvious that $RL \subseteq R \cap L$ always holds. In order to see that $R \cap L \subseteq RL$, let a be any element in $R \cap L$ and $a \notin RL$. Then $a \in R$ and $a \in L$ . Since the soft characteristic functions $\mathscr{S}_{R^c}$ and $\mathscr{S}_{L^c}$ is SU-right and SU-left ideal of S, respectively, we have:*

$$\mathscr{S}_{R^c}(a) = \mathscr{S}_{R^c}(a) = \emptyset$$

*and since $a \notin RL$, there do not exist $b \in R$ and $c \in L$ such that $a = bc$. Thus,*

$$(\mathscr{S}_{R^c} * \mathscr{S}_{L^c})(a) = U$$

*but this is a contradiction. So, $a \in RL$. Thus, $R \cap L \subseteq RL$ and $R \cap L = RL$. It follows that $S$ is weakly-regular. Hence (2) implies (1).*

**Theorem 10.3.** *For a monoid $S$, the following conditions are equivalent:*

*1) $S$ is weakly regular.*

*2) $f_S \widetilde{\cap} g_S \widetilde{\cup} h_S \widetilde{\supseteq} f_S * g_S * h_S$ for every SU-bi-ideal $f_S$ of $S$, for every SU-ideal $g_S$ of $S$ and for every SU-right ideal $h_S$ of $S$.*

**Proof.** *First assume that (1) holds. Let $x \in S$. Then, $x \in (xS)^2$. Thus, $x = xsxt$ for some $s,t \in S$. Hence,*

$$\begin{aligned}
(f_S * g_S * h_S)(x) &= [f_S * (g_S * h_S)](x) \\
&= \bigcap_{x = xsxt} [f_S(x) \cup (g_S * h_S)(sxt)] \\
&\subseteq f_S(x) \cup \{ \bigcap_{sxt = pv} (g_S(p) \cup h_S(v)) \} \\
&\subseteq f_S(x) \cup g_S(sxs) \cup h_S(xt^2) \\
&\subseteq f_S(x) \cup g_S(x) \cup h_S(x) \\
&= (f_S \widetilde{\cup} g_S \widetilde{\cup} h_S)(x)
\end{aligned}$$

*since $sxt = s(xsxt)t = (sxs)(xt^2)$. Thus, (1) implies (2).*

*Now, assume that (2) holds. Let $f_S$ be an SU-right ideal of $S$, $g_S$ be an SU-ideal of $S$ and let $h_S = \widetilde{\theta}$. Then, we have*

$$f_S \widetilde{\cup} g_S \widetilde{\cup} h_S = f_S \widetilde{\cup} g_S \widetilde{\cup} \widetilde{\theta} = f_S \widetilde{\cup} g_S$$

*and*

$$f_S * g_S * h_S = f_S * g_S * \widetilde{\theta} = f_S * (g_S * \widetilde{\theta}) \widetilde{\supseteq} f_S * g_S$$

*Then, $f_S \widetilde{\cup} g_S = f_S \widetilde{\cup} g_S \widetilde{\cup} h_S \widetilde{\supseteq} f_S * g_S * h_S \widetilde{\supseteq} f_S * g_S$ that is, $f_S \widetilde{\cup} g_S \widetilde{\supseteq} f_S * g_S$ for every SU-right ideal $f_S$ of $S$ and SU-ideal $g_S$ of $S$. Thus, $S$ is weakly regular. Hence (2) implies (1). This completes the proof.*

**Theorem 10.4.** *For a monoid S, the following conditions are equivalent:*

*1) S is weakly regular.*

*2)* $f_S \widetilde{\cup} g_S \widetilde{\supseteq} f_S * g_S$ *for every SU-bi-ideal* $f_S$ *of S and for every SU-ideal* $g_S$ *of S.*

**Proof.** *Similar to the the proof of Theorem 10.3.*

## 11. Quasi-regular semigroups

*In this section, we study a semigroup whose SU-left (right, two-sided) ideals are all idempotent. A semigroup S is called left (right) quasi-regular if every left (right) ideal of S is idempotent, and is called quasi-regular if every left ideal and right ideal of S is idempotent ([29]). It is easy to prove that S is left (right) quasi-regular if and only if* $a \in SaSa$ $(a \in aSaS)$*, this implies that there exist elements* $x, y \in S$ *such that* $a = xaya$ $(a = axay)$.

**Theorem 11.1.** *A semigroup S is left (right) quasi-regular if and only if every SU-left (right) ideal is idempotent.*

**Proof.** *Assume that* $f_S$ *is an SI-left ideal. Then, there exist* $x, y \in S$ *such that* $a = xaya$*. So, we have;*

$$\begin{aligned}(f_S * f_S)(a) &= \bigcap_{a=xaya} (f_S(xa) \cup f_S(ya)) \\ &\subseteq f_S(xa) \cup f_S(ya) \\ &\subseteq f_S(a) \cup f_S(a) \\ &= f_S(a)\end{aligned}$$

*and so,* $f_S * f_S \widetilde{\subseteq} f_S$*. Thus,* $f_S * f_S = f_S$ *and* $f_S$ *is idempotent.*

*Conversely, assume that every SU-left ideal of S is idempotent. Let* $a \in S$*. Then, since* $L[a]$ *is a principal left ideal of S, the soft characteristic function* $\mathscr{S}_{(L[a])^c}$ *is an SU-left ideal of S. It is known that* $a \in L[a]$ *and let* $a \notin L[a]L[a]$ *and so there do not exist* $y, z \in L[a]$ *such that* $a = yz$*. Then,*

$$\mathscr{S}_{(L[a])^c}(a) = \emptyset$$

*and*

$$(\mathscr{S}_{(L[a])^c} * \mathscr{S}_{(L[a])^c})(a) = U,$$

*but this is a contradiction. So*

$$a \in L[a]L[a] = (\{a\} \cup Sa)(\{a\} \cup Sa) = \{a^2\} \cup aSa \cup Sa^2 \cup SaSa \subseteq SaSa$$

*Hence, S is left quasi-regular. The case when S is right quasi-regular can be similarly proved.*

**Theorem 11.2.** *Let S be a semigroup. If $f_S = (f_S * \widetilde{\theta})^2 \widetilde{\cup} (\widetilde{\theta} * f_S)^2$ for every SU-ideal $f_S$ of S, then S is quasi-regular.*

**Proof.** *Let $f_S$ be any SU-right ideal of S. Thus, we have*

$$f_S = (f_S * \widetilde{\theta})^2 \widetilde{\cup} (\widetilde{\theta} * f_S)^2 \widetilde{\supseteq} (f_S * \widetilde{\theta})^2 \widetilde{\supseteq} f_S * f_S \widetilde{\supseteq} f_S * \widetilde{\theta} \widetilde{\supseteq} f_S$$

*and so $f_S = (f_S)^2$. It follows that S is right quasi-regular by Theorem 11.1. One can similarly show that S is left quasi-regular.*

**Theorem 11.3.** *For a semigroup S, the following conditions are equivalent:*

1) *S is both intra-regular and left quasi-regular.*
2) *$g_S \widetilde{\cup} h_S \widetilde{\cup} f_S = g_S * h_S * f_S$ for every SU-bi-ideal $f_S$, for every SU-left ideal $g_S$ and every SU-right ideal $h_S$ of S.*

**Proof.** *Assume that (1) holds. Let $f_S$ be any SU-bi-ideal, $g_S$ be any SU-left ideal and $h_S$ be any SU-right ideal of S. Let a be any element of S. Since S is intra-regular, there exist elements $x, y \in S$ such that $a = xa^2y$. Since S is left quasi-regular, there exist elements $u, v \in S$ such that $a = uava$. Hence*

$$a = uava = u(xaay)va = ((ux)a)((a(yv)a)$$

*Thus,*

$$\begin{aligned}(g_S * h_S * f_S)(a) &= [g_S * (h_S * f_S)](a) \\ &= \bigcap_{a=((ux)a)((a(yv)a)} [g_S((ux)a)) \cup (h_S * f_S)(a(yv)a))] \\ &\subseteq g_S((ux)a)) \cup (h_S * f_S)(a(yv)a)) \\ &\subseteq g_S(a) \cup (\bigcap_{(a(yv))a=mn)} h_S(m) \cup f_S(n)) \\ &\subseteq g_S(a) \cup (h_S(a(yv)) \cup f_S(a)) \\ &\subseteq g_S(a) \cup h_S(a) \cup f_S(a) \\ &= (g_S \widetilde{\cup} h_S \widetilde{\cup} f_S)(a)\end{aligned}$$

*and so* $g_S * h_S * f_S \subseteq g_S \widetilde{\cup} h_S \widetilde{\cup} f_S$. *Thus, (1) implies (2). Assume that (2) holds. Let* $g_S$ *be any SU-left ideal and* $f_S$ *be any SU-right ideal of S. Then, since SU-left ideal* $g_S$ *is a bi-ideal of S, and since* $\widetilde{\theta}$ *itself is an SU-right ideal of S, we have*

$$g_S = g_S \widetilde{\cup} \widetilde{\theta} \widetilde{\cup} g_S = g_S * \widetilde{\theta} * g_S = g_S * (\widetilde{\theta} * g_S) \widetilde{\supseteq} g_S * g_S \widetilde{\supseteq} \widetilde{\theta} * g_S \widetilde{\supseteq} g_S$$

*Hence* $g_S = g_S * g_S$. *Thus, by Theorem 11.1, S is left quasi-regular.*

*Now, since SU-right ideal* $f_S$ *is an SU-bi-ideal of S, and since* $\widetilde{\theta}$ *itself is an SU-right ideal of S, we have:*

$$g_S \widetilde{\cup} f_S = g_S \widetilde{\cup} \widetilde{\theta} \widetilde{\cup} f_S = g_S * \widetilde{\theta} * f_S = g_S * (\widetilde{\theta} * f_S) \widetilde{\supseteq} g_S * f_S$$

*Thus, by Theorem 8.2., S is intra-regular. Hence (2) implies (1). This completes the proof.*

## 12. Conclusion

*Throughout this paper, we have studied the concepts of soft union product of soft sets, soft characteristic function, soft union semigroup, soft union left (right, two-sided) ideals, soft union bi-ideals and soft union semiprime ideals. Moreover, we have characterized regular, intra-regular, completely regular, weakly regular and quasi-regular semigroups by the properties of these ideals. Based on these results, some further work can be done on the properties of soft* *union semigroups, which may be useful to characterize the classical semigroups in the following studies.*

**Conflict of Interests**

*The authors declare that there is no conflict of interests.*

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