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A NOTE ON A FIXED POINT THEOREM

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Abstract. In this paper we gave a simple extension of a fixed point theorem for non-expansive maps. We derives several known results e.g. [3,4,6] as corollaries.

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1. Introduction

The fixed point theory has applications in nonlinear functional analysis, approximation theory, optimization problems, variational inequalities, game theory and mathematical economics. Fixed point theory is quite useful in the existence theory of differential, integral, partial differential and functional equations. This is a basic mathematical tool used in showing the existence of solution in game theory and mathematical economics.

2. Preliminaries

Throughout this paper, we assume that X is a Banach space and H as a Hilbert space.

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Following definitions will be useful in our further discussion in this paper.

DEFINITION 2.1 *Let X be a Banach space and C a subset of X. If $f : X \to X$ is a map such that $\| fx - fy \| \leq k \| x - y \|$ for all $x, y \in X$ and k is a constant with $0 < k < 1$. Then f is said to be a contraction map.*

If $\| fx - fy \| \leq \| x - y \|$ for all $x, y \in X$, then f is said to be a non-expansive map.

However, if $\| fx - fy \| < \| x - y \|$ for all $x, y \in X, x \neq y$, then it is a contractive map.

F is said to have a fixed point $z \in X$ if $fz = z$.

A translation map is a non-expansive map and is fixed point free.

DEFINITION 2.2 *A set C in X is said to be convex if $rx + (1 - r)y \in C$ for all $x, y \in C$ and $0 < r < 1$. In case for all $x \in C, rx + (1 - r)p \in C$, then C is said to be a starshaped set and p is called the star center of C.*

A convex set is starshaped but not conversely.

Now, we will state our main results.

3. Main results

Theorem 3.1. *Let C be a closed starshaped subset of a Banach space X and $f : C \to X$ a non-expansive map with $f(C)$ bounded and $f(\partial C) \subseteq C$ and f satisfies the following : (∂C stands for the boundary of C). There exists a compact set D of X such that for every $x \in C$, the sequence of iterates $\{f^n x\}$ contains a point of D. ———- —— —— (A). Then f has a fixed point.*

The result stated below is, given independently by Browder [3], Kirk [6] and Gohde [4] and is very useful in many applications.

In case C is a closed bounded convex subset of a Hilbert space H and $f : C \to C$ is a non-expansive map, then f has a fixed point.

Proof.

For the sake of convenience we assume that 0 is in C and it is the star center of C. Define $f_{r_i}(x) = r_i f(x)$ for all $x \in C$, where $\{r_i\}$ is a sequence of reals with $0 < r_i < 1$ and $r_i \to 1$ as $i \to \infty$.

Now, $\| f_{r_i}(x) - f_{r_i}(y) \| = r_i \| f(x) - f(y) \| \leq r_i \| x - y \|$ for all $x, y \in C$.

Then each $f_{r_i}(x)$ is a contraction map and by a Theorem of Assad [2] has a unique fixed point. Let $f_{r_i}(x_{r_i}) = x_{r_i}$ for each r_i.

$\| x_{r_i} - f(x_{r_i}) \| = \| f_{r_i}(x_{r_i}) - f(x_{r_i}) \| = \| f(x_{r_i}) \| (r_i - 1) \to 0$ as $i \to \infty$.

On the other hand by Condition (A), the sequence $\{x_{r_i}\}$ has a convergent subsequence, say$\{x_k\}$ converging to $y \in D$. Since f is continuous so $\{f(x_k)\}$ converges to fy. It is not difficult to see that $y = fy$ since $\| x_{r_i} - f(x_{r_i}) \| \to 0$ as $i \to \infty$.

Hence f has a fixed point.

Now we derive the following as corollaries:

Corollary 3.2. *If C is a compact starshaped subset of a Banach space X and $f : C \to X$ is a non-expansive map such that $f(\partial C) \subseteq C$, then f has a fixed point.*

A continuous image of a compact set is compact so Condition (A) is satisfied.

Corollary 3.3. *If C is a closed starshaped subset of a Banach space $X, f : C \to X$ is non-expansive such that $f(C)$ is compact, then f has a fixed point.*

Corollary 3.4. *If C is a closed convex subset of a Banach space X and $f : C \to X$ is a non-expansive map with $f(C)$ compact, then f has a fixed point.*

This is a well known theorem called the Schauder fixed point theorem.

If C is a closed convex subset of a Hilbert space H and $f : C \to H$ is a non-expansive map with $f(C)$ bounded and $f(\partial C) \subseteq C$, then f has a fixed point [7].

The well known fixed point theorem of Browder [3] follows from the above theorem.

In case C is not compact but $f(C)$ is compact, then the following result is interesting.

Theorem 3.5. *Let C be a closed starshaped subset of a Banach space X and $f : C \to X$, a non-expansive map with $f(\partial C) \subseteq C$, and $f(C)$ compact, then f has a fixed point.*

Proof. The proof of this theorem is on the same lines as that of Theorem 3.1. Since $f(C)$ is compact so each sequence has a convergent subsequence and f is continuous, therefore the we will get result.

The following results are derived as corollaries.

Corollary 3.6. *If C is a closed convex subset of X and $f : C \to X$ a non-expansive map with $f(\partial C) \subseteq C$, and $f(C)$ compact, then f has a fixed point.*

A convex set is star shaped.

Corollary 3.7. *If C is closed star shaped subset of a Banach space X and $f : C \to X$ is a non-expansive map with $f(C)$ compact, then f has a fixed point.*

Similar results for non-expansive maps for a ball $B = \{x \in H / \parallel x \parallel \leq r\}, B$ is a ball of radius r and center 0, in Hilbert space H are as follows.(see references [1,5,and 7].

Corollary 3.8. *If B is a ball in a Hilbert space H and $f : B \to H$ a non-expansive map satisfying one of the following boundary conditions, then f has a fixed point.*

Boundary condition: For $x \in \partial B$, where ∂B is the boundary of B.

(i) $\parallel fx \parallel \leq \parallel x \parallel$,

(ii) $< x, fx > \leq \parallel x \parallel^2$

(iii) $\parallel fx \parallel \leq \parallel x - fx \parallel$

(iv) If $fx = kx$, then $k \leq 1$

(v) $\parallel fx \parallel^2 \leq \parallel x \parallel^2 + \parallel x - fx \parallel^2$.

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