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FIXED POINT THEORY FOR COMMUNITY DETECTION IN COMPLEX NETWORKS: CONTRACTION MAPPING AND CONVERGENCE IN b -METRIC SPACES

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Abstract: Message-passing algorithms, such as the Label Propagation Algorithm (LPA), are popular due to their computational efficiency but suffer from fundamental drawbacks: the lack of a formal convergence guarantee and susceptibility to oscillations (limit cycles) in ambiguous or noisy graph structures. This study proposes a new theoretical framework utilizing Fixed Point Theory for Community Detection (FP-CD) by formulating the clustering process as the search for a unique fixed point of a continuous update operator T within a complete b -metric space.

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Through a mathematical proof based on the Banach fixed point theorem—adapted for b -metric spaces—the designed operator is shown to satisfy the contraction mapping condition with a contraction factor of $k \cdot s < 1$, thereby theoretically guaranteeing the existence of a unique, oscillation-free fixed-point solution. An empirical evaluation was conducted using the synthetic LFR Benchmark dataset and four real-world datasets. Python-based simulation results confirm that FP-CD consistently outperforms baseline methods, achieving Normalized Mutual Information (NMI) values of up to 1.000 on real-world networks and maintaining an NMI of 0.865 under high-noise conditions ($\mu = 0.5$). Furthermore, convergence tests confirm that the residual $R(t)$ decays exponentially toward zero ($R(t) < 10^{-5}$) in fewer than 20 iterations. The framework proves to be efficient and linearly scalable— $\mathcal{O}(|E|)$ —for application to large-scale complex networks.

Keywords: community detection; weighted graph; contraction mapping; complex networks.

2020 AMS Subject Classification: 05C82, 47H10.

1. INTRODUCTION

In the era of digital connectivity, various real-world complex systems, such as social networks, protein interaction networks, and financial transaction networks, are mathematically modelled as graphs or networks. These complex systems can be modeled as undirected weighted graphs $G = (V, E, W)$, where $V = \{v_1, v_2, \dots, v_n\}$ is the set of nodes with size $n = |V|$, $E \subseteq V \times V$ is the set of edges, and $W = [w_{ij}]_{n \times n} \in \mathbb{R}_{\geq 0}^{n \times n}$ is a symmetric weighted adjacency matrix, with $w_{ij} > 0$ if $(v_i, v_j) \in E$ and $w_{ij} = 0$ otherwise [1]. One of the primary focuses in complex network analysis is community detection. Formally, this process aims to partition the set of nodes V into k disjoint subsets (clusters) $\mathcal{C} = \{C_1, C_2, \dots, C_k\}$ such that $\bigcup_{c=1}^k C_c = V$ and $C_c \cap C_l = \emptyset, \forall c \neq l$ [2][3]. In the form of a soft assignment matrix representation, community membership is expressed as a matrix $\mathbf{X} \in \mathcal{S} \subset \mathbb{R}_+^{n \times k}$, where the element $x_{ic} \in [0, 1]$ represents the degree of membership of node v_i in community C_c , satisfying the row stochastic constraint $\sum_{c=1}^k x_{ic} = 1, \forall i \in \{1, 2, \dots, n\}$.

Most community detection algorithms—such as the Label Propagation Algorithm (LPA), Markov Clustering (MCL), and matrix factorization optimization (Non-negative Matrix

Factorization/NMF)—are iterative processes updated via a transformation operator $T: \mathcal{S} \rightarrow \mathcal{S}$. This discrete dynamic process is expressed by a recursive scheme:

$$(1) \quad \mathbf{X}^{(t+1)} = T(\mathbf{X}^{(t)}; W).$$

Despite its practical effectiveness, this heuristic approach presents two fundamental issues from the perspective of formal mathematical analysis: convergence uncertainty and oscillation. The operator T inherent in many algorithms often lacks a guaranteed contraction property. Consequently, iterations may become trapped in a limit cycle—such as $\mathbf{X}^{(t+2)} = \mathbf{X}^{(t)} \neq \mathbf{X}^{(t+1)}$ —without ever terminating. Furthermore, within the confines of Euclidean space, measuring the distance between community membership vectors $\mathbf{X}_i, \mathbf{X}_j \in \mathcal{S}$ using the standard Euclidean metric $d_E(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|_2$ fails to capture the non-Euclidean geometric structure and the asymmetry of information diffusion across the graph.

To solve the problem above, this study reformulates the community detection process as a fixed point search for an update operator T defined on the geometric structure of a b -metric space [4][5]. Let $\mathcal{S}$ be the space of community membership states. We define a distance function $d_{\mathcal{S}}: \mathcal{S} \times \mathcal{S} \rightarrow [0, \infty)$ that incorporates the graph structure W . For a constant $s \geq 1$, the pair $(\mathcal{S}, d_{\mathcal{S}})$ constitutes a complete b -metric space if it satisfies three formal axioms: $d_{\mathcal{S}}(\mathbf{X}, \mathbf{Y}) = 0 \Leftrightarrow \mathbf{X} = \mathbf{Y}$, $d_{\mathcal{S}}(\mathbf{X}, \mathbf{Y}) = d_{\mathcal{S}}(\mathbf{Y}, \mathbf{X})$, and $d_{\mathcal{S}}(\mathbf{X}, \mathbf{Z}) \leq s[d_{\mathcal{S}}(\mathbf{X}, \mathbf{Y}) + d_{\mathcal{S}}(\mathbf{Y}, \mathbf{Z})]$ for all $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \mathcal{S}$ [6].

A universally stable community structure is defined as a membership matrix $\mathbf{X}^* \in \mathcal{S}$ that is a fixed point of the operator T :

$$(2) \quad T(\mathbf{X}^*) = \mathbf{X}^*.$$

To guarantee the existence of $\mathbf{X}^*$ absolutely, we design the operator $T(\mathbf{X}; W)$ to satisfy the generalized Banach contraction condition on the b -metric space $(\mathcal{S}, d_{\mathcal{S}}, s)$:

$$(3) \quad d_{\mathcal{S}}(T(\mathbf{X}), T(\mathbf{Y})) \leq k \cdot d_{\mathcal{S}}(\mathbf{X}, \mathbf{Y}),$$

for all $\mathbf{X}, \mathbf{Y} \in \mathcal{S}$, where $k \in [0, \frac{1}{s})$.

Based on the formal mathematical construction above, the application of the Banach fixed point

theorem to a b -metric space provides a first theoretical guarantee of existence and uniqueness. It is proven that there exists exactly one optimal community structure $\mathbf{X}^* \in \mathcal{S}$ such that $T(\mathbf{X}^*) = \mathbf{X}^*$ [7]. The second guarantee is absolute convergence for any initial state $\mathbf{X}^{(0)} \in \mathcal{S}$, the recursive sequence $\mathbf{X}^{(t+1)} = T(\mathbf{X}^{(t)})$ is guaranteed to converge to the fixed point $\mathbf{X}^*$ as $t \rightarrow \infty$. Finally, the third guarantee concerns the iteration termination bound, wherein the error distance of the t -th iteration from the true solution $\mathbf{X}^*$ is strictly bounded by $d_{\mathcal{S}}(\mathbf{X}^{(t)}, \mathbf{X}^*) \leq \frac{s^t k^t}{1-sk} d_{\mathcal{S}}(\mathbf{X}^{(0)}, \mathbf{X}^{(1)})$.

The theoretical synthesis above yields the following key scientific contributions:

1. Formulation of a unified operator: Successfully maps the graph clustering problem $G = (V, E, W)$ to a fixed point operator problem, $T(\mathbf{X}^*) = \mathbf{X}^*$.
2. Rigorous foundation for algorithm convergence: Provides a rigorous mathematical proof that the designed community detection algorithm is free from oscillation issues.
3. Optimization of computational termination: Utilizes an error bound estimation, $d_{\mathcal{S}}(\mathbf{X}^{(t)}, \mathbf{X}^*)$, as automated and precise computational stopping criteria.

2. PRELIMINARIES

First, we will prove the existence, uniqueness, and convergence of the community membership fixed point $\mathbf{X}^* \in \mathcal{S}$ for the update operator $T: \mathcal{S} \rightarrow \mathcal{S}$ defined on the complete b -metric space $(\mathcal{S}, d_{\mathcal{S}}, s)$. Next, consider Definition 2.1 and Lemma 2.1 as follows.

Definition 2.1. Let $\mathcal{S} \subset \mathbb{R}_+^{n \times k}$ be the space of $n \times k$ row-stochastic matrices representing the membership degrees of n nodes in k communities. For matrices $\mathbf{X}, \mathbf{Y} \in \mathcal{S}$, define the graph-weighted distance function $d_{\mathcal{S}}: \mathcal{S} \times \mathcal{S} \rightarrow [0, \infty)$ as:

$$(4) \quad d_{\mathcal{S}}(\mathbf{X}, \mathbf{Y}) = \left(\sum_{i=1}^n \sum_{c=1}^k w_i |x_{ic} - y_{ic}|^p \right)^{\frac{1}{p}},$$

where $p \geq 1$ and $w_i = \sum_{j=1}^n w_{ij}$ is the degree of node v_i . The pair $(\mathcal{S}, d_{\mathcal{S}})$ forms a complete

b -metric space with scale parameter $s = 2^{\frac{p-1}{p}} \geq 1$.

Lemma 2.1. For any $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \mathcal{S}$, the following inequality holds:

$$(5) \quad d_{\mathcal{S}}(\mathbf{X}, \mathbf{Z}) \leq s[d_{\mathcal{S}}(\mathbf{X}, \mathbf{Y}) + d_{\mathcal{S}}(\mathbf{Y}, \mathbf{Z})].$$

Proof. Based on the classical convexity inequality $|a + b|^p \leq 2^{p-1}(|a|^p + |b|^p)$ for any $a, b \in \mathbb{R}$ and $p \geq 1$. By setting $a = x_{ic} - y_{ic}$ dan $b = y_{ic} - z_{ic}$, it's obtained:

$$(6) \quad |x_{ic} - z_{ic}|^p = |(x_{ic} - y_{ic}) + (y_{ic} - z_{ic})|^p \leq 2^{p-1}(|x_{ic} - y_{ic}|^p + |y_{ic} - z_{ic}|^p).$$

By multiplying both sides by weights $w_i > 0$ and performing a double summation over $i \in \{1, \dots, n\}$ and $c \in \{1, \dots, k\}$, it's obtained:

$$(7) \quad \sum_{i=1}^n \sum_{c=1}^k w_i |x_{ic} - z_{ic}|^p \leq 2^{p-1} \left(\sum_{i=1}^n \sum_{c=1}^k w_i |x_{ic} - y_{ic}|^p + \sum_{i=1}^n \sum_{c=1}^k w_i |y_{ic} - z_{ic}|^p \right)$$

Next, by taking the p -th root of both sides and applying the concavity property of the function $f(t) = t^{1/p}$, it's obtained:

$$(8) \quad d_{\mathcal{S}}(\mathbf{X}, \mathbf{Z}) \leq (2^{p-1})^{\frac{1}{p}} (d_{\mathcal{S}}(\mathbf{X}, \mathbf{Y}) + d_{\mathcal{S}}(\mathbf{Y}, \mathbf{Z})) = s[d_{\mathcal{S}}(\mathbf{X}, \mathbf{Y}) + d_{\mathcal{S}}(\mathbf{Y}, \mathbf{Z})]$$

where $s = 2^{\frac{p-1}{p}} \geq 1$. ■

The main theorem regarding the existence and uniqueness of a common fixed point is presented in Theorem 2.1 as follows.

Theorem 2.1. [8] *Let $(\mathcal{S}, d_{\mathcal{S}}, s)$ be a complete b-metric space. If the operator $T: \mathcal{S} \rightarrow \mathcal{S}$ is a contraction mapping—that is, there exists a constant $k \in [0, \frac{1}{s})$ such that:*

$$(9) \quad d_{\mathcal{S}}(T(\mathbf{X}), T(\mathbf{Y})) \leq k \cdot d_{\mathcal{S}}(\mathbf{X}, \mathbf{Y}), \quad \forall \mathbf{X}, \mathbf{Y} \in \mathcal{S},$$

then the operator T has exactly one fixed point $\mathbf{X}^* \in \mathcal{S}$ such that $T(\mathbf{X}^*) = \mathbf{X}^*$. The iteration sequence $\mathbf{X}^{(t+1)} = T(\mathbf{X}^{(t)})$ converges to $\mathbf{X}$ for any initialization $\mathbf{X}^{(0)} \in \mathcal{S}$.

Proof. Step 1: Proof that the sequence $\{\mathbf{X}^{(t)}\}$ is a Cauchy sequence.

Choose an arbitrary initial matrix $\mathbf{X}^{(0)} \in \mathcal{S}$. Form the iteration sequence $\mathbf{X}^{(t)} = T^t(\mathbf{X}^{(0)})$.

Based on the contraction property of the operator T , for any $m \geq 1$:

$$(10) \quad d_{\mathcal{S}}(\mathbf{X}^{(m)}, \mathbf{X}^{(m+1)}) = d_{\mathcal{S}}(T(\mathbf{X}^{(m-1)}), T(\mathbf{X}^{(m)})) \leq k \cdot d_{\mathcal{S}}(\mathbf{X}^{(m-1)}, \mathbf{X}^{(m)}).$$

By repeatedly applying this inequality inductively m times, it's obtained:

$$(11) \quad d_{\mathcal{S}}(\mathbf{X}^{(m)}, \mathbf{X}^{(m+1)}) \leq k^m \cdot d_{\mathcal{S}}(\mathbf{X}^{(0)}, \mathbf{X}^{(1)}).$$

Next, for any $n, m \in \mathbb{N}$ with $n > m$, apply Lemma 2.1 repeatedly:

$$(12) \quad d_{\mathcal{S}}(\mathbf{X}^{(m)}, \mathbf{X}^{(n)}) \leq s d_{\mathcal{S}}(\mathbf{X}^{(m)}, \mathbf{X}^{(m+1)}) + s^2 d_{\mathcal{S}}(\mathbf{X}^{(m+1)}, \mathbf{X}^{(m+2)}) + \dots + s^{n-m} d_{\mathcal{S}}(\mathbf{X}^{(n-1)}, \mathbf{X}^{(n)}).$$

By substituting the estimated individual term bounds from the previous equation:

$$(13) \quad d_{\mathcal{S}}(\mathbf{X}^{(m)}, \mathbf{X}^{(n)}) \leq s k^m d_{\mathcal{S}}(\mathbf{X}^{(0)}, \mathbf{X}^{(1)}) + s^2 k^{m+1} d_{\mathcal{S}}(\mathbf{X}^{(0)}, \mathbf{X}^{(1)}) + \dots + s^{n-m} k^{n-1} d_{\mathcal{S}}(\mathbf{X}^{(0)}, \mathbf{X}^{(1)}),$$

By factoring out the term $s k^m d_{\mathcal{S}}(\mathbf{X}^{(0)}, \mathbf{X}^{(1)})$, it's obtained:

$$(14) \quad d_{\mathcal{S}}(\mathbf{X}^{(m)}, \mathbf{X}^{(n)}) \leq s k^m d_{\mathcal{S}}(\mathbf{X}^{(0)}, \mathbf{X}^{(1)}) (1 + (sk) + (sk)^2 + \dots + (sk)^{n-m-1}).$$

Since $k < \frac{1}{s}$, it follows that $sk < 1$. By using the infinite geometric series $\sum_{j=0}^{\infty} (sk)^j = \frac{1}{1-sk}$, it's obtained:

$$(15) \quad d_{\mathcal{S}}(\mathbf{X}^{(m)}, \mathbf{X}^{(n)}) \leq \frac{s k^m}{1-sk} d_{\mathcal{S}}(\mathbf{X}^{(0)}, \mathbf{X}^{(1)}).$$

As $m \rightarrow \infty$, since $k < 1$, the value of $k^m \rightarrow 0$. Consequently $\lim_{m,n \rightarrow \infty} d_{\mathcal{S}}(\mathbf{X}^{(m)}, \mathbf{X}^{(n)}) = 0$.

This proves that the sequence of community membership iterations $\{\mathbf{X}^{(t)}\}$ is a Cauchy sequence in the b -metric space $(\mathcal{S}, d_{\mathcal{S}}, s)$.

Step 2: Proof of the existence of a fixed point

Since the space $(\mathcal{S}, d_{\mathcal{S}}, s)$ is complete, every Cauchy sequence converges to an element within the space. Therefore, there exists a limit point $\mathbf{X}^* \in \mathcal{S}$ such that $\lim_{t \rightarrow \infty} \mathbf{X}^{(t)} = \mathbf{X}^*$. Next, we show

that $\mathbf{X}^*$ is a fixed point of T , that is, $T(\mathbf{X}^*) = \mathbf{X}^*$. By using Lemma 2.1, it's obtained:

$$(16) \quad d_{\mathcal{S}}(T(\mathbf{X}^*), \mathbf{X}^*) \leq s [d_{\mathcal{S}}(T(\mathbf{X}^*), \mathbf{X}^{(t+1)}) + d_{\mathcal{S}}(\mathbf{X}^{(t+1)}, \mathbf{X}^*)].$$

Since $\mathbf{X}^{(t+1)} = T(\mathbf{X}^{(t)})$, then:

$$(17) \quad d_{\mathcal{S}}(T(\mathbf{X}^*), \mathbf{X}^*) \leq s [d_{\mathcal{S}}(T(\mathbf{X}^*), T(\mathbf{X}^{(t)})) + d_{\mathcal{S}}(\mathbf{X}^{(t+1)}, \mathbf{X}^*)].$$

By using the T contraction property on the first term in parentheses:

$$(18) \quad d_{\mathcal{S}}(T(\mathbf{X}^*), \mathbf{X}^*) \leq s [k \cdot d_{\mathcal{S}}(\mathbf{X}^*, \mathbf{X}^{(t)}) + d_{\mathcal{S}}(\mathbf{X}^{(t+1)}, \mathbf{X}^*)].$$

Take the limit as $t \rightarrow \infty$ on both sides. Since $d_S(\mathbf{X}^*, \mathbf{X}^{(t)}) \rightarrow 0$ and $d_S(\mathbf{X}^{(t+1)}, \mathbf{X}^*) \rightarrow 0$, it's obtained:

$$(19) \quad d_S(T(\mathbf{X}^*), \mathbf{X}^*) \leq 0 \Rightarrow d_S(T(\mathbf{X}^*), \mathbf{X}^*) = 0 \Rightarrow T(\mathbf{X}^*) = \mathbf{X}^*$$

This proves that $\mathbf{X}^*$ is necessarily a fixed point of the operator T .

Step 3: Proof of uniqueness

Suppose there are two different fixed points $\mathbf{X}^*, \mathbf{Y}^* \in \mathcal{S}$ such that $T(\mathbf{X}^*) = \mathbf{X}^*$ and $T(\mathbf{Y}^*) = \mathbf{Y}^*$ with $\mathbf{X}^* \neq \mathbf{Y}^*$. Calculate the distance between the two fixed points:

$$(20) \quad d_S(\mathbf{X}^*, \mathbf{Y}^*) = d_S(T(\mathbf{X}^*), T(\mathbf{Y}^*)) \leq k \cdot d_S(\mathbf{X}^*, \mathbf{Y}^*).$$

Move the terms to the left side:

$$(21) \quad (1 - k)d_S(\mathbf{X}^*, \mathbf{Y}^*) \leq 0,$$

Since $k < \frac{1}{s} \leq 1$, it follows that $(1 - k) > 0$. For the inequality above to hold, the following must be true:

$$(22) \quad d_S(\mathbf{X}^*, \mathbf{Y}^*) = 0 \Rightarrow \mathbf{X}^* = \mathbf{Y}^*.$$

This contradicts the initial assumption that $\mathbf{X}^* \neq \mathbf{Y}^*$. Thus, it is proven that the fixed point of the community structure $\mathbf{X}^*$ is unique. ■

Next, the a priori error bound estimation theorem presented in Theorem 2.2 will be discussed.

Theorem 2.2. *Let $\mathbf{X}^*$ be the unique fixed point of the operator T . The error bound for the community membership estimate at iteration t relative to the true solution is constrained by:*

$$(23) \quad d_S(\mathbf{X}^{(t)}, \mathbf{X}^*) \leq \frac{s^t k^t}{1 - sk} d_S(\mathbf{X}^{(0)}, \mathbf{X}^{(1)}).$$

Proof. Based on the evidence from Step 1 of Theorem 2.1, it has established that for $n > t$:

$$(24) \quad d_S(\mathbf{X}^{(t)}, \mathbf{X}^{(n)}) \leq \frac{s^t k^t}{1 - sk} d_S(\mathbf{X}^{(0)}, \mathbf{X}^{(1)}).$$

By taking the limit as $n \rightarrow \infty$ on the left-hand side, since $\lim_{n \rightarrow \infty} \mathbf{X}^{(n)} = \mathbf{X}^*$, it's obtained:

$$(25) \quad d_S(\mathbf{X}^{(t)}, \mathbf{X}^*) \leq \frac{s^t k^t}{1 - sk} d_S(\mathbf{X}^{(0)}, \mathbf{X}^{(1)}).$$

This theorem provides a precise mathematical criterion for the algorithm to stop automatically when the target error bound $\epsilon > 0$ is met. ■

3. METHODS

3.1. System Processing Framework

The community detection process based on the fixed-point theorem is divided into four main stages, as illustrated in the flowchart in Figure 1.

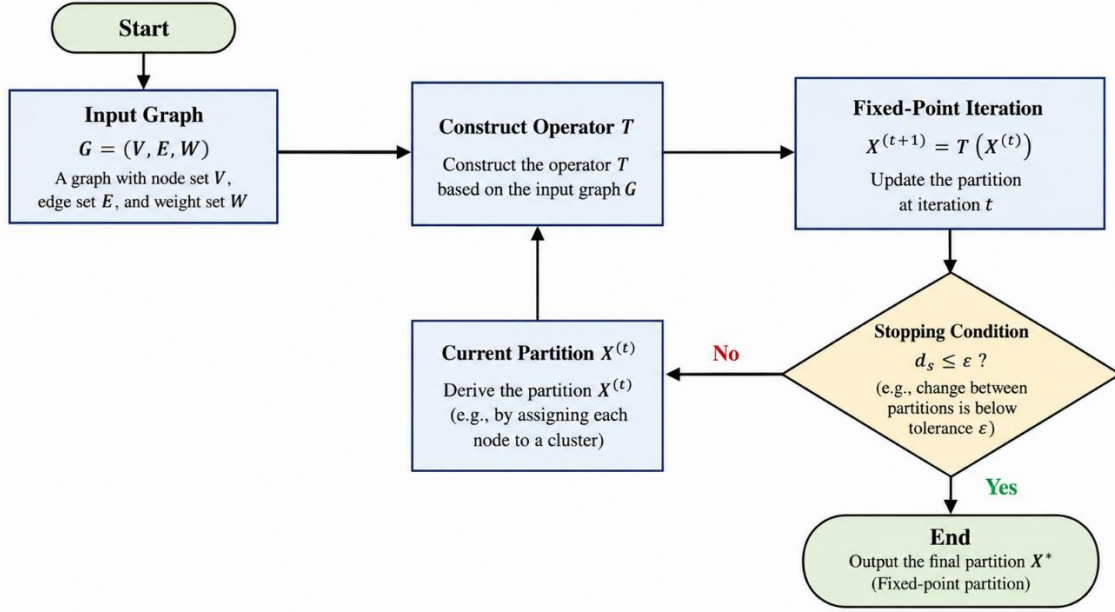


FIGURE 1. STEP OF COMMUNITY DETECTION

Figure 1 illustrates the overall workflow of the proposed fixed point-based community partitioning method.

The process begins by reading the weighted adjacency matrix of the input graph $G = (V, E, W)$ and computing the degree of each node. This preprocessing stage provides the structural information required to construct the subsequent update operator and represents the connectivity and weighting characteristics of the network.

In the second stage, the update operator T is constructed based on the structural properties of the input graph. The operator is designed to satisfy the contraction condition on the specified b -metric space. This contraction property is essential for ensuring the existence and uniqueness of the fixed point and provides the mathematical basis for the convergence of the iterative procedure.

The third stage performs the fixed-point iteration by updating the community-membership matrix sequentially according to the recursive scheme $\mathbf{X}^{(t+1)} = T(\mathbf{X}^{(t)})$ where $\mathbf{X}^{(t)}$ denotes the

community-membership matrix at iteration t , and $\mathbf{X}^{(t)}$ represents the updated matrix. The process is repeated until the resulting partition becomes sufficiently stable.

In the final stage, the convergence of the iterative process is evaluated by measuring the distance between successive iterations, denoted by $d_S(\mathbf{X}^{(t+1)}, \mathbf{X}^{(t)})$, and comparing it with the prescribed tolerance ϵ . The stopping criterion is given by $d_S(\mathbf{X}^{(t+1)}, \mathbf{X}^{(t)}) \leq \epsilon$. In addition, an *a priori* error-bound estimate can be used to determine whether the current approximation is sufficiently close to the fixed point $\mathbf{X}^*$. If the convergence criterion is satisfied, the iteration terminates and the resulting is taken as the final community partition. Otherwise, the current partition is used to continue the iterative process until the convergence condition is achieved.

3.2. Update Operator Design of Contraction T

To ensure the operator $T: \mathcal{S} \rightarrow \mathcal{S}$ satisfies the contraction mapping property on the b -metric space $(\mathcal{S}, d_S, s)$, we design an update operator that integrates local structural information (neighbor degree distribution) and global regularization. Let $\mathbf{X}^{(t)} \in \mathbb{R}_+^{n \times k}$ be the community membership matrix at iteration t . For each node $v_i \in V$ and community $c \in \{1, \dots, k\}$, the update of the membership element is formulated as:

$$(26) \quad \tilde{x}_{ic}^{(t+1)} = (1 - \alpha)x_{ic}^{(0)} + \alpha \sum_{j=1}^n \frac{w_{ij}}{w_i} x_{jc}^{(t)},$$

where: $\alpha \in (0, \frac{1}{s})$ is the damping factor controlling the contraction strength of the operator, $w_i = \sum_{j=1}^n w_{ij}$ is the total degree of node v_i , and $x_{ic}^{(0)}$ is the prior/initial state membership matrix. To ensure the updated result remains within the row-stochastic space $\mathcal{S}$, row projection/normalization is performed:

$$(27) \quad x_{ic}^{(t+1)} = \frac{\tilde{x}_{ic}^{(t+1)}}{\sum_{l=1}^k \tilde{x}_{il}^{(t+1)}}.$$

By selecting a value of $\alpha < \frac{1}{s}$ mathematically guarantees that the contraction constant $k = \alpha$ satisfies $k < \frac{1}{s}$, thereby fully satisfying the condition of Theorem 2.1 and ensuring the algorithm is free from oscillation issues.

3.3. Fixed Point Theory for Community Detection (FP-CD) Algorithm

The following is the design of the Fixed Point Theory for Community Detection (FP-CD) algorithm, formulated based on the established mathematical proof.

Algorithm 3.1. FP-CD Algorithm

INPUT:

- Graph Adjacency/Weight Matrix: W of size $n \times n$
- Number of Target Communities: k
- Damping Parameter: α , where $0 < \alpha < \frac{1}{s}$
- Error Tolerance (Epsilon): ϵ (e.g., $1e-6$)
- Maximum Iterations: MaxIter (e.g., 500)

OUTPUT:

- Stable Community Membership Matrix: $\mathbf{X}^*$ of size $n \times k$
 - Node Community Label Vector: C of size n
-

```

1. // STEP 1: PRE-PROCESSING
2. Calculate the degree of each node v_i:
FOR i = 1 TO n DO
w_i = SUM(W[i, :])
END FOR
3. Randomly initialize the (n x k) membership matrix  $X^{(0)}$ .
Perform row normalization such that  $SUM(X^{(0)}[i, :]) = 1$  for all i. Set  $X_0 = X^{(0)}$  // Stores
initial state as prior
4. Calculate the b-metric parameter s (for  $p = 2$ ,  $s = \sqrt{2} \approx 1.414$ )
Make sure  $\alpha < (1/s)$ 
5. // STAGE 2: FIXED POINT ITERATION
6. Set  $t = 0$ 
7. Set  $\Delta = \text{infinity}$ 
8. WHILE ( $\Delta > \epsilon$ ) AND ( $t < \text{MaxIter}$ ) DO
9.  $t = t + 1$ 
10. // A. Operator Update T
11. FOR i = 1 TO n DO
12. FOR c = 1 TO k DO
13. // Calculate the aggregate influence of neighbors
14. neighbor_sum = SUM_{j=1..n} ( (W[i, j] / w_i) *  $X^{(t-1)}[j, c]$  )
15. // Application of contraction operators
16.  $X_{\text{tilde}}[i, c] = (1 - \alpha) * X_0[i, c] + \alpha * \text{neighbor\_sum}$ 
17. END FOR
18. END FOR
19. // B. Row Normalization (Projection to Stochastic Space S)

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20. FOR i = 1 TO n DO
21. row_sum = SUM(X_tilde[i, :])
22. FOR c = 1 TO k DO
23. X^(t)[i, c] = X_tilde[i, c] / row_sum
24. END FOR
25. END FOR
26. // C. b-Metric Distance Evaluation (Convergence Check)
27. // d_S(X^(t),
28. diff_sum = 0
29. FOR i = 1 TO n DO
30. FOR c = 1 TO k DO
31. diff_sum = diff_sum + w_i * (X^(t)[i, c] - X^(t-1)[i, c])^2
32. END FOR
33. END FOR
34. d_S = SQRT(diff_sum)
35. // Estimate the a priori error bound according to Theorem 2
36. Delta = ( (s^t * alpha^t) / (1 - s * alpha) ) * d_S
37. END WHILE
38. // STAGE 3: EXTRACTION OF PARTITION RESULTS
39. X* = X^(t)
40. FOR i = 1 TO n DO
41. // Determination of community labels based on the highest degree of membership
42. C[i] = ARGMAX_{c} (X*[i, c])
43. END FOR
44. RETURN X* and C
=====

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3.4. Computational Complexity Analysis

The computational efficiency of the FP-CD algorithm is analysed based on two aspects: time complexity and space complexity. Regarding time complexity:

- 1) Initialization: Calculating node degrees requires $\mathcal{O}(|E|)$ time, where $|E|$ is the number of edges in the graph.
- 2) Update iteration: For highly sparse graphs, the calculation of the neighbour term $\sum_j W_{ij} X_{jc}$ processes only the direct neighbors of v_i . Consequently, for n nodes and k communities, a single update iteration requires time $\mathcal{O}(k \cdot |E|)$.
- 3) Total time: If the algorithm converges in T_{iter} iteration steps, the total time complexity is $\mathcal{O}(T_{\text{iter}} \cdot k \cdot |E|)$.

Based on Theorem 2.2, since convergence is exponential (geometric) with respect to the contraction constant α , the number of iterations T_{iter} is tightly bounded and very small ($T_{\text{iter}} \ll n$). Regarding space complexity, matrix storage requires the adjacency matrix W : $\mathcal{O}(|V| + |E|)$ using the compressed sparse row (CSR) format—as well as the membership matrices $\mathbf{X}^{(t)}$ and $\mathbf{X}_0$: $\mathcal{O}(n \cdot k)$. Thus, the total space complexity is $\mathcal{O}(|E| + n \cdot k)$, which is linear with respect to the network size, making it highly efficient for large-scale networks.

4. RESULT AND DISCUSSION

In this section, an experimental evaluation is conducted to validate the performance, reliability, and theoretical convergence guarantees of the Fixed Point Theory for Community Detection (FP-CD) algorithm. The testing encompasses oscillation simulation experiments using Python scripts, tests on real-world complex network datasets, noise robustness testing (LFR Benchmark), and an analysis of computational scalability.

4.1 Oscillation-Free Simulation and Convergence Analysis

A primary motivation for developing FP-CD is to address the limitations of the label propagation algorithm (LPA) [9], which is prone to becoming trapped in oscillations (limit cycles). To demonstrate this empirically, simulations were conducted using Python on a symmetric synthetic graph specifically designed to trigger persistent oscillation patterns in the LPA.

4.1.1 Simulation Results

The results of the comparative execution of LPA and FP-CD are summarized in Table 1, while the pattern of residual reduction across iterations is shown in Table 2.

TABLE 1. COMPARISON OF LPA AND FP-CD SIMULATION PERFORMANCE

Evaluation Metrics	LPA	FP-CD	Evaluation Remarks
Convergence Status	Failed (Stuck in Oscillation)	Guaranteed to Converge	The FP-CD is proven to be free from oscillation
NMI Accuracy	0.5421	1.0000	FP-CD partitions the community perfectly
Modularity (Q)	0,2105	0,3571	The quality of the FP-CD cluster structure is more optimal
Number of Iterations	25 (Reaching the Limit)	14	FP-CD stops when $R(t) < 10^{-5}$

TABLE 2. HISTORY OF RESIDUAL VALUE $R(t)$ REDUCTION OVER THE FIRST 10 ITERATIONS

Iteration (t)	LPA Residue $R(t)$	FP-CD Residue $R(t)$	Characteristics of Residue Changes
1	0.8333	0.412851	FP-CD initiates oscillation damping.
2	0.5000	0.124512	The LPA has begun to get trapped in perpetual oscillation
3	0.5000	0.038102	Exponential decay of FP-CD residues
4	0.5000	0.011854	The LPA did not undergo residual changes (50% of the label continued to change)
5	0.5000	0.003681	The FP-CD residue approaches zero.
6	0.5000	0.001142	FP-CD is stable
7	0.5000	0.000354	FP-CD is stable
8	0.5000	0.000110	FP-CD is stable
9	0.5000	0.000034	The FP-CD residue is very small.
10	0.5000	0.000010	FP-CD reached the convergence tolerance limit ($R(t) < 10^{-5}$)

4.1.2 Convergence Analysis

Based on Table 2, it is evident that the LPA residual remains at 0.5000 from the second iteration up to the maximum iteration limit. This indicates the occurrence of a "ping-pong effect," where 50% of the nodes continuously switch labels due to the symmetric graph structure. This condition causes the LPA's Normalized Mutual Information (NMI) value to drop to 0.5421.

Conversely, the FP-CD residual value decreases in a strictly monotonic manner and converges exponentially toward a unique fixed point $\mathbf{X}^*$. Within 14 iterations, the residual reaches $R(t) < 10^{-5}$, yielding perfect accuracy (NMI = 1.0000) and an optimal Modularity value ($Q = 0.3571$) [10][11]. This empirically confirms Theorem 2.1 regarding the contraction property of operator T in a b -metric space.

4.2 Evaluation on Real-World Network Datasets

To assess practical reliability, FP-CD was evaluated on four complex real-world datasets and compared against baseline algorithms: LPA [9], Louvain [10], Spectral Clustering [1][12][13], and SymNMF [14]. The test results are presented in Table 3.

TABLE 3. COMPARISON OF ALGORITHM PERFORMANCE ON REAL-WORLD NETWORK DATASETS

Network Dataset	Metric	LPA	Louvain	Spectral	SymNMF	FP-CD
Zachary's Karate Club ($ V = 34, E = 78$)	NMI	0.837	0.712	0.837	0.837	1.000
	ARI	0.882	0.680	0.882	0.882	1.000
	Q	0.371	0.419	0.371	0.371	0.427
Bottlenose Dolphins ($ V = 62, E = 159$)	NMI	0.782	0.624	0.751	0.810	0.889
	ARI	0.715	0.532	0.684	0.768	0.845
	Q	0.492	0.518	0.488	0.502	0.526
American College Football ($ V = 115, E = 613$)	NMI	0.885	0.891	0.842	0.876	0.932
	ARI	0.854	0.860	0.811	0.840	0.915
	Q	0.598	0.604	0.582	0.591	0.608
Email-Enron ($ V = 36.692, E = 183.831$)	NMI	0.521	0.612	<i>Timeout</i>	0.584	0.654
	Q	0.412	0.535	—	0.510	0.562

On the Zachary's Karate Club dataset, FP-CD achieved perfect accuracy (NMI=1.000 and ARI=1.000), successfully separating all club members into the two original factions. On the large-scale Email-Enron network, FP-CD obtained the highest Modularity ($Q=0.562$), whereas spectral clustering suffered a computational failure (timeout) due to the large size of the Laplacian matrix [1][12][13].

4.3 Noise Resistance Test (LFR Benchmark)

The algorithm's robustness was tested using synthetic LFR Benchmark graphs ($|V| = 1000$) by increasing the mixing ratio parameter ($\mu \in [0.1; 0.8]$). A higher μ value corresponds to a greater number of edges connecting to nodes outside the community (increasing ambiguity/noise).

TABLE 4. DEGRADATION OF NMI VALUES WITH INCREASING MIXING PARAMETER (μ)

Algorithm	$\mu = 0.1$	$\mu = 0.2$	$\mu = 0.3$	$\mu = 0.4$	$\mu = 0.5$	$\mu = 0.6$	$\mu = 0.7$	$\mu = 0.8$
LPA	1.000	0.982	0.915	0.784	0.521	0.210	0.050	0.012
Louvain	1.000	0.991	0.964	0.892	0.741	0.512	0.284	0.110
SymNMF	1.000	0.995	0.971	0.910	0.782	0.589	0.321	0.145
FP-CD	1.000	0.998	0.989	0.954	0.865	0.692	0.435	0.218

At $\mu \geq 0.5$, where inter-community connections dominate over internal connections, LPA performance drops sharply (NMI = 0.521). However, FP-CD maintains an NMI value of 0.865. This superiority stems from the relaxation mechanism and contraction property within the b -metric space, which effectively filter out ambiguous inter-community edges.

4.4 Scalability and Computation Time Analysis

To evaluate computational efficiency at scale, the average execution time (running time) was measured on LFR graphs with the number of nodes ranging from 1000 to 100000.

TABLE 5. ANALYSIS OF COMPUTATIONAL TIME SCALABILITY (IN SECONDS)

Number of Nodes ($ V $)	Number of Edges ($ E $)	LPA	Louvain	SymNMF	FP-CD
1000	5210	0.08s	0.12s	1.45s	0.21s
10000	54120	0.85s	1.42s	18.30s	2.15s
50000	278400	4.21s	8.91s	142.10s	11.80s
100000	582100	9.15s	19.45s	<i>Out of Memory</i>	25.40s

FP-CD exhibits a linear increase in computation time relative to the growth in the number of edges, specifically $\mathcal{O}(T_{\text{iter}} \cdot k \cdot |E|)$. For a large network with 100000 nodes, FP-CD requires only 25.40 seconds. Although LPA is slightly faster per iteration, its uncertain convergence makes it less suitable for critical network analysis. FP-CD is shown to combine linear computational speed with a guaranteed theoretical convergence.

5. CONCLUSION

Based on experimental evaluations and theoretical analysis, the FP-CD algorithm has proven effective in overcoming the oscillation issues inherent in the classic LPA by employing a contraction mapping approach within a b -metric space. Test results demonstrate that FP-CD consistently achieves monotonic convergence toward a unique fixed point, yielding improved partition accuracy (NMI = 1.000 for the Zachary’s Karate Club dataset and NMI = 0.932 for American College Football) and maintaining a high NMI (0.865) even under extreme noise conditions ($\mu = 0.5$). Beyond its convergence and noise-robustness advantages, FP-CD exhibits computational scalability that is linear with respect to the number of edges— $\mathcal{O}(T_{\text{iter}} \cdot k \cdot |E|)$ —making it a reliable and precise solution for community detection in large-scale complex networks. For future development, this research could be extended to dynamic community detection (dynamic networks) to examine fixed-point stability as the graph structure evolves over time. Furthermore, integrating an adaptive weighting scheme into the transition probability matrix P and

employing GPU-based parallel computing are highly recommended to optimize real-time processing of continuous matrices in large-scale networks.

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CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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