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A FIXED POINT APPROACH TO ORTHOGONAL STABILITY OF AN ADDITIVE - QUADRATIC FUNCTIONAL EQUATION

MUAADH ALMAHALEBI* AND SAMIR KABBAJ

Laboratory LAMA, Harmonic Analysis and functional equations Team, Department of Mathematics - Faculty of Sciences - University of Ibn Tofail , Kenitra, Morocco

Abstract. Using the fixed point method, we prove the Hyers-Ulam stability of the following orthogonally additive-quadratic functional equation

$$f(x+ay) = f(x) + a^2 f(y) - \frac{(a^2-a)}{2}\,(f(x+y) - f(x-y))$$

where $a \in \mathbb{N} - \{0,1\}$, in orthogonality spaces.

Keywords: Hyers-Ulam stability; fixed point; orthogonally additive-quadratic functional equation; orthogonality space.

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1. Introduction and preliminaries

Assume that X is a real inner product space and $f : X \longrightarrow \mathbb{R}$ is a solution of the orthogonal Cauchy functional equation $f(x+y) = f(x) + f(y)$ with $\langle x, y\rangle = 0$. By the Pythagorean theorem, $f(x) = \|x\|^2$ is a solution of the conditional equation. Of course, this function does not satisfy the additivity equation everywhere. Thus orthogonal Cauchy equation is not equivalent to the classic Cauchy equation on the whole inner product space.

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*Corresponding author

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Pinsker [23] characterized orthogonally additive functionals on an inner product space when the orthogonality is the ordinary one in such spaces. Sundaresan [32] generalized this result to arbitrary Banach spaces equipped with the Birkhoff-James orthogonality. The orthogonally Cauchy functional equation $f(x+y) = f(x)+f(y)$, with $x \perp y$, in which $\perp$ is an abstract orthogonality relation, was first investigated by Gudder and Strawther [12]. They defined $\perp$ by a system consisting of five axioms and described the general semi-continuous real-valued solution of conditional Cauchy functional equation. In 1985, Rätz [29] introduced a new definition of orthogonality by using more restrictive axioms than of Gudder and Strawther. Moreover, he investigated the structure of orthogonally additive mappings. Rätz and Szabó [30] investigated the problem in a rather more general framework.

Let us recall the orthogonality in the sense of Rätz [29]. Suppose X is a real vector space with $dim X \geq 2$ and $\perp$ is a binary relation on X with the following properties:

- totality of $\perp$ for zero: $x \perp 0$, $0 \perp x$ for all $x \in X$;
- independence: if $x, y \in X - \{0\}$, $x \perp y$, then x, y are linearly independent;
- homogeneity: if $x, y \in X$, $x \perp y$, then $\alpha x \perp \beta y$ for all $\alpha, \beta \in \mathbb{R}$;
- the Thalesian property: if P is a 2-dimensional subspace of X, $x \in P$ and $\lambda \in \mathbb{R}_+$, which is the set of nonnegative real numbers, then there exists $y_0 \in P$ such that $x \perp y_0$ and $x + y_0 \perp \lambda x - y_0$.

The pair $(X, \perp)$ is called *an orthogonality space*. By an orthogonality normed space, we mean an orthogonality space having a normed structure.

The stability problem of functional equations is that *when the solutions of an equation differing slightly from a given one must be close to an exact solution of the given equation*? In 1941, S. M. Ulam [34] posed the first question on the subject concerning the stability of group homomorphisms. In 1940, Hyers [13] gave a partial affirmative answer to the question of Ulam in the context of Banach spaces. In 1978, Th. M. Rassias [24] extended the theorem of Hyers by considering the unbounded Cauchy difference $\|f(x+y) - f(x) - f(y)\| \leq \varepsilon\left(\|x\|^p + \|y\|^p\right)$, $(\varepsilon > 0, p \in [0, 1))$. The result of Th. M. Rassias has provided a lot of influence in the development of what we now call generalized

Hyers-Ulam stability or Hyers-Ulam stability of functional equations. During the last two decades, several stability problems of functional equations have been investigated in the spirit of Hyers-Ulam-Rassias. The reader is referred to [6,7,14,16,28] and references there in for detailed information on stability of functional equations.

Ger and Sikorskà [11] investigated the orthogonal stability of the Cauchy functional equation $f(x+y) = f(x) + f(y)$, namely, they showed that if f is a mapping from an orthogonality space X into a real Banach space Y and $\|f(x+y) - f(x) - f(y)\| \leq \varepsilon$ for all $x, y \in X$ with $x \perp y$ and some $\varepsilon > 0$, then there exists exactly one orthogonally additive mapping $g : X \longrightarrow Y$ such that $\|f(x) - g(x)\| \leq \frac{16}{3}\varepsilon$ for all $x \in X$.

The first author treating the stability of the quadratic equation was Skof [31] by proving that if f is a mapping from a normed space X into a Banach space Y satisfying $\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \varepsilon$ for some $\varepsilon > 0$, then there is a unique quadratic mapping $g : X \longrightarrow Y$ such that $\|f(x) - g(x)\| \leq \frac{\varepsilon}{2}$. Cholewa [4] extended the Skof's theorem by replacing X by an abelian group G. The Skof's result was later generalized by Czerwik [5] in the spirit of Hyers-Ulam-Rassias. The stability problem of functional equations has been extensively investigated by some mathematicians (see [22,25,26,27]).

The orthogonally quadratic equation $f(x+y) + f(x-y) = 2f(x) + 2f(y)$, $x \perp y$ was first investigated by Vajzović [35] when X is a Hilbert space, Y is the scalar field, f is continuous and $\perp$ means the Hilbert space orthogonality. Later, Drljevic [9], Fochi [10], Moslehian [18,19], Szabo [33], Moslehian and Th. M. Rassias [20] and Paganoni and Rätz [21] have investigated the orthogonal stability of functional equations. We recall a fundamental result in fixed point theory. Let X be a set. A function $d : X \times X \to [0, \infty)$ is called a *generalized metric* on X if d satisfies :

- $d(x, y) = 0$ if and only if $x = y$,
- $d(x, y) = d(y, x)$ for all $x, y \in X$,
- $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 1.1. [8] *Suppose we are given a complete generalized metric space* (X, d) *and a strictly contractive mapping* $J : X \to X$*, with the Lipshitz constant* $L < 1$*. If there exists a nonnegative integer* k *such that*

$$d(J^k x, J^{k+1} x) < \infty$$

for some $x \in X$, *then the following are true:*

(1) the sequence $J^n x$ *converges to a fixed point* x^* *of* J*;*

(2) x^* *is the unique fixed point of* J *in the set* $Y = \{y \in X : d(J^k x, y) < \infty\}$*;*

(3) $d(y, x^*) \leq \frac{1}{1-L} d(y, Jy)$ *for all* $y \in Y$.

In 1996, Isac and Th. M. Rassias [33] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. In this paper, we prove the Hyers-Ulam stability of the following orthogonally additive-quadratic functional equation

$$f(x+ay) = f(x) + a^2 f(y) - \frac{(a^2-a)}{2}\left(f(x+y) - f(x-y)\right) \tag{1}$$

where $a \in \mathbb{N} - \{0, 1\}$, by using fixed point method.

It is easy to show that the function $f(x) = bx^2$, $b \in \mathbb{R}$ satisfies the functional equation (1), which is called *a quadratic functional equation* and every solution of the quadratic functional equation is said to be *a quadratic mapping.*

It is easy to show that the function $f(x) = cx$, $c \in \mathbb{R}$ satisfies the functional equation (1), which is called *an additive functional equation* and every solution of the additive functional equation is said to be *an additive mapping.*

Throughout this paper, assume that $(X, \perp)$ is an orthogonality space and that $(Y, ||.||)$ is a real Banach space.

2. Main results

Throughout this section, we will apply the fixed point method to prove the Hyers-Ulam-Rassias stability of the orthogonally additive-quadratic functional equation (1). For convenience, we use the following abbreviation. For a given function $f : X \longrightarrow Y$, we define

$$Df(x,y) := f(x+ay) - f(x) - a^2 f(y) + \frac{(a^2-a)}{2}\left(f(x+y) - f(x-y)\right) \tag{2}$$

for all $x, y \in X$ with $x \perp y$, where$\perp$ is the orthogonality in the sense of Rätz.

Using the fixed point method and applying some ideas from [11,14,16,28], we prove the Hyers-Ulam stability of the additive-quadratic functional equation $Df(x, y) = 0$ in orthogonality spaces.

Theorem 2.1. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq a^2 L \varphi(\frac{x}{a}, \frac{y}{a}) \tag{3}$$

for all $x, y \in X$ with $x \perp y$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and

$$\|Df(x, y)\| \leq \varphi(x, y) \tag{4}$$

for all $x, y \in X$ with $x \perp y$. Then, there exists a unique orthogonally quadratic mapping $Q : X \longrightarrow Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{1}{a^2 - a^2 L} \varphi(0, x) \tag{5}$$

for all $x \in X$.

Proof. *Let us consider the set $S := \{g : X \to Y\}$ and introduce the generalized metric on S as follows:*

$$d(g, h) = inf \left\{K \in [0, \infty) : \|g(x) - h(x)\| \leq K\varphi(0, x), \forall x \in X\right\}.$$

where, as usual, $inf \emptyset = +\infty$. It is easy to show that (S, d) is complete (see for example [17]*, Lemma* 2.1*). Now, we consider the linear mapping $J : S \to S$ such that*

$$Jg(x) := \frac{1}{a^2} g(ax)$$

for all $g \in S$ and $x \in X$. First we assert that J is strictly contractive on S.

For given $g, h \in S$, let $K \in [0, \infty)$ be an arbitrary constant with $d(g, h) \leq K$, that is $\| g(x) - h(x) \| \leq K\varphi(0, x)$. So we have

$$\| Jg(x) - Jh(x) \| = \frac{1}{a^2} \| g(ax) - h(ax) \| \leq \frac{1}{a^2} K\varphi(0, ax) \leq KL\varphi(0, x)$$

for all $x \in X$*, that is,* $d(Jg, Jh) \leq Ld(g, h)$ *for any* $g, h \in S$*. Letting* $x = 0$ *and* $y = x$ *in* (4)*, we get*

$$\left\| f(x) - \frac{1}{a^2} f(ax) \right\| \leq \frac{1}{a^2} \varphi(0, x)$$

for all $x \in X$*. Hence,*

$$d(f, Jf) \leq \frac{1}{a^2} < \infty$$

By Theorem 1.1, there exists a mapping $Q : X \to Y$ *satisfying the following:*

- *Q is fixed point of J, that is,*

$$Q(ax) = a^2 Q(x) \tag{6}$$

  for all $x \in X$*. The mapping* Q *is a unique fixed point of* J *in the set*

$$M = \{g \in S : d(f, g) \leq \infty\}.$$

  This implies that Q *is a unique mapping such that there exists* $K \in (0, \infty)$ *satisfying*

$$\|f(x) - Q(x)\| \leq K\varphi(0, x)$$

  for all $x \in X$*.*

- $d(J^n, Q) \longrightarrow 0$*,* $n \longrightarrow \infty$*. This implies the equality*

$$\lim_{n \to +\infty} J^n f(x) = \lim_{n \to +\infty} \frac{f(a^n x)}{a^{2n}} = Q(x)$$

  for all $x \in X$*.*

- $$d(f, Q) \leq \frac{1}{1 - L} d(f, Jf),$$

  which implies the inequality

$$d(f, Q) \leq \frac{1}{a^2 - a^2 L}$$

This implies that the inequality (5) *holds. From* (3) *and* (4)*, we get*

$$\| DQ(x, y) \| = \lim_{n \to +\infty} \frac{1}{a^{2n}} \|Df(a^n x, a^n y)\| \leq \lim_{n \to +\infty} \frac{1}{a^{2n}} \varphi(a^n x, a^n y)$$

$$\leq \lim_{n \to +\infty} \frac{a^{2n} L^n}{a^{2n}} \varphi(x, y) = 0$$

for all $x, y \in X$ *with* $x \perp y$*. So,* $DQ(x,y) = 0$ *for all* $x, y \in X$ *with* $x \perp y$*. Hence,* $Q : X \longrightarrow Y$ *is an orthogonally quadratic mapping, as desired.*

Corollary 2.2. *Assume that* $(X, \perp)$ *is an orthogonality normed space. Let* θ *be a positive real number and* p *a real number with* $0 < p < 2$ *and let* $f : X \longrightarrow Y$ *be an even mapping satisfying* $f(0) = 0$ *and*

$$\|Df(x,y)\| \leq \theta \left(\|x\|^p + \|y\|^p\right) \tag{7}$$

for all $x, y \in X$ *with* $x \perp y$*. Then, there exists a unique orthogonally quadratic mapping* $Q : X \longrightarrow Y$ *such that*

$$\|f(x) - Q(x)\| \leq \frac{\theta}{a^2 - a^p}\|x\|^p \tag{8}$$

for all $x \in X$*.*

proof. *We get the result from Theorem 2.1 by taking* $\varphi(x,y) = \theta(||x||^p + ||y||^p)$ *for all* $x, y \in X$ *with* $x \perp y$ *and choosing* $L = a^{p-2}$*.*

Theorem 2.3. *Let* $\varphi : X^2 \to [0, \infty)$ *be a function such that there exists an* $L < 1$ *with*

$$\varphi(x,y) \leq aL\varphi(\frac{x}{a}, \frac{y}{a}) \tag{9}$$

for all $x, y \in X$ *with* $x \perp y$*. Let* $f : X \to Y$ *be an odd mapping satisfying* $f(0) = 0$ *and* (4)*. Then, there exists a unique orthogonally additive mapping* $A : X \longrightarrow Y$ *such that*

$$\|f(x) - A(x)\| \leq \frac{1}{a - aL}\varphi(0,x) \tag{10}$$

for all $x \in X$*.*

proof. *Let us consider the set* $S := \{g : X \to Y\}$ *and introduce the generalized metric on* S *as follows:*

$$d(g,h) = inf\left\{K \in [0,\infty) : \|g(x) - h(x)\| \leq K\varphi(0,x), \forall x \in X\right\}.$$

where, as usual, $inf\emptyset = +\infty$*. It is easy to show that* (S,d) *is complete (see for example* [17]*, Lemma 2.1). Now, we consider the linear mapping* $J : S \to S$ *such that*

$$Jg(x) := \frac{1}{a}g(ax)$$

for all $g \in S$ *and* $x \in X$. *For given* $g, h \in S$ *and* $K \in [0, \infty)$ *such that* $d(g, h) \leq K$, *so we get*

$$\parallel Jg(x) - Jh(x) \parallel = \frac{1}{a} \parallel g(ax) - h(ax) \parallel \leq \frac{1}{a} K\varphi(0, ax) \leq KL\varphi(0, x)$$

for all $x \in X$. *Hence we see that* $(Jg, Jh) \leq Ld(g, h)$ *for all* $g, h \in S$. *So* J *is a strictly contractive operator.*
Letting $x = 0$ *and* $y = x$ *in* (4), *we get*

$$\left\| f(x) - \frac{1}{a} f(ax) \right\| \leq \frac{1}{a} \varphi(0, x)$$

for all $x \in X$. *Hence,*

$$d(f, Jf) \leq \frac{1}{a} < \infty$$

The rest of the proof is similar to the proof of Theorem 2.1.

Corollary 2.4. *Assume that* $(X, \perp)$ *is an orthogonality normed space. Let* θ *be a positive real number and* p *a real number with* $0 < p < 1$ *and let* $f : X \longrightarrow Y$ *be an odd mapping satisfying* (7). *Then, there exists a unique orthogonally additive mapping* $A : X \longrightarrow Y$ *such that*

$$\|f(x) - A(x)\| \leq \frac{\theta}{a - a^p} \|x\|^p \tag{11}$$

for all $x \in X$.

proof. *Taking* $\varphi(x, y) = \theta(||x||^p + ||y||^p)$ *for all* $x, y \in X$ *with* $x \perp y$ *in Theorem* 2.3 *and choosing* $L = a^{p-1}$, *we get the desired result.*

Theorem 2.5. *Let* $\varphi : X^2 \to [0, \infty)$ *be a function such that there exists an* $L < 1$ *with*

$$\varphi(x, y) \leq aL\varphi(\frac{x}{a}, \frac{y}{a}) \tag{12}$$

for all $x, y \in X$ *with* $x \perp y$. *Let* $f : X \to Y$ *be a mapping satisfying* $f(0) = 0$ *and* (4). *Then, there exists a unique orthogonally additive mapping* $A : X \longrightarrow Y$ *and a unique quadratic mapping* $Q : X \longrightarrow Y$ *such that*

$$\|f(x) - A(x) - Q(x)\| \leq \frac{(a + 1 - 2L)}{2a(1 - L)(a - L)} \left(\varphi(0, x) + \varphi(0, -x)\right) \tag{13}$$

for all $x \in X$.

proof. *We decompose f into the odd part and the even part by putting $f_o(x) = \frac{f(x)-f(-x)}{2}$ and $f_e(x) = \frac{f(x)+f(-x)}{2}$ for all $x \in X$. It is clear that $f(x) = f_o(x) + f_e(x)$ for all $x \in X$. It follows from (4) that*

$$\|Df_o(x, y)\| \leq \frac{1}{2}\left(\varphi(x, y) + \varphi(-x, -y)\right),$$

$$\|Df_e(x, y)\| \leq \frac{1}{2}\left(\varphi(x, y) + \varphi(-x, -y)\right) \tag{14}$$

for all $x, y \in X$. By Theorem 2.3, there exists a unique orthogonally additive mapping $A : X \longrightarrow Y$ such that

$$\|f_o(x) - A(x)\| \leq \frac{1}{2a(1-L)}\left(\varphi(0, x) + \varphi(0, -x)\right) \tag{15}$$

for all $x \in X$.

Putting $L' = \frac{L}{a} < 1$ in (12), we get that

$$\varphi(x, y) \leq a^2 L' \varphi(\frac{x}{a}, \frac{y}{a}) \tag{16}$$

for all $x \in X$. So, by Theorem 2.1, there exists a unique orthogonally quadratic mapping $Q : X \longrightarrow Y$ such that

$$\|f_e(x) - Q(x)\| \leq \frac{1}{2a(a-L)}\left(\varphi(0, x) + \varphi(0, -x)\right) \tag{17}$$

for all $x \in X$.

By (15) and (17), we get

$$\|f(x) - A(x) - Q(x)\| = \|f_o(x) + f_e(x) - A(x) - Q(x)\|$$

$$\leq \|f_o(x) - A(x)\| + \|f_e(x) - Q(x)\| \leq \left(\frac{1}{2a(1-L)} + \frac{1}{2a(a-L)}\right)\left(\varphi(0, -x) + \varphi(0, x)\right)$$

$$= \frac{(a + 1 - 2L)}{2a(1-L)(a-L)}\left(\varphi(0, x) + \varphi(0, -x)\right)$$

for all $x \in X$.

Corollary 2.6. *Assume that $(X, \perp)$ is an orthogonality normed space. Let θ be a positive real number and p a real number with $0 < p < 1$ and let $f : X \longrightarrow Y$ be a mapping* *satisfying* $f(0) = 0$ *and* (7). *Then, there exists an orthogonally additive mapping* $A : X \longrightarrow Y$ *and an orthogonally quadratic mapping* $Q : X \longrightarrow Y$ *such that*

$$\|f(x) - A(x) - Q(x)\| \leq \frac{\theta(a+1-2a^{p-1})}{a^2(1-a^{p-1})(1-a^{p-2})}\|x\|^p \tag{18}$$

for all $x \in X$.

proof. *The proof follows from Theorem* 2.5 *by taking* $\varphi(x,y) = \theta(||x||^p + ||y||^p)$ *for all* $x, y \in X$ *with* $x \perp y$ *and by choosing* $L = a^{p-1}$, *we get the desired result.*

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