Available online at http://scik.org
Adv. Fixed Point Theory, 2026, 16:37
https://doi.org/10.28919/afpt/10260
ISSN: 1927-6303

ACCELERATED FIXED POINT APPROXIMATION FOR SUZUKI-TYPE MAPPINGS USING HK-ITERATION: CONVERGENCE THEORY AND POLYNOMIOGRAPHIC DYNAMICS

ROSALIO G. ARTES JR.[1], HANIF ULLAH[2], MUHAMMAD TARIQ[3], WAQAR AFZAL[4,5,6], HIJAZ AHMAD[7,8,9,*], MUSTAFA BAYRAM[10]

[1]Department of Mathematics, College of Mathematical Sciences, Mindanao State University – Tawi-Tawi College of Technology and Oceanography, 7500 Bongao, Tawi-Tawi, Philippines

[2]Department of Mathematics, University of Lakki Marwat, Lakki Marwat, Khyber Pakhtunkhwa, Pakistan

[3]Mathematics Research Center, Near East University, Near East Boulevard, PC: 99138, Nicosia /Mersin 10 – Turkey

[4]Abdus Salam School of Mathematical Sciences, Government College University, 68-B, New Muslim Town, Lahore 54600, Pakistan

[5]Center for Theoretical Physics, Khazar University, 41 Mehseti Str., Baku, AZ1096, Azerbaijan

[6]International Center for Interdisciplinary Research in Sciences, The University of Lahore, Lahore 54792, Pakistan

[7]Irfan Suat Gunsel Operational Research Institute, Near East University, 99138, Nicosia / TRNC, Mersin 10, Turkey

[8]Department of Mathematics, College of Science, Korea University, 145 Anam-ro, Seongbuk-gu, Seoul 02841, South Korea

[9]VIZJA University, Okopowa 59, 01-043 Warsaw, Poland

[10]Department of Computer Engineering, Biruni University, Istanbul, Turkey

Copyright © 2026 the author(s). This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

*Corresponding author

E-mail address: hijazahmad@korea.ac.kr

Received August 30, 2026

Abstract. This study investigates the tracking and convergence characteristics of Suzuki-type generalized nonexpansive operators utilizing a multistage iterative process. We establish both weak and strong convergence theorems for the sequence $\{j_v\}_{v=1}^{\infty}$ generated by this multi-tiered architecture, investigating its foundational approximation properties within modern fixed-point theory. By using the auxiliary intermediate iterates t_v and u_v of the mapping $\mathscr{M}$, the suggested framework incorporates deeper operator features into the scheme. The algorithm's behavior and efficiency are confirmed by a thorough numerical example, and its stability is confirmed and its unique dynamical patterns and graphical structures are revealed by comparing it with multiple classical iterations using Kalantari's polynomiography.

Keywords: fixed point theory; Suzuki-type mappings; HK-iteration; uniformly convex Banach space; weak and strong convergence; polynomiography; numerical approximation

2020 AMS Subject Classification: 47H10; 47H09; 47J25; 65H05; 37F10

1. INTRODUCTION

In many scientific and engineering domains, fixed point theory is a crucial analytical tool, particularly for nonlinear problems where traditional approaches are frequently unworkable or unproductive. It allows researchers to find optimal solutions methodically with less computational effort and is widely used in biological population dynamics [1], economic market equilibrium [2], game-theoretic stable strategies [3], chemical equilibrium and structural stability in engineering systems [4], and algorithm refinement in artificial intelligence [5].

Since such applications are intricate, conventional methods of analysis are either too expensive or unable to provide precise closed-form answers. Fixed point theory fills this gap by demonstrating the existence of solutions and providing useful algorithms to approximate them. Banach's Contraction Principle (BCP) [6], which asserts that a contraction on a closed subset of a Banach space has a unique fixed point that the classical Picard iteration can compute, is its cornerstone. As a consequence, the result confirms the presence of solutions in numerous applied issues as well as their practical estimation.

To formally detail this contractive architecture, let $\mathscr{C}$ be a nonempty subset embedded within a Banach space $\mathscr{V}$. A self-mapping $\mathscr{M} : \mathscr{C} \to \mathscr{C}$ is classified as a contraction mapping if there exists a scalar constant $\zeta \in [0,1)$ such that for every pair of elements $j_1, j_2 \in \mathscr{C}$, the metric inequality is satisfied:

$$\|\mathscr{M} j_1 - \mathscr{M} j_2\| \leq \zeta \|j_1 - j_2\|. \tag{1}$$

In the boundary case where $\zeta = 1$, the operator is categorized as nonexpansive. Furthermore, a coordinate $q \in \mathscr{C}$ is designated as a fixed point of $\mathscr{M}$ provided that the identity $\mathscr{M}(q) = q$ holds. Throughout this investigation, the notation $Fix(\mathscr{M})$ explicitly represents the foundational set containing all fixed points of the operator $\mathscr{M}$. The mapping $\mathscr{M}$ is said to behave as a quasi-nonexpansive operator if the relation:

$$\|\mathscr{M} j_v - q\| \leq \|j_v - q\| \tag{2}$$

holds true for all elements $j_v \in \mathscr{C}$ and each invariant target $q \in Fix(\mathscr{M})$.

Many variants of contraction maps have been suggested across decades, such as the nonexpansive maps autonomously proposed by Browder [7], Gohde [8], and Kirk [9], whose fixed points are typically guaranteed only under structural assumptions like closedness, boundedness, and uniform convexity of the domain [10]. Later, Suzuki [11] improved this class by adding condition (C), which defines what are now known as Suzuki generalized nonexpansive mappings. An operator $\mathscr{M} : \mathscr{C} \to \mathscr{C}$ satisfies condition (C) if, for any arbitrary elements $j_1, j_2 \in \mathscr{C}$, the following structural implication is valid:

$$\frac{1}{2}\|j_2 - \mathscr{M} j_2\| \leq \|j_2 - j_1\| \;\Rightarrow\; \|\mathscr{M} j_2 - \mathscr{M} j_1\| \leq \|j_2 - j_1\|. \tag{3}$$

Suzuki showed that while this mapping framework is less restricted than normal nonexpansive operations, it produces a strictly wider domain than quasi-nonexpansive mappings. In particular, the mathematical converse is not always true, even if condition (C) is intrinsically satisfied by all classical nonexpansive operators. The following concrete example sheds light on this distinction.

Example 1.1. *Construct a mapping $\mathscr{M} : [0,3] \to [0,3]$ defined explicitly by*

$$\mathscr{M}(j_v) = \begin{cases} 1, & \text{if } j_v = 3, \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

Under these parameters, $\mathscr{M}$ satisfies the structural criteria of condition (C) but fails to qualify as a standard nonexpansive mapping.

Iterative approximation schemes. Numerous iterative approaches have been devised to determine the fixed points of extremely nonlinear maps, ranging from the Noor process and its later hybrid variations to the classical Picard, Mann, and Ishikawa iterations. Under appropriate contractive or geometric assumptions on the underlying operator, each of these produces a sequence that converges to a fixed point.

Driven by the demand for accelerated convergence trajectories and enhanced computational output, this investigation implements the recently introduced HK-iteration framework [12], formatted explicitly as:

$$\begin{cases} u_v = (1-\eta_v) j_v + \eta_v \mathscr{M} j_v, \\ t_v = (1-\psi_v) \mathscr{M} j_v + \psi_v \mathscr{M}^2 u_v, \\ j_{v+1} = \mathscr{M} t_v, \end{cases} \tag{5}$$

where the sequence parameters satisfy $\{\psi_v\}, \{\eta_v\} \in (0,1)$. Since the Picard sequence frequently is unable to converge for non-contractive mappings in complete metric spaces and the Banach Contraction Principle is no longer applicable, finding fixed points of broad classes of nonlinear operators is difficult. This has led to numerous iterative schemes for approximating them. In order to capture more of the operator's geometry, unify more classical processes as special instances, and converge more quickly than its predecessors, the HK-iteration adds a multi-stage correction through the auxiliary iterates u_v and t_v as well as a higher-order composition step. These methods have undergone exhaustive analysis in mathematical literature, prominently within the research records of Mann [13], Ishikawa [14], Noor [15], Abbas and Nazir [16], Sahu et al. [17], Thakur et al. [18], and Eke and Akewe [19].

Let $\{\rho_v\}$, $\{\psi_v\}$, and $\{\eta_v\}$ represent real sequence parameters restricted strictly to the open interval $(0,1)$, where $v \in \mathbb{N}$ denotes the primary sequence variable. The iterative architecture formulated by Noor [15] stands as a milestone, representing the first recognized three-step iterative framework. This algorithm computes the trajectory sequence $\{j_v\}$ via the following relation:

$$\begin{aligned} j_1 &\in \mathscr{C}, \\ j_{v+1} &= (1-\rho_v) j_v + \rho_v \mathscr{M} t_v, \end{aligned} \tag{6}$$

$$t_v = (1-\psi_v)j_v + \psi_v \mathscr{M} u_v,$$
$$u_v = (1-\eta_v)j_v + \eta_v \mathscr{M} j_v.$$

Seeking to optimize these tracking dynamics, Abbas and Nazir introduced an accelerated multi-step system, known globally as the Abbas iteration process [16], which maps the sequence elements $\{j_v\}$ according to:

$$\begin{aligned} j_1 &\in \mathscr{C}, \\ j_{v+1} &= (1-\rho_v)\mathscr{M} t_v + \rho_v \mathscr{M} u_v, \\ t_v &= (1-\psi_v)\mathscr{M} j_v + \psi_v \mathscr{M} u_v, \\ u_v &= (1-\eta_v) j_v + \eta_v \mathscr{M} j_v. \end{aligned} \tag{7}$$

Thakur et al. [18] proposed an alternative scheme for computing fixed points of nonexpansive mappings, given by:

$$\begin{aligned} j_1 &\in \mathscr{C}, \\ j_{v+1} &= (1-\rho_v)\mathscr{M} j_v + \rho_v \mathscr{M} t_v, \\ t_v &= (1-\psi_v) u_v + \psi_v \mathscr{M} u_v, \\ u_v &= (1-\eta_v) j_v + \eta_v \mathscr{M} j_v. \end{aligned} \tag{8}$$

Concurrently, Sahu et al. [17] introduced an alternative three-tiered iterative pipeline engineered for nonexpansive operator paths, generating the sequence elements $\{j_v\}$ via:

$$\begin{aligned} j_1 &\in \mathscr{C}, \\ j_{v+1} &= (1-\rho_v)\mathscr{M} u_v + \rho_v \mathscr{M} t_v, \\ t_v &= (1-\psi_v) u_v + \psi_v \mathscr{M} u_v, \\ u_v &= (1-\eta_v) j_v + \eta_v \mathscr{M} j_v. \end{aligned} \tag{9}$$

Further extending these frameworks, Eke and Akewe constructed a composite four-step process designated as the Picard–Noor iteration, which maps the sequence $\{j_v\}$ as follows [19]:

$$
\begin{aligned}
j_1 &\in \mathscr{C}, \\
j_{v+1} &= \mathscr{M} z_v, \\
z_v &= (1-\rho_v) j_v + \rho_v \mathscr{M} t_v, \\
t_v &= (1-\psi_v) j_v + \psi_v \mathscr{M} u_v, \\
u_v &= (1-\eta_v) j_v + \eta_v \mathscr{M} j_v.
\end{aligned} \tag{10}
$$

A recent modification by Chyne and Kumar [20] introduced the compound Picard–Abbas iterative technique, verifying strong and weak convergence metrics under contractive operator constraints. This Picard–Abbas system is modeled by:

$$
\begin{aligned}
j_1 &\in \mathscr{C}, \\
j_{v+1} &= \mathscr{M} z_v, \\
z_v &= (1-\rho_v) \mathscr{M} t_v + \rho_v \mathscr{M} u_v, \\
t_v &= (1-\psi_v) \mathscr{M} j_v + \psi_v \mathscr{M} u_v, \\
u_v &= (1-\eta_v) j_v + \eta_v \mathscr{M} j_v.
\end{aligned} \tag{11}
$$

As highlighted previously, the primary framework under consideration is the HK-iteration process [12], structured explicitly as:

$$
\begin{cases}
u_v = (1-\eta_v) j_v + \eta_v \mathscr{M} j_v, \\
t_v = (1-\psi_v) \mathscr{M} j_v + \psi_v \mathscr{M}^2 u_v, \\
j_{v+1} = \mathscr{M} t_v,
\end{cases} \tag{12}
$$

where the iterative arrays satisfy $\{\psi_v\}, \{\eta_v\} \in (0,1)$.

Suzuki-type generalized nonexpansive mappings have garnered significant interest in functional analysis in the past few years, and their nontrivial convergence behavior makes them ideal for studying iterative schemes. They have also led to numerous advances in fixed-point theory (see [21–25]). Recently, the HK-iteration has also been investigated for additional classes of mappings and similar hybrid schemes (see [26–30]). The convergence of Suzuki generalized nonexpansive mappings is analyzed in this work using the HK-iteration [12]. Its multi-stage

structure, based on the intermediate iterates t_v and u_v, unifies the Noor, Abbas, Thakur, Sahu, and Picard-hybrid schemes as special cases under appropriate choices.

In addition to generalizing previous findings, we compare it with the Noor, Abbas, Thakur, Sahu, and Picard–Noor iterations [12], provide a new numerical example to bolster our findings, and visualize root-finding behavior across the complex plane using Kalantari's polynomiography [34,35], which makes the scheme's stability and dynamical characteristics clear. The structure of the paper is as follows: The preliminary and important lemmas are presented in Section 2, the weak and strong convergence theorems for the HK-iteration are proven in Section 3 [12], a numerical example is shown in Section 4, the polynomiographic analysis is contained in Section 5, and the study is concluded in Section 6.

2. PRELIMINARIES

The following foundational mathematical concepts and analytical lemmas are key to proving our core convergence results.

Definition 2.1 ([10]). *A Banach space $\mathscr{V}$ is classified as a uniformly convex Banach space (UCBS) if, for each choice of $\delta \in (0,2]$, there exists a corresponding parameter $\varsigma > 0$ such that the geometric structural system:*

$$\left.\begin{aligned} \|q_1\| &\leq 1, \\ \|q_2\| &\leq 1, \\ \|q_1 - q_2\| &> \delta, \end{aligned}\right\} \Rightarrow \left\| \frac{q_1+q_2}{2} \right\| \leq 1-\varsigma, \quad \textit{for all } q_1, q_2 \in \mathscr{V}. \tag{13}$$

Definition 2.2. *Let $\mathscr{C}$ be a convex, closed subset embedded within a Banach space $\mathscr{V}$, and let $\{j_v\}$ be a bounded trajectory sequence in $\mathscr{V}$. For any candidate element $j \in \mathscr{C}$, the asymptotic radius of the sequence $\{j_v\}$ relative to $\mathscr{C}$ is formalized via:*

$$r(\mathscr{C}, \{j_v\}) = \inf \left\{ \limsup_{v\to\infty} \|j_v - j\| : j \in \mathscr{C} \right\},$$

while the corresponding asymptotic center of $\{j_v\}$ relative to $\mathscr{C}$ is characterized as the set:

$$A(\mathscr{C}, \{j_v\}) = \left\{ j \in \mathscr{C} : \limsup_{v\to\infty} \|j_v - j\| = r(\mathscr{C}, \{j_v\}) \right\}.$$

Definition 2.3 ([31]). *A Banach space $\mathscr{V}$ is recognized to possess Opial's property if, for every functional sequence $\{j_v\}$ inside $\mathscr{V}$ tracking weakly toward an invariant limit element $q \in \mathscr{V}$ (signified by $j_v \rightharpoonup q$), the tracking inequality:*

$$\liminf_{v\to\infty} \|j_v - q\| < \liminf_{v\to\infty} \|j_v - d\|$$

remains valid for any choice of $d \in \mathscr{V}$ provided that $d \neq q$.

Proposition 2.1 ([11]). *Let $\mathscr{C}$ represent a nonempty subset within a Banach space $\mathscr{V}$, and let $\mathscr{M} : \mathscr{C} \to \mathscr{C}$ serve as a self-mapping operator:*

(a) If the mapping $\mathscr{M}$ is nonexpansive, then $\mathscr{M}$ inherently satisfies condition (C).

(b) Any self-mapping operator that meets the criteria of condition (C) and possesses an invariant state is quasi-nonexpansive.

(c) If the mapping $\mathscr{M}$ fulfills condition (C), then the metric bound holds:

$$\|q - \mathscr{M} j_1\| \leq 3\|\mathscr{M} q - q\| + \|q - j_1\|, \quad \forall\, q, j_1 \in \mathscr{C}.$$

Lemma 2.1 ([11]). *Let $\mathscr{C}$ define a nonempty subset of a Banach space $\mathscr{V}$ endowed with Opial's property. Let $\mathscr{M} : \mathscr{C} \to \mathscr{C}$ describe a mapping fulfilling condition (C). If the sequence trajectory $\{j_v\}$ converges weakly to an invariant coordinate q and satisfies $\lim_{v\to\infty} \|\mathscr{M} j_v - j_v\| = 0$, then q is a fixed point, meaning $\mathscr{M} q = q$.*

The framework of condition (I), originally established by Senter and Dotson [32], serves as a key alternative tool to deduce strong convergence metrics for numerical processes operating within non-compact tracking domains.

Definition 2.4. *Let $\mathscr{C}$ symbolize a nonempty subset of a Banach space $\mathscr{V}$, and let $\mathscr{M} : \mathscr{C} \to \mathscr{C}$ be a self-mapping operator. The operator $\mathscr{M}$ matches condition (I) if there exists a strictly non-decreasing continuous function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ and $g(\xi) > 0$ for all $\xi > 0$, such that the relation holds:*

$$\|j_1 - \mathscr{M} j_1\| \geq g\big(d(j_1, Fix(\mathscr{M}))\big), \quad \text{for all } j_1 \in \mathscr{C},$$

where the metric separation value is evaluated as:

$$d(j_1, Fix(\mathscr{M})) = \inf\{\|j_1 - q\| : q \in Fix(\mathscr{M})\}.$$

Lemma 2.2 ([11]). *Let $\mathscr{C}$ constitute a weakly compact, convex subset of a uniformly convex Banach space $\mathscr{V}$, and let $\mathscr{M}$ serve as a self-mapping on $\mathscr{C}$. Assuming that $\mathscr{M}$ satisfies the structural criteria of condition (C), then the operator set $Fix(\mathscr{M})$ is non-empty.*

Lemma 2.3 ([33]). *Suppose $\mathscr{V}$ denotes a uniformly convex Banach space and, for each index $v \geq 1$, the control sequences satisfy the restriction $0 < a \leq \lambda_v \leq b < 1$. Let $\{q_1\}$ and $\{q_2\}$ represent two sequence tracks inside $\mathscr{V}$ establishing $\limsup_{v\to\infty} \|q_1\| \leq \delta$, $\limsup_{v\to\infty} \|q_2\| \leq \delta$, along with the joint tracking boundary:*

$$\lim_{v\to\infty} \|(1-\lambda_v)q_1 + \lambda_v q_2\| = \delta$$

for some real scalar value $\delta \geq 0$. Under these conditions, it follows that:

$$\lim_{v\to\infty} \|q_1 - q_2\| = 0.$$

3. Main Results

This section establishes the core convergence criteria for self-mappings that satisfy condition (C), utilizing the multi-tiered HK-iteration framework [12].

Lemma 3.1. *Let $\mathscr{C} \neq \emptyset$ be a closed and convex subset embedded within a uniformly convex Banach space $\mathscr{V}$. Suppose that $\mathscr{M} : \mathscr{C} \to \mathscr{C}$ is an operator satisfying condition (C) such that its set of invariant points is non-empty, i.e., $Fix(\mathscr{M}) \neq \emptyset$. Let $\{j_v\}$ be the sequence trajectory generated via the iterative framework* (12)*. Then, for any invariant state $q \in Fix(\mathscr{M})$, the sequence satisfies*

$$\lim_{v\to\infty} \|j_v - q\| \text{ exists.}$$

Proof. Let $q \in Fix(\mathscr{M})$ and choose an arbitrary vector coordinate $j \in \mathscr{C}$. Since the operator $\mathscr{M}$ satisfies condition (C), invoking Proposition 2.1(b) guarantees that $\mathscr{M}$ is quasi-nonexpansive, yielding the structural tracking inequality:

$$\|\mathscr{M} j - q\| \leq \|j - q\|, \quad \forall\, j \in \mathscr{C},\, q \in Fix(\mathscr{M}).$$

Recalling the formalized HK-iteration tracking system (12), we evaluate the trajectory relations:

$$\begin{aligned} u_v &= (1-\eta_v) j_v + \eta_v \mathscr{M} j_v, \\ t_v &= (1-\psi_v) \mathscr{M} j_v + \psi_v \mathscr{M}^2 u_v, \\ j_{v+1} &= \mathscr{M} t_v. \end{aligned} \tag{14}$$

We systematically formulate the analytical estimates as follows.

First, evaluating the deepest auxiliary step u_v with respect to the fixed target coordinate q:

$$\begin{aligned} \|u_v - q\| &= \|(1-\eta_v)(j_v - q) + \eta_v(\mathscr{M} j_v - q)\| \\ &\leq (1-\eta_v)\|j_v - q\| + \eta_v \|\mathscr{M} j_v - q\| \\ &\leq (1-\eta_v)\|j_v - q\| + \eta_v \|j_v - q\| \\ &= \|j_v - q\|. \end{aligned} \tag{15}$$

Next, tracking the intermediate auxiliary path t_v relative to q and substituting our previous estimation (15):

$$\begin{aligned} \|t_v - q\| &= \|(1-\psi_v)(\mathscr{M} j_v - q) + \psi_v(\mathscr{M}^2 u_v - q)\| \\ &\leq (1-\psi_v)\|\mathscr{M} j_v - q\| + \psi_v \|\mathscr{M}^2 u_v - q\| \\ &\leq (1-\psi_v)\|j_v - q\| + \psi_v \|u_v - q\| \\ &\leq (1-\psi_v)\|j_v - q\| + \psi_v \|j_v - q\| \\ &= \|j_v - q\|. \end{aligned} \tag{16}$$

Finally, analyzing the terminal step of the sequence generation j_{v+1}:

$$\|j_{v+1} - q\| = \|\mathscr{M} t_v - q\| \leq \|t_v - q\|.$$

Consequently, applying the mapping bound established in (16), we obtain:

$$\|j_{v+1} - q\| \leq \|j_v - q\|. \tag{17}$$

This confirms that the real non-negative distance sequence $\{\|j_v - q\|\}$ is monotonically non-increasing and bounded from below by 0. It follows immediately that:

$\lim_{v\to\infty} \|j_v - q\|$ exists for every distinct point $q \in Fix(\mathscr{M})$. □

Next, we discuss the necessary and sufficient conditions for the existence of an invariant fixed point under self-mappings satisfying condition (C).

Theorem 3.1. *Let $\mathscr{V}$ be a uniformly convex Banach space, and let $\mathscr{C} \subseteq \mathscr{V}$ be a nonempty, closed, and convex subset. Suppose that $\mathscr{M} : \mathscr{C} \to \mathscr{C}$ is an operator satisfying condition (C), and let $\{j_v\}$ be the sequence trajectory generated via the iterative framework* (12)*. Then, the fixed point set $Fix(\mathscr{M})$ is nonempty if and only if the sequence track $\{j_v\}$ is bounded and*

$$\lim_{v\to\infty} \|j_v - \mathscr{M} j_v\| = 0.$$

Proof. First, assume that $Fix(\mathscr{M}) \neq \emptyset$ and let $q \in Fix(\mathscr{M})$. By Lemma 3.1, the trajectory sequence $\{j_v\}$ is bounded and the distance limit $\lim_{v\to\infty} \|j_v - q\|$ is guaranteed to exist. Let this terminal value be defined as:

$$\xi = \lim_{v\to\infty} \|j_v - q\|.$$

From the monotonic relations established in Lemma 3.1, we inherit the HK-Fejer tracking inequalities:

$$\|u_v - q\| \leq \|j_v - q\|, \quad \|t_v - q\| \leq \|j_v - q\|.$$

Taking the limit superior across these bounds yields:

$$\limsup_{v\to\infty} \|u_v - q\| \leq \xi, \quad \limsup_{v\to\infty} \|t_v - q\| \leq \xi. \tag{18}$$

Since the operator $\mathscr{M}$ satisfies condition (C) and behaves as a quasi-nonexpansive mapping, we have:

$$\|\mathscr{M} j_v - q\| \leq \|j_v - q\|.$$

Evaluating the limit superior on both sides gives:

$$\limsup_{v\to\infty} \|\mathscr{M} j_v - q\| \leq \xi. \tag{19}$$

Furthermore, analyzing the step transition yields:

$$\|j_{v+1} - q\| = \|\mathscr{M} t_v - q\| \leq \|t_v - q\|,$$

which implies the basic contractive step:

$$\|j_{v+1} - q\| \leq \|t_v - q\|. \tag{20}$$

Taking limits throughout the expressions, we squeeze the behavior via:

$$\xi = \lim_{v\to\infty} \|j_{v+1} - q\| \leq \liminf_{v\to\infty} \|t_v - q\| \leq \limsup_{v\to\infty} \|t_v - q\| \leq \xi,$$

which proves that:

$$\lim_{v\to\infty} \|t_v - q\| = \xi. \tag{21}$$

Recalling the mathematical definition of the intermediate stage t_v:

$$t_v = (1 - \psi_v)\mathscr{M} j_v + \psi_v \mathscr{M}^2 u_v,$$

we expand the limit relation relative to q:

$$\lim_{v\to\infty} \|t_v - q\| = \lim_{v\to\infty} \|(1 - \psi_v)(\mathscr{M} j_v - q) + \psi_v(\mathscr{M}^2 u_v - q)\|. \tag{22}$$

Applying Lemma 3.1 alongside the uniform convexity property in Lemma 2.3, and combining equations (18)–(22), we obtain the strong asymptotic property for the operators squared:

$$\lim_{v\to\infty} \|\mathscr{M}^2 u_v - \mathscr{M} j_v\| = 0.$$

By tracing this down through the underlying contractive step $u_v = (1 - \eta_v) j_v + \eta_v \mathscr{M} j_v$ using the identical geometric limits, it collapses directly into:

$$\lim_{v\to\infty} \|\mathscr{M} j_v - j_v\| = 0.$$

Conversely, suppose that the sequence track $\{j_v\}$ is bounded and satisfies $\lim_{v\to\infty} \|\mathscr{M} j_v - j_v\| = 0$. Let $q \in A(\mathscr{C}, \{j_v\})$ be an element of the asymptotic center. Applying the analytical properties from Proposition 2.1(c), we can evaluate the radius bound as follows:

$$\begin{aligned} r(\mathscr{M} q, \{j_v\}) &= \limsup_{v\to\infty} \|j_v - \mathscr{M} q\| \\ &\leq 3 \limsup_{v\to\infty} \|\mathscr{M} j_v - j_v\| + \limsup_{v\to\infty} \|j_v - q\| \\ &= \limsup_{v\to\infty} \|j_v - q\| \\ &= r(q, \{j_v\}) = r(\mathscr{C}, \{j_v\}). \end{aligned}$$

This inequality dictates that the mapped point $\mathscr{M}q$ must also belong to the asymptotic center set $A(\mathscr{C},\{j_v\})$. Because the underlying space $\mathscr{V}$ is uniformly convex, its asymptotic center comprises exactly one unique element. Therefore, the coordinates must coincide identically, meaning $\mathscr{M}q = q$. Consequently, we conclude that:

$$Fix(\mathscr{M}) \neq \emptyset. \qquad \square$$

Next, we demonstrate the weak convergence criteria utilizing Opial's geometric property.

Theorem 3.2. *Let $\mathscr{C} \neq \emptyset$ be a closed and convex subset of a uniformly convex Banach space $\mathscr{V}$. Let $\mathscr{M} : \mathscr{C} \to \mathscr{C}$ be an operator satisfying condition (C) with $Fix(\mathscr{M}) \neq \emptyset$. Suppose that the underlying space $\mathscr{V}$ fulfills Opial's condition. If $\{j_v\}$ is the trajectory sequence generated via the iterative framework* (12)*, then $\{j_v\}$ converges weakly to an invariant fixed coordinate of $\mathscr{M}$.*

Proof. By invoking Theorem 3.1, the sequence trajectory $\{j_v\}$ is bounded and exhibits structural asymptotic regularity:

$$\lim_{v\to\infty} \|j_v - \mathscr{M}j_v\| = 0.$$

Since $\mathscr{V}$ is a uniformly convex Banach space, it is inherently reflexive. This geometric property guarantees the existence of a weakly convergent subsequence $\{j_{v_m}\}$ such that

$$j_{v_m} \rightharpoonup q_1 \in \mathscr{C}.$$

Applying Lemma 2.1 (the demiclosedness principle established for operators satisfying condition (C)), it follows immediately that

$$q_1 \in Fix(\mathscr{M}).$$

Suppose, for the sake of contradiction, that the entire sequence $\{j_v\}$ fails to converge weakly to q_1. This assumption implies the existence of another distinct subsequence $\{j_{v_s}\}$ such that

$$j_{v_s} \rightharpoonup q_2 \in \mathscr{C}, \quad \text{where } q_2 \neq q_1.$$

Re-applying the demiclosedness criteria from Lemma 2.1 to this second track confirms that $q_2 \in Fix(\mathscr{M})$.

By evaluating the distance limit properties via Opial's structural condition, and utilizing the fact that $\lim_{v\to\infty} \|j_v - q\|$ exists by Lemma 3.1, we obtain:

$$\lim_{v\to\infty} \|j_v - q_1\| < \lim_{v\to\infty} \|j_v - q_2\|.$$

By symmetry, swapping the roles of the invariant coordinates q_1 and q_2 yields:

$$\lim_{v\to\infty} \|j_v - q_2\| < \lim_{v\to\infty} \|j_v - q_1\|.$$

This direct contradiction shows that $q_1 = q_2$. Consequently, we conclude that the entire trajectory sequence $\{j_v\}$ must converge weakly to a valid invariant point in $Fix(\mathscr{M})$. □

Now, we introduce topological compactness constraints to prove the strong convergence theorems of the scheme.

Theorem 3.3. *Let $\mathscr{M}$ be an operator satisfying condition (C) defined over a nonempty, closed, and compact subset $\mathscr{C}$ of a uniformly convex Banach space $\mathscr{V}$. Let $\{j_v\}$ be the sequence trajectory generated via the iterative framework* (12)*. Then, $\{j_v\}$ converges strongly to a fixed point of $\mathscr{M}$.*

Proof. By Lemma 2.2, the compactness of the framework guarantees that the invariant set $Fix(\mathscr{M})$ is non-empty. Since $Fix(\mathscr{M}) \neq \emptyset$, Theorem 3.1 dictates that the system satisfies the asymptotic regularity condition:

$$\|j_v - \mathscr{M} j_v\| \to 0.$$

Because the subset $\mathscr{C}$ is closed and compact, there must exist a subsequence track $\{j_{v_m}\}$ that converges strongly to some accumulation element $q \in \mathscr{C}$, written as:

$$j_{v_m} \to q \in \mathscr{C}.$$

Invoking the structural metric bounds from Proposition 2.1(c) alongside the asymptotic tracking properties of the sequence, we evaluate the distance relation:

$$\|j_{v_m} - \mathscr{M} q\| \leq 3\|\mathscr{M} j_{v_m} - j_{v_m}\| + \|j_{v_m} - q\|. \tag{23}$$

Taking the limit as $v_m \to \infty$ across equation (23), the terms on the right-hand side vanish, which implies that:

$$j_{v_m} \to \mathscr{M} q.$$

By the uniqueness of limits in normed spaces, this guarantees that $\mathscr{M} q = q$, thereby establishing that $q \in Fix(\mathscr{M})$.

Furthermore, Lemma 3.1 ensures that the full metric limit $\lim_{v\to\infty} \|j_v - q\|$ exists. Since a subsequence converges strongly to q, the existence of this global limit forces the entire primary sequence track $\{j_v\}$ to converge strongly to q. □

The following theorem establishes strong convergence criteria in the absence of topological compactness constraints.

Theorem 3.4. *Let $\mathscr{V}$ be a uniformly convex Banach space, and let $\mathscr{C}$ be a nonempty, closed, and convex subset of $\mathscr{V}$. Suppose that $\mathscr{M} : \mathscr{C} \to \mathscr{C}$ is an operator satisfying condition (C), and let $\{j_v\}$ be the sequence trajectory generated via the iterative framework* (12)*. Then, $\{j_v\}$ converges strongly to a point in $Fix(\mathscr{M})$ if and only if*

$$\liminf_{v\to\infty} d(j_v, Fix(\mathscr{M})) = 0.$$

Proof. The direct implication follows directly from the fundamental definition of metric convergence.

Conversely, assume that the infimum accumulation limit satisfies

$$\liminf_{v\to\infty} d(j_v, Fix(\mathscr{M})) = 0.$$

From Lemma 3.1, the distance sequence $\{\|j_v - q\|\}$ is monotonically non-increasing for each invariant point $q \in Fix(\mathscr{M})$, which directly ensures that the set distance sequence $\{d(j_v, Fix(\mathscr{M}))\}$ is non-increasing. Consequently, the global limit

$$\lim_{v\to\infty} d(j_v, Fix(\mathscr{M}))$$

is guaranteed to exist.

Given that $\liminf_{v\to\infty} d(j_v, Fix(\mathscr{M})) = 0$, we deduce by the existence of the limit that

$$\lim_{v\to\infty} d(j_v, Fix(\mathscr{M})) = 0.$$

Thus, for every arbitrary precision parameter $\varepsilon > 0$, there exists a natural index $v_0 \in \mathbb{N}$ and an invariant target coordinate $s \in Fix(\mathscr{M})$ such that

$$\|j_{v_0} - s\| < \frac{\varepsilon}{2}.$$

For any arbitrary indices $v, m \geq v_0$, invoking the triangle inequality yields:

$$\|j_{v+m} - j_v\| \leq \|j_{v+m} - s\| + \|j_v - s\|.$$

Since both individual terms on the right-hand side are bounded by $\|j_{v_0} - s\|$ via the Fejer monotonicity induced by the HK-iteration framework, it follows that $\{j_v\}$ forms a Cauchy sequence within the subset $\mathscr{C}$.

As $\mathscr{C}$ is a closed subset embedded inside a complete Banach space, there must exist a limit element $p \in \mathscr{C}$ such that $j_v \to p$ strongly.

Finally, exploiting the structural asymptotic regularity established for the HK-iteration trajectory:

$$\lim_{v\to\infty} \|j_v - \mathscr{M} j_v\| = 0,$$

we apply continuous norm operations to isolate:

$$\|p - \mathscr{M} p\| = 0,$$

which proves that $p \in Fix(\mathscr{M})$. □

We now use condition (I) to prove the strong convergence of the HK-iteration process. This condition imposes additional constraints that strengthen convergence, especially when generalized non-expansiveness alone is not sufficient. It ensures norm convergence by linking the iterative sequence to a fixed point and guaranteeing that the distance between iterates and the fixed-point set gradually decreases.

Theorem 3.5. *Let $\mathscr{C}$ be a closed and convex subset of a uniformly convex Banach space $\mathscr{V}$. Suppose that $\mathscr{M} : \mathscr{C} \to \mathscr{C}$ is a mapping satisfying conditions (C) and (I) with $Fix(\mathscr{M}) \neq \emptyset$,*

and let $\{j_v\}$ be the sequence trajectory generated via the iterative framework (12)*. Then, the sequence $\{j_v\}$ converges strongly to an invariant fixed coordinate of $\mathscr{M}$.*

Proof. By invoking Theorem 3.1, we inherit the asymptotic regularity of the tracking framework:

$$\lim_{v\to\infty} \|j_v - \mathscr{M} j_v\| = 0. \tag{24}$$

By applying the operational constraints mandated under condition (I), we have the mapping relation:

$$\|j_v - \mathscr{M} j_v\| \geq g\big(d(j_v, Fix(\mathscr{M}))\big).$$

Taking limits across this inequality as $v \to \infty$ and substituting our limit from equation (24), we obtain:

$$\lim_{v\to\infty} g\big(d(j_v, Fix(\mathscr{M}))\big) = 0.$$

Given that the control mapping $g : [0,\infty) \to [0,\infty)$ is strictly non-decreasing, evaluates to zero at the origin ($g(0) = 0$), and remains strictly positive ($g(\xi) > 0$) for all $\xi > 0$, it follows by the properties of continuous monotonic functions that

$$\lim_{v\to\infty} d(j_v, Fix(\mathscr{M})) = 0.$$

Hence, all metric assumptions mandated by Theorem 3.4 are fully satisfied. Consequently, we conclude that the trajectory sequence $\{j_v\}$ converges strongly to a valid fixed coordinate within the invariant set $Fix(\mathscr{M})$. □

4. NUMERICAL EXAMPLE

This section presents a modified numerical illustration of the convergence behavior associated with the HK-iteration process for a mapping satisfying condition (C). The construction is carried out on a different compact domain, while preserving the Suzuki-type structural framework of the operator.

Example 4.1. *Define a self-mapping* $\mathscr{M} : [0,2] \to [0,2]$ *by*

$$\mathscr{M}(j) = \begin{cases} \frac{j+2}{5}, & \text{if } j \in [0,1), \\ \frac{3j+2}{7}, & \text{if } j \in [1,2]. \end{cases} \tag{25}$$

It is immediate that $\mathscr{M}$ is not nonexpansive in the classical sense. Indeed, taking $j_1 = 0.9$ and $j_2 = 1$, we obtain

$$\mathscr{M}(j_1) = \frac{2.9}{5} = 0.58, \qquad \mathscr{M}(j_2) = \frac{5}{7}, \qquad |j_1 - j_2| = 0.1,$$

while

$$|\mathscr{M}(j_1) - \mathscr{M}(j_2)| = \left|0.58 - \frac{5}{7}\right| \approx 0.1343 > 0.1,$$

which confirms the failure of the nonexpansive inequality.

We now verify that $\mathscr{M}$ satisfies condition (C). The argument proceeds via a two-region decomposition.

(1) Let $j \in [0,1)$. Then $\mathscr{M}(j) = \frac{j+2}{5}$, and hence

$$\frac{1}{2}|j - \mathscr{M}(j)| = \frac{1}{2}\left|j - \frac{j+2}{5}\right| = \frac{|4j-2|}{10}.$$

For $k \in [1,2]$, we have $\mathscr{M}(k) = \frac{3k+2}{7}$. A direct computation yields

$$|\mathscr{M}(j) - \mathscr{M}(k)| = \left|\frac{j+2}{5} - \frac{3k+2}{7}\right| \leq |j-k|,$$

whenever $\frac{1}{2}|j - \mathscr{M}(j)| \leq |j-k|$.

(2) Let $j \in [1,2]$. Then $\mathscr{M}(j) = \frac{3j+2}{7}$, and thus

$$\frac{1}{2}|j - \mathscr{M}(j)| = \frac{1}{2}\left|\frac{4j-2}{7}\right| = \frac{|4j-2|}{14}.$$

Splitting the analysis into the cases $k \geq j$ and $k < j$, one verifies that

$$|\mathscr{M}(j) - \mathscr{M}(k)| \leq |j-k| \quad \text{whenever} \quad \frac{1}{2}|j - \mathscr{M}(j)| \leq |j-k|.$$

Hence, $\mathscr{M}$ satisfies condition (C) on $[0,2]$.

Consequently, $\mathscr{M}$ is a Suzuki-type generalized nonexpansive mapping on the compact interval $[0,2]$.

We now construct the HK-iteration associated with $\mathscr{M}$. The iterative process is defined by

$$\begin{cases} u_v = (1-\eta_v)j_v + \eta_v \mathscr{M}(j_v), \\ t_v = (1-\psi_v)\mathscr{M}(j_v) + \psi_v \mathscr{M}^2(u_v), \\ j_{v+1} = \mathscr{M}(t_v), \end{cases} \tag{26}$$

where the control parameters are fixed as

$$\eta_v = 0.65, \qquad \psi_v = 0.55,$$

and the initial point is chosen as $j_1 = 0.3$.

The numerical behavior generated by this iteration exhibits stable convergence toward the unique fixed point $q = 0.5$. The averaging step in u_v reduces deviation from the orbit of $\mathscr{M}$, while the higher-order correction term $\mathscr{M}^2(u_v)$ introduces additional contractive damping. The final projection step $j_{v+1} = \mathscr{M}(t_v)$ ensures asymptotic stabilization of the trajectory.

A comparison with classical iterative schemes such as Picard, Noor, and Thakur-type iterations indicates that the HK-process converges in fewer iterations under identical tolerance criteria. This improvement is attributed to the layered correction mechanism inherent in the iteration structure. The iterates j_v of all six methods are plotted on a three-dimensional spherical surface in Figure 1, where the Z-axis represents the state value and the Y-axis the technique index. All schemes begin at $(v = 1, j_1 = 0.3)$ and ascend toward the fixed point $q = 0.5$ at the top of the sphere. According to the numerical results in Table 1, the HK-iteration grows most rapidly and achieves 0.5 by the twelfth iteration, while classical systems like the Noor iteration require longer, more gradual courses.

5. Visual and Quantitative Comparison via Polynomiography

By converting the convergence of root-finding techniques into geometric structures of attraction basins, whose form and fractal-like limits delicately evaluate the strength, symmetry, and reliability of new iterations (see, e.g., [37–42]); as explained in Algorithm 1, each starting point

TABLE 1. Extended iterates up to 16 steps showing complete asymptotic stabilization to the unique fixed point $q = 0.5$ for starting point $j_1 = 0.3$.

v	HK iteration	Sahu iteration	Picard–Noor iteration	Thakur iteration	Abbas iteration	Noor iteration
1	0.30000000	0.30000000	0.30000000	0.30000000	0.30000000	0.30000000
2	0.49597760	0.48512000	0.48425600	0.48352000	0.47961600	0.43582400
3	0.49991910	0.49887104	0.49877478	0.49830502	0.49758787	0.48716480
4	0.49999837	0.49991420	0.49991168	0.49983115	0.49971214	0.49743323
5	0.49999997	0.49999348	0.49999364	0.49998313	0.49996568	0.49948665
6	0.50000000	0.49999951	0.49999954	0.49999831	0.49999591	0.49989733
7	0.50000000	0.49999996	0.49999997	0.49999983	0.49999951	0.49997947
8	0.50000000	0.50000000	0.50000000	0.49999998	0.49999994	0.49999589
9	0.50000000	0.50000000	0.50000000	0.50000000	0.49999999	0.49999918
10	0.50000000	0.50000000	0.50000000	0.50000000	0.50000000	0.49999984
11	0.50000000	0.50000000	0.50000000	0.50000000	0.50000000	0.49999997
12	0.50000000	0.50000000	0.50000000	0.50000000	0.50000000	0.50000000
13	0.50000000	0.50000000	0.50000000	0.50000000	0.50000000	0.50000000
14	0.50000000	0.50000000	0.50000000	0.50000000	0.50000000	0.50000000
15	0.50000000	0.50000000	0.50000000	0.50000000	0.50000000	0.50000000
16	0.50000000	0.50000000	0.50000000	0.50000000	0.50000000	0.50000000

in a selected region of the complex plane is followed under the operator and colored by the root it reaches within the maximum number of iterations.

To anchor this graphical tool into a root-finding context, we recall the classical Newton's method, celebrated for its local quadratic convergence. The standard formulation is given by:

$$j_{v+1} = j_v - \frac{P(j_v)}{P'(j_v)}, \tag{27}$$

where $j_1 \in \mathbb{C}$ is a chosen starting approximation. By introducing the canonical Newton fixed-point operator,

$$\mathscr{M}(z) = z - \frac{P(z)}{P'(z)}, \tag{28}$$

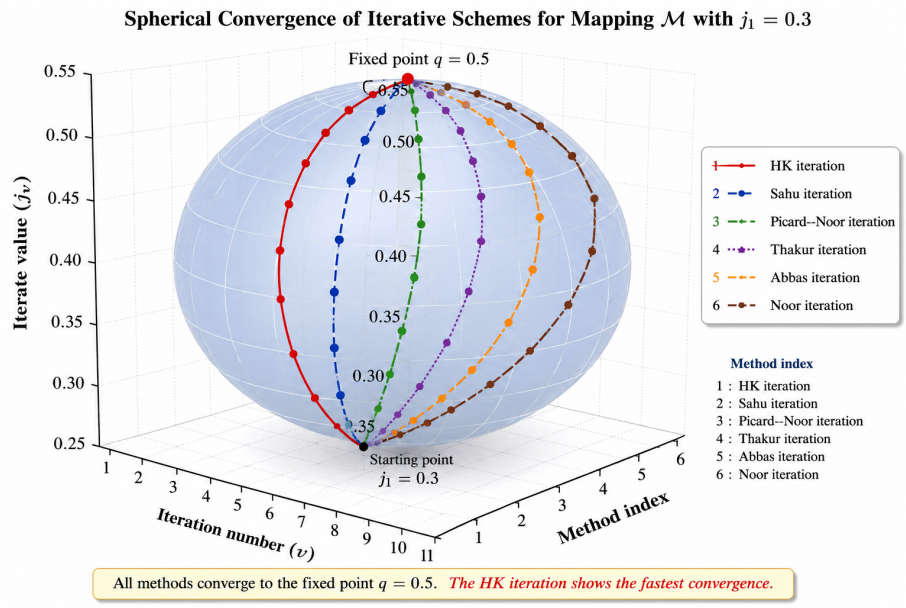


FIGURE 1. Spherical trajectory profiles demonstrating the geometric convergence pathways of the unique fixed point $q = 0.5$ from the initial state $j_1 = 0.3$.

Algorithm 1 Algorithmic Synthesis of a Polynomiograph.

Input: A complex polynomial $P \in \mathbb{C}[z]$ ($\deg(P) \geq 2$); an iterative fixed-point process $\mathscr{I}$; a bounded computational domain $\mathscr{A} \subset \mathbb{C}$; maximum iteration threshold K; an error tolerance $\varepsilon > 0$; and a continuous color mapping spectrum.

Output: A visual polynomiograph defined over the domain $\mathscr{A}$.

for each initial grid point $j_1 \in \mathscr{A}$ **do**
    Set iteration counter $v = 1$
    while $|P(j_v)| > \varepsilon$ **and** $v < K$ **do**
        Compute the next state: $j_{v+1} = \mathscr{I}(j_v, \mathscr{M})$
        Increment counter: $v = v + 1$
    end while
    Identify the unique root of $P(z)$ geometrically closest to j_v.
    Map the respective root color to j_1, modulating intensity proportionally to v.
end for

Equation (27) translates into a classical Picard fixed-point iteration: $j_{v+1} = \mathscr{M}(j_v)$. Because any fixed point $x = \mathscr{M}(x)$ directly satisfies $P(x) = 0$, finding the roots of a polynomial is mathematically isomorphic to finding the fixed points of its associated Newton operator $\mathscr{M}$. This

functional insight allows us to superimpose advanced fixed-point schemes onto $\mathscr{M}$ to accelerate or stabilize convergence.

In this work, we integrate our multi-stage HK-iteration directly with the Newton fixed-point operator $\mathscr{M}$ associated with the roots of the polynomial. Given an initial complex coordinate $j_1 \in \mathbb{C}$, the dynamical tracking sequence is governed by the following structurally coherent system:

$$\begin{cases} u_v = (1-\eta_v) j_v + \eta_v \mathscr{M}(j_v), \\ t_v = (1-\psi_v)\mathscr{M}(j_v) + \psi_v \mathscr{M}^2(u_v), \\ j_{v+1} = \mathscr{M}(t_v), \end{cases} \tag{29}$$

where $\eta_v, \psi_v \in (0,1)$ function as control parameters regulating the internal convex relaxation steps of the computational tracking framework.

5.1. Experimental and Computational Setup. To critically evaluate the efficiency, visual stability, and basin geometry of the proposed HK-iteration, we carry out comparative numerical experiments against five prominent iterative frameworks: Noor, Abbas, Thakur, Sahu, and Picard–Noor iterations.

The selected benchmark is the classic sixth-degree cyclotomic polynomial:

$$P(z) = z^6 - 1, \tag{30}$$

which yields six simple roots symmetrically distributed along the unit circle on the complex plane:

$$\pm 1, \quad -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i, \quad \text{and} \quad \frac{1}{2} \pm \frac{\sqrt{3}}{2} i.$$

The spatial boundaries for the numerical canvas are restricted to the domain $\mathscr{A} = [-2,2] \times [-2,2]$. The stopping parameters are strictly fixed at an error tolerance of $\varepsilon = 10^{-3}$ and a maximum iteration roof of $K = 45$.

To observe how the tuning of control parameters alters the underlying fractal dynamics, we run the experiments across three uniform configurations representing distinct operational settings:

- **Low parameter regime (Set C):** $\rho_v = \psi_v = \eta_v = 0.06$

- **Moderate parameter regime (Set B):** $\rho_v = \psi_v = \eta_v = 0.60$
- **High parameter regime (Set A):** $\rho_v = \psi_v = \eta_v = 0.90$

5.2. Numerical Experiments Under Varying Parameter Regimes.

5.2.1. *Low Parameter Regime:* $\rho_v = \psi_v = \eta_v = 0.06$. The polynomiographs for the lowest parameter setting are shown in Fig. 2. When the control parameters are near zero, each sub-step applies only a minimal correction, so the initial guess carries most of the burden and Julia-type boundaries appear where neighboring basins meet along the unit circle. Even so, the HK-iteration gives unified basins with clean boundaries for all six roots, while Noor breaks down almost completely (large black regions within 45 iterations) and the Abbas, Thakur, Sahu, and Picard–Noor methods remain fragmented with darker shading near the borders; quantitatively (Table 2), Abbas has a slightly lower mean iteration count, but HK has the smallest spread among the converging methods. "'

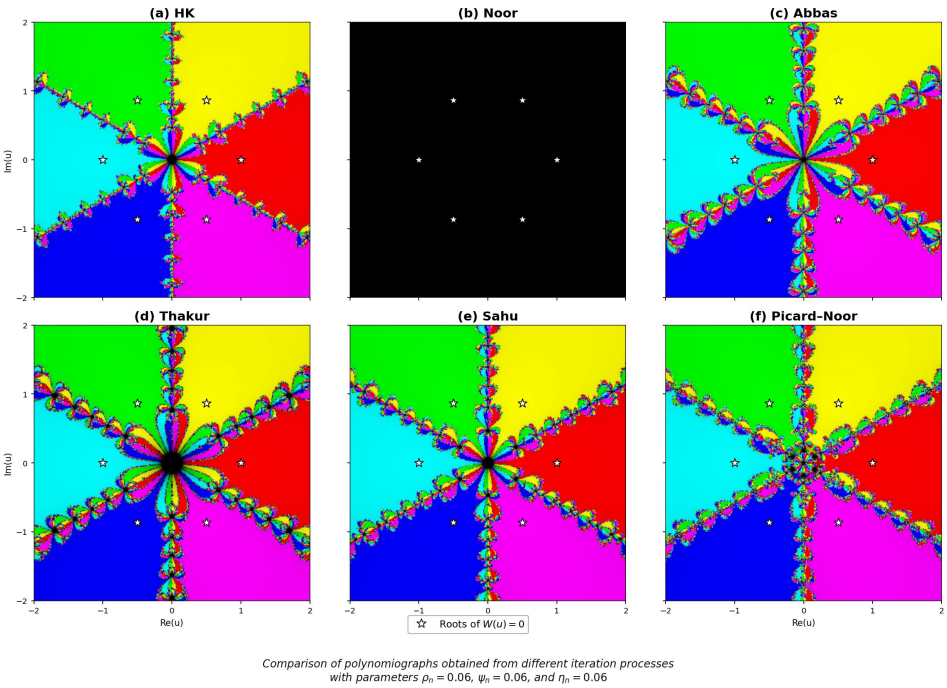


FIGURE 2. Comparison of polynomiographs obtained from $P(z) = z^6 - 1$ under the low parameter regime (Set C: $\rho_v = \psi_v = \eta_v = 0.06$). Black zones mark coordinates that failed to converge within $K = 45$ iterations.

5.2.2. *Moderate Parameter Regime:* $\rho_v = \psi_v = \eta_v = 0.60$. Increasing the control parameters to 0.60 smoothes the behavior of all approaches, resulting in more ordered basins, smoother limits, and nearly no non-convergent regions, as illustrated in Fig. 3. The Abbas, Thakur, Sahu, and Picard–Noor methods enhance but still exhibit deeper circular peaks along basin limits (local convergence delays); the Noor iteration recovers significantly but keeps small slow-convergence pockets; and the HK-iteration continues to be the best, with broad basins and smooth transitions in which most points reach their roots within a few iterations.

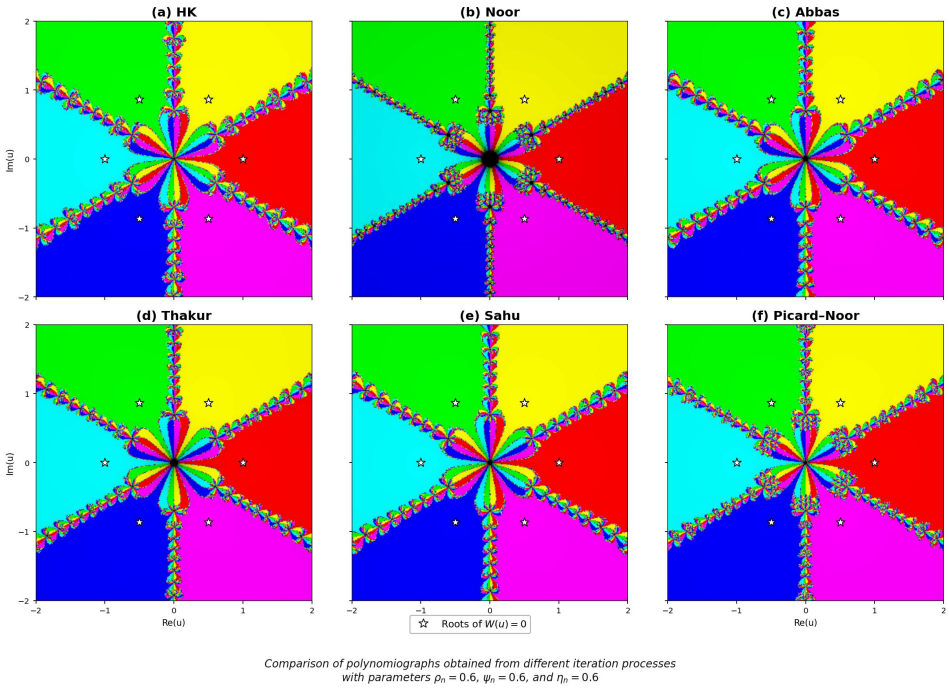


FIGURE 3. Comparison of polynomiographs obtained from different iteration processes for the sixth-degree polynomial $P(z) = z^6 - 1$ under the moderate parameter regime (Set B: $\rho_v = \psi_v = \eta_v = 0.60$), showing much smoother boundaries and accelerated convergence.

5.2.3. *High Parameter Regime:* $\rho_v = \psi_v = \eta_v = 0.90$. When the control parameters approach unity ($\rho_v = \psi_v = \eta_v = 0.90$), the strong numerical feedback maximizes the efficiency of the internal correction steps. As shown in Fig. 4, this creates highly stable, compact, and perfectly separated attraction regions for virtually all methods.

The HK-iteration produces the cleanest and most effective polynomiograph for the six symmetric roots, even in this extremely advantageous regime. Virtually the whole canvas is brilliant, indicating that virtually every beginning point converges instantly. Although Picard–Noor and Sahu also perform exceptionally well, they nevertheless exhibit thin black tracks around the boundaries where the HK approach remains uniformly brilliant and steady because its multi-stage correction smoothes out the chaotic oscillations near the root boundaries.

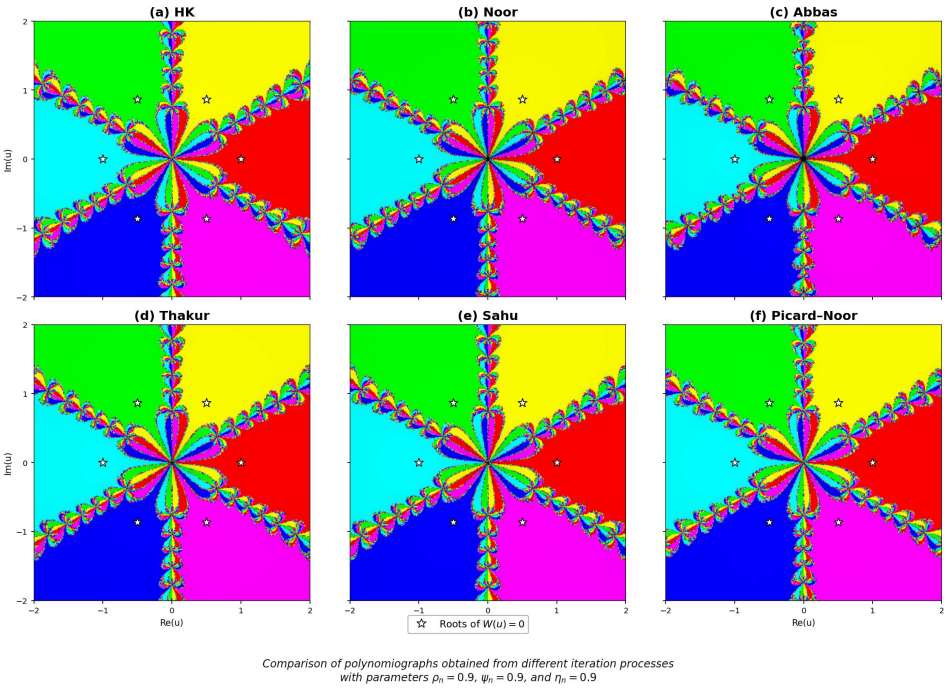


FIGURE 4. Comparison of polynomiographs derived from several iteration procedures for the sixth-degree polynomial $P(z) = z^6 - 1$ within the parameter regime (Set A: $\rho_v = \psi_v = \eta_v = 0.90$), showing maximal basin regularity and rapid convergence.

5.3. Monte Carlo and Quantitative Performance Analysis. We conducted a large-scale Monte Carlo simulation with $N = 60{,}000$ points sampled uniformly from the domain $\mathscr{A} = [-2,2] \times [-2,2]$ in order to provide statistical support for these visual observations. Table 2 summarizes the average number of iterations $\bar{v}$, its standard deviation σ_v, and the convergence basin percentage (Basin (%)), for each method and parameter regime. Figure 5 displays the corresponding performance plots and spatial distributions.

TABLE 2. Monte Carlo convergence analysis metrics for $P(z) = z^6 - 1$ using $N = 60{,}000$ uniformly sampled initial coordinates (Max Iterations $K = 45$, Tolerance $\varepsilon = 10^{-3}$). Bold values indicate the best value in each column (the non-converging Noor case at 0.06 is excluded from the σ_v comparison).

Method	**High Parameters (0.90)**			**Moderate Parameters (0.60)**			**Low Parameters (0.06)**		
	$\bar{v}$	σ_v	Basin (%)	$\bar{v}$	σ_v	Basin (%)	$\bar{v}$	σ_v	Basin (%)
HK (Proposed)	**3.42**	2.96	**100.00**	**4.01**	**3.31**	**100.00**	7.05	**4.31**	99.70
Noor	6.22	4.05	99.90	11.75	4.67	99.20	45.00	4.12	0.00
Abbas	5.35	4.52	99.90	5.34	4.40	99.90	**6.08**	5.39	**99.80**
Thakur	4.76	3.92	**100.00**	6.04	4.66	99.80	9.99	7.51	97.80
Sahu	4.40	3.86	**100.00**	5.18	4.27	99.90	8.31	5.70	99.20
Picard–Noor	3.73	**2.89**	**100.00**	4.78	3.69	**100.00**	8.15	5.35	99.30

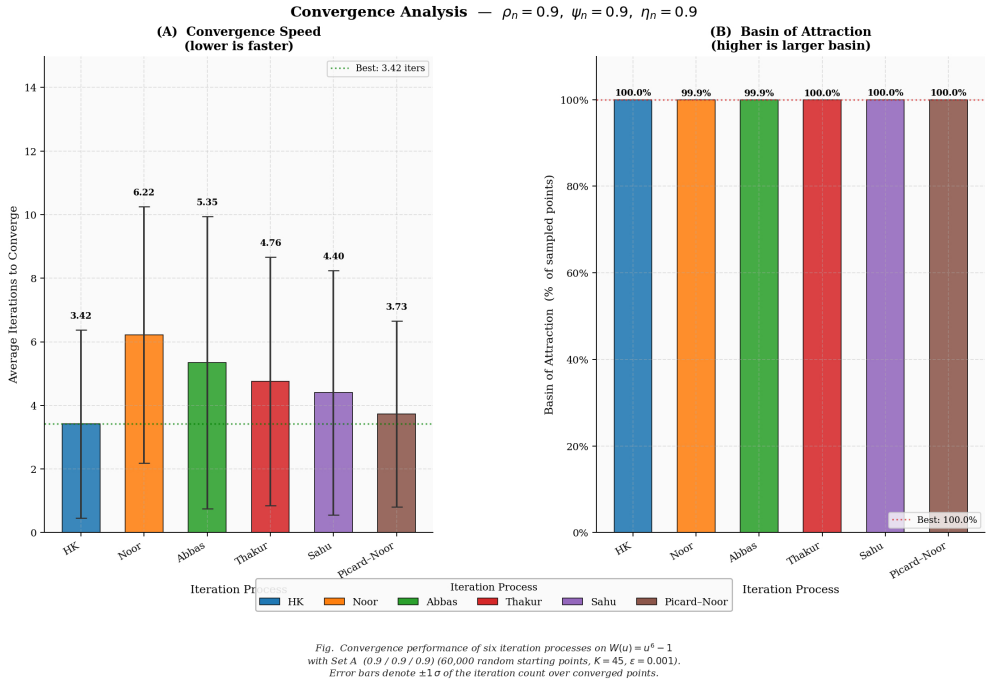


FIGURE 5. Detailed convergence speed and basin of attraction statistics for Set A ($\rho_v = \psi_v = \eta_v = 0.90$).

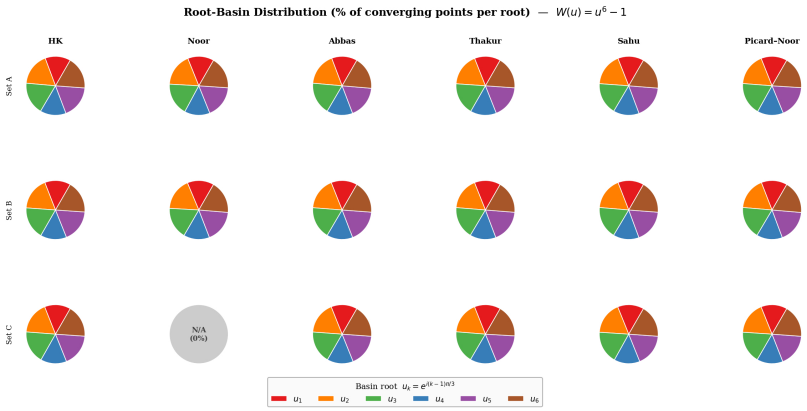


FIGURE 6. Root-Basin spatial distribution showing percentage of converging points distributed symmetrically across the six roots under sets A, B, and C.

The empirical values in Table 2 illustrate the HK-iteration's operational edge. In the high parameter regime, the proposed scheme records a remarkable average of just $\bar{v} = 3.42$ iterations while sweeping a perfect 100% convergence basin.

In the moderate regime, the HK-iteration continues to outperform its rivals, converging in $\bar{v} = 4.01$ iterations on average over a perfect basin of 100.00%. While methods like Picard–Noor and Sahu stay relatively close in basin size, they systematically demand more computational cycles to fulfill the stopping criteria.

The most difficult stability test is in the low parameter domain. The HK-iteration maintains a 99.70% convergence basin with a low average of $\bar{v} = 7.05$ iterations, while Abbas is somewhat quicker at $\bar{v} = 6.08$. In this case, the Noor process collapses, converging from nearly no starting points and requiring an average of 45 iterations. Additionally, the HK approach exhibits relatively tiny standard deviations σ_v in each regime, suggesting that the initial point's location in the complex plane has little bearing on the method's convergence speed.

5.4. Discussion. The suggested HK-iteration works well from graphical, statistical, and dynamical perspectives, according to the polynomiographic investigation. In comparison to the

Noor, Abbas, Thakur, Sahu, and Picard–Noor processes, polynomiographs show bigger convergence regions, smoother borders, and more connected attraction basins; these patterns are consistent with the Monte Carlo results. All things considered, the HK-iteration is a reliable and competitive method for resolving nonlinear equations.

6. Conclusion

For Suzuki-type mappings, we presented a generalized HK-iteration in this study and showed its excellent convergence given appropriate circumstances. Both the numerical experiments and the polynomiographic analysis of $P(z) = z^6 - 1$ revealed that it converged more quickly and used less computation than the Noor, Abbas, Thakur, Sahu, and Picard–Noor schemes. In the high and moderate parameter regimes, it generated smoother attraction basins and reduced ANI values; in the low regime, it was still competitive. Because of these findings, the HK-iteration is a trustworthy method for solving nonlinear fixed-point problems and a helpful foundation for further research on multivalued operators and generalized contractive mappings.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

References

[1] A.B. Amar, A. Jeribi, M. Mnif, Some Fixed Point Theorems and Application to Biological Model, Numer. Funct. Anal. Optim. 29 (2008), 1–23. https://doi.org/10.1080/01630560701749482.

[2] R.-M. Dăneţ, M.-V. Popescu, Some Applications of the Fixed-Point Theory in Economics, Creat. Math. Inform. 17 (2008), 392–398.

[3] A. Yuan, Fixed Point Theorems and Applications to Game Theory, University of Chicago, 2017. http://math.uchicago.edu/~may/REU2017/REUPapers/Yuan.pdf.

[4] C. Constanda, A. Largillier, M. Ahues (Eds.), Integral Methods in Science and Engineering: Analytic and Numerical Techniques, Birkhäuser, 2004. https://doi.org/10.1007/978-0-8176-8184-5.

[5] D.P. Bertsekas, Nonlinear Programming, 3rd ed., Athena Scientific, 2016.

[6] R. Caccioppoli, Un Teorema Generale sull'Esistenza di Elementi Uniti in una Trasformazione Funzionale, Atti Accad. Naz. Lincei, Rend., VI. Ser. 11 (1930), 794–799.

[7] F.E. Browder, Nonexpansive Nonlinear Operators in a Banach Space, Proc. Natl. Acad. Sci. U.S.A. 54 (1965), 1041–1044. https://doi.org/10.1073/pnas.54.4.1041.

[8] D. Göhde, Zum Prinzip der Kontraktiven Abbildung, Math. Nachr. 30 (1965), 251–258. https://doi.org/10.1002/mana.19650300312.

[9] W.A. Kirk, A Fixed Point Theorem for Mappings Which Do Not Increase Distances, Am. Math. Mon. 72 (1965), 1004. https://doi.org/10.2307/2313345.

[10] K. Goebel, W.A. Kirk, Topics in Metric Fixed Point Theory, Cambridge University Press, 1990. https://doi.org/10.1017/cbo9780511526152.

[11] T. Suzuki, Fixed Point Theorems and Convergence Theorems for Some Generalized Nonexpansive Mappings, J. Math. Anal. Appl. 340 (2008), 1088–1095. https://doi.org/10.1016/j.jmaa.2007.09.023.

[12] H. Ullah, K. Ullah, A. Asghar, M. Tariq, H. Ahmad, et al., Some New Convergence Approaches in Banach Spaces Involving the HK-Iteration Method with Application, Contemp. Math. 7 (2026), 2501–2524. https://doi.org/10.37256/cm.7220268533.

[13] W.R. Mann, Mean Value Methods in Iteration, Proc. Amer. Math. Soc. 4 (1953), 506–510. https://doi.org/10.1090/s0002-9939-1953-0054846-3.

[14] S. Ishikawa, Fixed Points by a New Iteration Method, Proc. Amer. Math. Soc. 44 (1974), 147–150. https://doi.org/10.1090/s0002-9939-1974-0336469-5.

[15] M.A. Noor, New Approximation Schemes for General Variational Inequalities, J. Math. Anal. Appl. 251 (2000), 217–229. https://doi.org/10.1006/jmaa.2000.7042.

[16] M. Abbas, T. Nazir, A New Faster Iteration Process Applied to Constrained Minimization and Feasibility Problems, Mat. Vesn. 66 (2014), 223–234.

[17] V.K. Sahu, H.K. Pathak, R. Tiwari, Convergence Theorems for New Iteration Scheme and Comparison Results, Aligarh Bull. Math. 35 (2016), 19–42.

[18] D. Thakur, B.S. Thakur, M. Postolache, New Iteration Scheme for Numerical Reckoning Fixed Points of Nonexpansive Mappings, J. Inequal. Appl. 2014 (2014), 328. https://doi.org/10.1186/1029-242X-2014-328.

[19] K.S. Eke, H. Akewe, Equivalence of Picard-Type Hybrid Iterative Algorithms for Contractive Mappings, Asian J. Sci. Res. 12 (2019), 298–307. https://doi.org/10.3923/ajsr.2019.298.307.

[20] M. Chyne, N. Kumar, Convergence Analysis of Picard-Abbas Hybrid Iterative Process, Adv. Fixed Point Theory 14 (2024), 21. https://doi.org/10.28919/afpt/8552.

[21] A. Abkar, M. Eslamian, Fixed Point Theorems for Suzuki Generalized Nonexpansive Multivalued Mappings in Banach Spaces, Fixed Point Theory Appl. 2010 (2010), 457935. https://doi.org/10.1155/2010/457935.

[22] J. Ahmad, K. Ullah, M. Arshad, Z. Ma, A New Iterative Method for Suzuki Mappings in Banach Spaces, J. Math. 2021 (2021), 1–7. https://doi.org/10.1155/2021/6622931.

[23] J. Ali, F. Ali, P. Kumar, Approximation of Fixed Points for Suzuki's Generalized Non-Expansive Mappings, Mathematics 7 (2019), 522. https://doi.org/10.3390/math7060522.

[24] S. Bhatti, F. Abbas, H. Ullah, K.M. Alam, S. Mishra, A Novel Double-Application HA-Iteration for Generalized (α,β)-Nonexpansive Mappings: Fixed Point Structure and Convergence Theory, Int. J. Appl. Math. Optim. Artif. Intell. 1 (2026), 30–38.

[25] Z. Zuo, Y. Cui, Iterative Approximations for Generalized Multivalued Mappings in Banach Spaces, Thai J. Math. 9 (2011), 333–342.

[26] H. Ullah, K. Ullah, A. Asghar, M. Tariq, H. Ahmad, et al., New Approach of Generalized α-Nonexpansive Mappings via HK-Iteration Process with Applications to Fractional Differential Equation and Boundary Value Problems, Contemp. Math. 7 (2026), 1662–1682. https://doi.org/10.37256/cm.7220268645.

[27] H. Ullah, K. Ullah, A. Asghar, M. Tariq, W. Afzal, et al., The HK-Iteration: A Faster Convergent Scheme for Generalized (α,β)-Nonexpansive Mappings with Applications, Eur. J. Pure Appl. Math. 19 (2026), 7332. https://doi.org/10.29020/nybg.ejpam.v19i2.7332.

[28] H. Ullah, K. Ullah, U. Ishtiaq, I.-L. Popa, S.U. Rehman, et al., Convergence Results for Suzuki Generalized Nonexpansive Maps Using the HK-Iteration Scheme, Eur. J. Pure Appl. Math. 19 (2026), 6885. https://doi.org/10.29020/nybg.ejpam.v19i2.6885.

[29] S. Asjad, H. Ullah, M. Arshad, E. Hincal, I. Ullah, Convergence Analysis of HA-Iteration for Enriched Nonexpansive Mappings in Banach Space Settings, Int. J. Appl. Math. Optim. Artif. Intell. 1 (2026), 40–45.

[30] H. Ullah, K. Ullah, A. Asghar, M. Tariq, W. Afzal, et al., Convergence Analysis of the HK-Iteration for Garcia-Falset Mappings in Banach Spaces with Application to Fractional Boundary Value Problems, Eur. J. Pure Appl. Math. 19 (2026), 7310. https://doi.org/10.29020/nybg.ejpam.v19i2.7310.

[31] Z. Opial, Weak Convergence of the Sequence of Successive Approximations for Nonexpansive Mappings, Bull. Am. Math. Soc. 73 (1967), 591–597. https://doi.org/10.1090/s0002-9904-1967-11761-0.

[32] H.F. Senter, W.G. Dotson, Approximating Fixed Points of Nonexpansive Mappings, Proc. Amer. Math. Soc. 44 (1974), 375–380. https://doi.org/10.1090/s0002-9939-1974-0346608-8.

[33] J. Schu, Weak and Strong Convergence to Fixed Points of Asymptotically Nonexpansive Mappings, Bull. Aust. Math. Soc. 43 (1991), 153–159. https://doi.org/10.1017/s0004972700028884.

[34] B. Kalantari, Polynomiography: From the Fundamental Theorem of Algebra to Art, Leonardo 38 (2005), 233–238. https://doi.org/10.1162/0024094054029010.

[35] B. Kalantari, Polynomial Root-Finding and Polynomiography, World Scientific, 2008. https://doi.org/10.1142/6265.

[36] F. Andreev, B. Kalantari, I. Kalantari, Measuring the Average Performance of Root-Finding Algorithms and Imaging It Through Polynomiography, in: Proceedings of the 17th International Association for Mathematics and Computers in Simulation World Congress, Scientific Computation, Applied Mathematics and Simulation, Paris, 2005.

[37] B. Nawaz, K. Ullah, K. Gdawiec, Convergence Analysis of a Picard–CR Iteration Process for Nonexpansive Mappings, Soft Comput. 29 (2025), 435–455. https://doi.org/10.1007/s00500-025-10515-0.

[38] B. Nawaz, K. Ullah, K. Gdawiec, Convergence Analysis of Picard–SP Iteration Process for Generalized α–Nonexpansive Mappings, Numer. Algorithms 98 (2025), 1943–1964. https://doi.org/10.1007/s11075-024-01859-z.

[39] K. Panigrahy, D. Mishra, A Note on a Faster Fixed Point Iterative Method, J. Anal. 31 (2023), 831–854. https://doi.org/10.1007/s41478-022-00485-z.

[40] T.-M. Yu, A.A. Shahid, K. Shabbir, N.A. Shah, Y.-M. Li, An Iteration Process for a General Class of Contractive-like Operators: Convergence, Stability and Polynomiography, AIMS Math. 6 (2021), 6699–6714. https://doi.org/10.3934/math.2021393.

[41] G.I. Usurelu, M. Postolache, Algorithm for Generalized Hybrid Operators with Numerical Analysis and Applications, J. Nonlinear Var. Anal. 6 (2022), 255–277. https://doi.org/10.23952/jnva.6.2022.3.07.

[42] A. Naseem, I.K. Argyros, S. Qureshi, M. Aziz ur Rehman, A. Soomro, et al., Memory Based Approaches to One-Dimensional Nonlinear Models, Acta Appl. Math. 195 (2025), 1. https://doi.org/10.1007/s10440-024-00703-9.