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SOLVABILITY OF A SYSTEM OF GENERALIZED EXTENDED VARIATIONAL INEQUALITIES

BALWANT SINGH THAKUR*, AND SUJA VARGHESE

School of Studies in Mathematics, Pt.Ravishankar Shukla University, Raipur, 492010, India

Abstract. In this paper, we proposed a system of generalized extended nonlinear variational inequality problem (SGENVI) involving three different trivariate mappings and six different nonlinear mappings. We also proposed three steps relaxed parallel projection algorithm for convergence analysis of the approximate solvability of the SGENVI problem. Further, we propose relaxed parallel projection algorithm, which converges to common element of solution set of the SGENVI, and the fixed point set of three nonexpansive mappings.

Keywords: system of generalized extended variational inequalities, fixed point problem, projection method, relaxed cocoercive mapping, Lipschitz continuous mapping.

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1. Introduction

Variational inequality theory has appeared as an effective and powerful tool to study and investigate a wide class of problems arising in pure and applied sciences. Verma [16] introduced a system of nonlinear strongly monotone variational inequalities and studied its approximate

*Corresponding author

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solvability. Chang et al. [5] introduced a new system of nonlinear relaxed cocoercive variational inequalities and studied the approximation solvability of this system based on a system of projection methods. Shang et al. [13] studied a system of variational inequalities involving three different relaxed cocoercive mappings and studied iterative methods for finding common element of the set of the common fixed points of three different quasi-nonexpansive mappings and the set of solutions of the variational inequalities with three different cocoercive mappings. Cho et al. [6] introduced a system of general nonlinear variational inequalities with three different relaxed cocoercive mapping and three different nonlinear mappings and studied the approximate solvability using iterative schemes based on the projection methods.

However, these studies were based on sequential iterative methods, which are only suitable for implementing on the traditional single-processor computers. To satisfy practical requirements of modern multiprocessor systems, iterative methods having parallel characteristics need to be developed for the system of variational inequalities. Lions [11] has studied parallel algorithms for solution of parabolic variational inequalities. Bertsekas and Tsitsiklis [2, 3] developed parallel algorithms by using the metric projection. Recently, Yang et al. [18] studied parallel projection algorithm for a system of nonlinear variational inequalities.

Inspired and motivated by research works in this field, we introduce and study a system of variational inequalities involving three sets of three different nonlinear operators. Using the parallel projection technique, we suggest and analyze a parallel iterative method for solving this system. We also study a general three-step method for the projection methods, which can be applied to the approximation solvability of the system of variational inequalities and common fixed point of three different nonexpansive mappings.

2. Preliminaries

Let $T_i : H \times H \times H \to H$ and $g_i, h_i : K \to H$ be nonlinear mappings for $i = 1,2,3$. Consider a system of generalized extended nonlinear variational inequality problem (SGENVI) as follows

: Find $x^*, y^*, z^* \in H$ such that, for all $\rho, \eta, \sigma > 0$,

$$\begin{aligned} &\langle \rho T_1(y^*,z^*,x^*) + g_1(x^*) - h_1(y^*), h_1(x) - g_1(x^*)\rangle \geq 0, \ \forall\, h_1(x) \in K, \\ &\langle \eta T_2(z^*,x^*,y^*) + g_2(y^*) - h_2(z^*), h_2(x) - g_2(y^*)\rangle \geq 0, \ \forall\, h_2(x) \in K, \\ &\langle \sigma T_3(x^*,y^*,z^*) + g_3(z^*) - h_3(x^*), h_3(x) - g_3(z^*)\rangle \geq 0, \ \forall\, h_3(x) \in K. \end{aligned} \tag{1}$$

Here ρ, η, σ are constants and play an important part in the studies of the convergence analysis.

If $g_i = h_i$ for $i = 1,2,3$, then the SGENVI problem (1) is equivalent to finding $x^*, y^*, z^* \in H$ such that, for all $\rho, \eta, \sigma > 0$,

$$\begin{aligned} &\langle \rho T_1(y^*,z^*,x^*) + g_1(x^*) - g_1(y^*), g_1(x) - g_1(x^*)\rangle \geq 0, \ \forall\, g_1(x) \in K, \\ &\langle \eta T_2(z^*,x^*,y^*) + g_2(y^*) - g_2(z^*), g_2(x) - g_2(y^*)\rangle \geq 0, \ \forall\, g_2(x) \in K, \\ &\langle \sigma T_3(x^*,y^*,z^*) + g_3(z^*) - g_3(x^*), g_3(x) - g_3(z^*)\rangle \geq 0, \ \forall\, g_3(x) \in K, \end{aligned} \tag{2}$$

where $\rho, \eta, \sigma > 0$ are constants. System (2) is studied by Cho et al. [6].

If $g_i = h_i =$ identity operator for $i = 1,2,3$, then the SGENVI problem (1) is equivalent to finding $x^*, y^*, z^* \in H$ such that, for all $\rho, \eta, \sigma > 0$,

$$\begin{aligned} &\langle \rho T_1(y^*,z^*,x^*) + x^* - y^*, x - x^*\rangle \geq 0, \ \forall\, x \in K, \\ &\langle \eta T_2(z^*,x^*,y^*) + y^* - z^*, x - y^*\rangle \geq 0, \ \forall\, x \in K, \\ &\langle \sigma T_3(x^*,y^*,z^*) + z^* - x^*, x - z^*\rangle \geq 0, \ \forall\, x \in K, \end{aligned} \tag{3}$$

where $\rho, \eta, \sigma > 0$ are constants. System (3) is studied by Shang et al. [13].

We now discuss some more special cases of the SGENVI problem (1).

Special cases :

(1) If T_3, g_3, h_3 are zero operators, g_1, g_2, h_1, h_2 are identity operators and $T_1 = T_2 = T$ is a bivariate operator from $H \times H \to H$, then the SGENVI problem (1) is equivalent to system of nonlinear variational inequality problem, studied by Chang et al. [5] and Verma [16].

(2) If T_3, g_3, h_3 are zero operators, g_1, g_2, h_1, h_2 are identity operators and T_1, T_2 are bivariate operators from $H \times H \to H$, then the SGENVI problem (1) is equivalent to system of nonlinear variational inequality involving two different operators, studied by Huang et al. [9] .

(3) If T_3, g_3, h_3 are zero operators, g_1, g_2, h_1, h_2 are identity operators and T_1, T_2 are univariate operator from $H \to H$, then the SGENVI problem (1) is equivalent to system of variational inequality problem, studied by Ceng et al.[4]

(4) If T_3, g_3, h_3 are zero operators, g_1, g_2, h_1, h_2 are identity operators and $T_1 = T_2 = T$ is a univariate operator from $H \to H$, then the SGENVI problem (1) is equivalent to system of variational inequality problem, studied by Verma [14, 17].

(5) If T_3, g_3, h_3 are zero operators, $g_1 = g_2 =$ identity operator and T_1, T_2 are bivariate operators from $H \times H \to H$, then the SGENVI problem (1) is equivalent to system of general variational inequality problem, studied by Noor et al. [12].

(6) If $T_2, T_3, g_2, g_3, h_2, h_3$ are zero operators, $g_1 = h_1 =$ identity operator and T_1 is a bivariate operator from $H \times H \to H$, then the SGENVI problem (1) is equivalent to variational inequality problem studied by Verma [15].

Let us recall the following result, which is frequently used to study the solvability of variational inequality problem :

Lemma 1. [10] *For an element $z \in H$, we have $x \in K$ and $\langle x - z, y - x \rangle \geq 0$ for all $y \in K$ if and only if $x = P_K(z)$.*

Using Lemma 1 we can see, that the SGENVI problem (1) is equivalent to the following projection problem : Find $x^*, y^*, z^* \in H$ such that

$$
\begin{aligned}
g_1(x^*) &= P_K\left[h_1(y^*) - \rho\left(T_1(y^*, z^*, x^*)\right)\right], \ \ \rho > 0, \\
g_2(y^*) &= P_K\left[h_2(z^*) - \eta\left(T_2(z^*, x^*, y^*)\right)\right], \ \ \eta > 0, \\
g_3(z^*) &= P_K\left[h_3(x^*) - \sigma\left(T_3(x^*, y^*, z^*)\right)\right], \ \ \sigma > 0.
\end{aligned} \tag{4}
$$

3. Algorithm

In this section, we consider the general three-step parallel algorithms, which can be applied to the convergence analysis using projection methods in the context of the approximation solvability of the SGENVI problems. We can rewrite relation (4) in the following way : Find

$x^*, y^*, z^* \in H$ such that

$$
\begin{aligned}
& x^* = x^* - g_1(x^*) + P_K\left[h_1(y^*) - \rho\left(T_1(y^*, z^*, x^*)\right)\right], \quad \rho > 0, \\
(5) \qquad & y^* = y^* - g_2(y^*) + P_K\left[h_2(z^*) - \eta\left(T_2(z^*, x^*, y^*)\right)\right], \quad \eta > 0, \\
& z^* = z^* - g_3(z^*) + P_K\left[h_3(x^*) - \sigma\left(T_3(x^*, y^*, z^*)\right)\right], \quad \sigma > 0.
\end{aligned}
$$

Using the formulation (5), we now suggest following general iteration method for approximation solvability of system of generalized extended variational inequalities (1):

Algorithm 1. *For any $x_0, y_0, z_0 \in H$, compute the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by the iterative process*

$$
\begin{aligned}
& z_{n+1} = (1-\gamma_n)z_n + \gamma_n\left\{z_n - g_3(z_n) + P_K\left[h_3(x_n) - \sigma\left(T_3(x_n, y_n, z_n)\right)\right]\right\}, \\
(6) \qquad & y_{n+1} = (1-\beta_n)y_n + \beta_n\left\{y_n - g_2(y_n) + P_K\left[h_2(z_n) - \eta\left(T_2(z_n, x_n, y_n)\right)\right]\right\}, \\
& x_{n+1} = (1-\alpha_n)x_n + \alpha_n\left\{x_n - g_1(x_n) + P_K\left[h_1(y_n) - \rho\left(T_1(y_n, z_n, x_n)\right)\right]\right\},
\end{aligned}
$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$ for all $n \geq 0$.

If $g_i = h_i$ for $i = 1, 2, 3$, then the Algorithm 1 reduces to the following

Algorithm 2. *For any $x_0, y_0, z_0 \in H$, compute the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by the iterative process*

$$
\begin{aligned}
& z_{n+1} = (1-\gamma_n)z_n + \gamma_n\left\{z_n - g_3(z_n) + P_K\left[g_3(x_n) - \sigma\left(T_3(x_n, y_n, z_n)\right)\right]\right\}, \\
(7) \qquad & y_{n+1} = (1-\beta_n)y_n + \beta_n\left\{y_n - g_2(y_n) + P_K\left[g_2(z_n) - \eta\left(T_2(z_n, x_n, y_n)\right)\right]\right\}, \\
& x_{n+1} = (1-\alpha_n)x_n + \alpha_n\left\{x_n - g_1(x_n) + P_K\left[g_1(y_n) - \rho\left(T_1(y_n, z_n, x_n)\right)\right]\right\},
\end{aligned}
$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$ for all $n \geq 0$.

If $g_i = h_i =$identity operator, for $i = 1, 2, 3$, then the Algorithm 1 reduces to the following

Algorithm 3. *For any $x_0, y_0, z_0 \in H$, compute the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by the iterative process*

$$
\begin{aligned}
& z_{n+1} = (1-\gamma_n)z_n + \gamma_n P_K\left[x_n - \sigma\left(T_3(x_n, y_n, z_n)\right)\right], \\
(8) \qquad & y_{n+1} = (1-\beta_n)y_n + \beta_n P_K\left[z_n - \eta\left(T_2(z_n, x_n, y_n)\right)\right], \\
& x_{n+1} = (1-\alpha_n)x_n + \alpha_n P_K\left[y_n - \rho\left(T_1(y_n, z_n, x_n)\right)\right],
\end{aligned}
$$

where $\{\alpha_n\}, \{\beta_n\}$ *and* $\{\gamma_n\}$ *are sequences in* $[0,1]$ *for all* $n \geq 0$.

One of the attractive features of above Algorithms is, its suitability for implementing on computer having three different processor. Assume that x_n, y_n and z_n are given in Algorithm 1, in order to get the new iterative points, we can set one processor of computer to compute x_{n+1}, set second processor to compute y_{n+1} and set the other processor to compute z_{n+1}. In other words, x_{n+1}, y_{n+1} and z_{n+1} are solved in parallel. Algorithm 1 is called parallel projection method. The sequential iterative methods introduced in [5, 6, 13, 16] are only suitable for implementing on the traditional single-processor computer. That is, assume that x_n, y_n and z_n are given, in order to get the new iterative points, we need to solve x_{n+1}, y_{n+1} and z_{n+1} in sequence. Therefore, in order to satisfy practical requirements of modern multiprocessor systems, the parallel iterative methods are more attractive with respect to the sequential iterative methods. We refer the interested reader to the papers [1, 2, 3, 7, 8] and references therein for more examples and ideas of the parallel iterative methods.

We now recall some definitions:

Definition 2. *A mapping* $T : H \to H$ *is said to be :*

(i) *strongly monotone, if for each* $x \in H$, *there exists a constant* $\nu > 0$ *such that*

$$\langle T(x) - T(y), x - y \rangle \geq \nu \|x - y\|^2$$

holds, for all $y \in H$;

(ii) $\phi-$*cocoercive, if for each* $x \in H$, *there exists a constant* $\phi > 0$ *such that*

$$\langle T(x) - T(y), x - y \rangle \geq \phi \|T(x) - T(y)\|^2$$

holds, for all $y \in H$;

(iii) *relaxed* $\phi-$*cocoercive, if for each* $x \in H$, *there exists a constant* $\phi > 0$ *such that*

$$\langle T(x) - T(y), x - y \rangle \geq -\phi \|T(x) - T(y)\|^2$$

holds, for all $y \in H$;

(iv) *relaxed* $(\phi,\psi)-$*cocoercive, if for each* $x \in H$*, there exists constants* $\phi > 0$ *and* $\psi > 0$ *such that*

$$\langle T(x)-T(y),x-y\rangle \geq -\phi \|T(x)-T(y)\|^2+\psi \|x-y\|^2$$

holds, for all $y \in H$*;*

(v) $\mu-$*Lipschitz continuous, if for each* $x,y \in H$*, there exists a constant* $\mu > 0$ *such that*

$$\|T(x)-T(y)\| \leq \mu \|x-y\| ,$$

(vi) *nonexpansive, if for each* $x,y \in H$*,*

$$\|T(x)-T(y)\| \leq \|x-y\| .$$

A mapping $T : H\times H\times H \to H$ *is said to be*

(vii) *relaxed* $(\phi,\psi)-$*cocoercive in the first variable, if for each* $x,x' \in H$*, there exists constants* $\phi > 0$ *and* $\psi > 0$ *such that*

$$\left\langle T(x,y,z)-T(x',y',z'),x-x'\right\rangle \geq -\phi \left\|T(x,y,z)-T(x',y',z')\right\|^2+\psi \left\|x-x'\right\|^2$$

holds, for all $y,y',z,z' \in H$*;*

(viii) $\mu-$*Lipschitz continuous in the first variable, if for each* $x,x' \in H$*, there exists a constant* $\mu > 0$ *such that*

$$\left\|T(x,y,z)-T(x',y',z')\right\| \leq \mu \left\|x-x'\right\| ,$$

for all $y,y',z,z' \in H$*.*

We now recall a lemma useful to prove the next result :

Lemma 3. [19] *Let* $\{a_n\}$ *be a non negative sequence satisfying* $a_{n+1} \leq (1-c_n)a_n + b_n$ *, with* $c_n \in [0,1]$*,* $\sum_{n=0}^{\infty} c_n = \infty$*,* $b_n = o(c_n)$*. Then* $\lim_{n\to\infty} a_n = 0$*.*

4. Main results

Theorem 4. *Let K be a closed convex subset of a real Hilbert space H. Let $T_i : H \times H \times H \to H$ be a relaxed (ϕ_i, ψ_i)-cocoercive and μ_i-Lipschitz continuous in the first variable, $g_i : K \to H$ be a relaxed (ζ_i, ω_i)-cocoercive and τ_i-Lipschitz continuous mapping, $h_i : K \to H$ be a relaxed (δ_i, λ_i)-cocoercive and ν_i-Lipschitz continuous mapping. Suppose that $x^*, y^*, z^* \in H$ are solutions of the SGENVI problem* (1) *and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in $[0,1]$. Assume that the following conditions are satisfies:*

(i) $\theta_3,\ \theta_6,\ \theta_9 < 1$,

(ii) $\Theta_{1n} = \alpha_n(1-\theta_3) - \gamma_n(\theta_7+\theta_8) \geq 0$, $\Theta_{2n} = \beta_n(1-\theta_6) - \alpha_n(\theta_1+\theta_2) \geq 0$, *and* $\Theta_{3n} = \gamma_n(1-\theta_9) - \beta_n(\theta_4+\theta_5) \geq 0$,

(iii) $\sum_{n=0}^{\infty} \Theta_{1n} = \infty$, $\sum_{n=0}^{\infty} \Theta_{2n} = \infty$ *and* $\sum_{n=0}^{\infty} \Theta_{3n} = \infty$,

where

$$\theta_1 = \sqrt{1-2\rho\psi_1+2\rho\phi_1\mu_1^2+\rho^2\mu_1^2}\,, \qquad \theta_2 = \sqrt{1-2\lambda_1+2\delta_1\nu_1^2+\nu_1^2}\,,$$
$$\theta_3 = \sqrt{1-2\omega_1+2\zeta_1\tau_1^2+\tau_1^2}\,, \qquad \theta_4 = \sqrt{1-2\eta\psi_2+2\eta\phi_2\mu_2^2+\eta^2\mu_2^2}\,,$$
$$\theta_5 = \sqrt{1-2\lambda_2+2\delta_2\nu_2^2+\nu_2^2}\,, \qquad \theta_6 = \sqrt{1-2\omega_2+2\zeta_2\tau_2^2+\tau_2^2}\,,$$
$$\theta_7 = \sqrt{1-2\sigma\psi_3+2\sigma\phi_3\mu_3^2+\sigma^2\mu_3^2}\,, \qquad \theta_8 = \sqrt{1-2\lambda_3+2\delta_3\nu_3^2+\nu_3^2}\,,$$
$$\theta_9 = \sqrt{1-2\omega_3+2\zeta_3\tau_3^2+\tau_3^2}\,.$$

Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ generated by Algorithm 1, converges strongly to x^*, y^* and z^*, respectively.

Proof. Since x^*, y^* and z^* are solution to the SGENVI problem (1), we have from (5) that

$$z^* = (1-\gamma_n)z^* + \gamma_n\{z^* - g_3(z^*) + P_K[h_3(x^*) - \sigma(T_3(x^*,y^*,z^*))]\}\,,$$
$$y^* = (1-\beta_n)y^* + \beta_n\{y^* - g_2(y^*) + P_K[h_2(z^*) - \eta(T_2(z^*,x^*,y^*))]\}\,,$$
$$x^* = (1-\alpha_n)x^* + \alpha_n\{x^* - g_1(x^*) + P_K[h_1(y^*) - \rho(T_1(y^*,z^*,x^*))]\}\,.$$

In view of (6), we obtain that

$$\begin{aligned}
\|x_{n+1}-x^*\| &= \|(1-\alpha_n)x_n+\alpha_n\{x_n-g_1(x_n)+P_K[h_1(y_n)-\rho\,(T_1(y_n,z_n,x_n))]\}-x^*\| \\
&= \|(1-\alpha_n)x_n+\alpha_n\{x_n-g_1(x_n)+P_K[h_1(y_n)-\rho\,(T_1(y_n,z_n,x_n))]\} \\
&\qquad -[(1-\alpha_n)x^*+\alpha_n\{x^*-g_1(x^*)+P_K[h_1(y^*)-\rho\,(T_1(y^*,z^*,x^*))]\}]\| \\
&\le (1-\alpha_n)\|x_n-x^*\|+\alpha_n\|x_n-x^*-\{g_1(x_n)-g_1(x^*)\}\| \\
&\quad +\alpha_n\|P_K[h_1(y_n)-\rho\,(T_1(y_n,z_n,x_n))]-P_K[h_1(y^*)-\rho\,(T_1(y^*,z^*,x^*))]\| \\
&\le (1-\alpha_n)\|x_n-x^*\|+\alpha_n\|x_n-x^*-\{g_1(x_n)-g_1(x^*)\}\| \\
&\quad +\alpha_n\|h_1(y_n)-h_1(y^*)-\rho\{T_1(y_n,z_n,x_n)-T_1(y^*,z^*,x^*)\}\| \\
&\le (1-\alpha_n)\|x_n-x^*\|+\alpha_n\|x_n-x^*-\{g_1(x_n)-g_1(x^*)\}\| \\
&\quad +\alpha_n\|y_n-y^*-\{h_1(y_n)-h_1(y^*)\}\| \\
&\quad +\alpha_n\|y_n-y^*-\rho\{T_1(y_n,z_n,x_n)-T_1(y^*,z^*,x^*)\}\| \qquad (9)
\end{aligned}$$

By the assumption that T_1 is relaxed (ϕ_1,ψ_1)-cocoercive and μ_1-Lipschitz continuous in the first variable, we obtain

$$\begin{aligned}
&\|y_n-y^*-\rho\{T_1(y_n,z_n,x_n)-T_1(y^*,z^*,x^*)\}\|^2 \\
&\quad = \|y_n-y^*\|^2-2\rho\langle T_1(y_n,z_n,x_n)-T_1(y^*,z^*,x^*),y_n-y^*\rangle \\
&\qquad +\rho^2\|T_1(y_n,z_n,x_n)-T_1(y^*,z^*,x^*)\|^2 \\
&\quad \le \|y_n-y^*\|^2-2\rho\left\{-\phi_1\|T_1(y_n,z_n,x_n)-T_1(y^*,z^*,x^*)\|^2+\psi_1\|y_n-y^*\|^2\right\} \\
&\qquad +\rho^2\mu_1^2\|y_n-y^*\|^2 \\
&\quad \le \|y_n-y^*\|^2+2\rho\phi_1\mu_1^2\|y_n-y^*\|^2-2\rho\psi_1\|y_n-y^*\|^2+\rho^2\mu_1^2\|y_n-y^*\|^2 \\
&\quad = (1-2\rho\psi_1+2\rho\phi_1\mu_1^2+\rho^2\mu_1^2)\|y_n-y^*\|^2 \\
&\quad = \theta_1^2\|y_n-y^*\|^2, \qquad (10)
\end{aligned}$$

where $\theta_1=\sqrt{1-2\rho\psi_1+2\rho\phi_1\mu_1^2+\rho^2\mu_1^2}$.

Also, since h_1 is relaxed (δ_1, λ_1)-cocoercive and ν_1-Lipschitz continuous, we have

$$
\begin{aligned}
&\|y_n - y^* - (h_1(y_n) - h_1(y^*))\|^2 \\
&\quad = \|y_n - y^*\|^2 - 2\langle h_1(y_n) - h_1(y^*), y_n - y^*\rangle + \|h_1(y_n) - h_1(y^*)\|^2 \\
&\quad \leq \|y_n - y^*\|^2 - 2\left\{-\delta_1 \|h_1(y_n) - h_1(y^*)\|^2 + \lambda_1 \|y_n - y^*\|^2\right\} \\
&\qquad + \nu_1^2 \|y_n - y^*\|^2 \\
&\quad \leq \left(1 - 2\lambda_1 + 2\delta_1 \nu_1^2 + \nu_1^2\right) \|y_n - y^*\|^2 \\
&\quad = \theta_2^2 \|y_n - y^*\|^2, \qquad (11)
\end{aligned}
$$

where $\theta_2 = \sqrt{1 - 2\lambda_1 + 2\delta_1 \nu_1^2 + \nu_1^2}$.

Similarly, since g_1 is relaxed (ζ_1, ω_1)-cocoercive and τ_1-Lipschitz continuous, we have

$$
\begin{aligned}
&\|x_n - x^* - \{g_1(x_n) - g_1(x^*)\}\|^2 \\
&\quad = \|x_n - x^*\|^2 - 2\langle g_1(x_n) - g_1(x^*), x_n - x^*\rangle + \|g_1(x_n) - g_1(x^*)\|^2 \\
&\quad \leq \|x_n - x^*\|^2 - 2\left\{-\zeta_1 \|g_1(x_n) - g_1(x^*)\|^2 + \omega_1 \|x_n - x^*\|^2\right\} \\
&\qquad + \tau_1^2 \|x_n - x^*\|^2 \\
&\quad \leq \left(1 - 2\omega_1 + 2\zeta_1 \tau_1^2 + \tau_1^2\right) \|x_n - x^*\|^2 \\
&\quad = \theta_3^2 \|x_n - x^*\|^2, \qquad (12)
\end{aligned}
$$

where $\theta_3 = \sqrt{1 - 2\omega_1 + 2\zeta_1 \tau_1^2 + \tau_1^2}$.

Substituting (10), (11) and (12) into (9), we get

$$
\begin{aligned}
\|x_{n+1} - x^*\| &\leq (1 - \alpha_n + \alpha_n \theta_3) \|x_n - x^*\| + \alpha_n(\theta_1 + \theta_2) \|y_n - y^*\| \\
&= (1 - \alpha_n(1 - \theta_3)) \|x_n - x^*\| + \alpha_n(\theta_1 + \theta_2) \|y_n - y^*\|. \qquad (13)
\end{aligned}
$$

Again, using (6), we obtain

$$\begin{aligned}
\|y_{n+1}-y^*\| &\le \|(1-\beta_n)y_n+\beta_n\{y_n-g_2(y_n)+P_K[h_2(z_n)-\eta(T_2(z_n,x_n,y_n))]\}-y^*\| \\
&= \|(1-\beta_n)y_n+\beta_n\{y_n-g_2(y_n)+P_K[h_2(z_n)-\eta(T_2(z_n,x_n,y_n))]\} \\
&\quad -[(1-\beta_n)y^*+\beta_n\{y^*-g_2(y^*)+P_K[h_2(z^*)-\eta(T_2(z^*,x^*,y^*))]\}]\| \\
&\le (1-\beta_n)\|y_n-y^*\|+\beta_n\|y_n-y^*-\{g_2(y_n)-g_2(y^*)\}\| \\
&\quad +\beta_n\|P_K[h_2(z_n)-\eta(T_2(z_n,x_n,y_n))]-P_K[h_2(z^*)-\eta(T_2(z^*,x^*,y^*))]\| \\
&\le (1-\beta_n)\|y_n-y^*\|+\beta_n\|y_n-y^*-\{g_2(y_n)-g_2(y^*)\}\| \\
&\quad +\beta_n\|h_2(z_n)-h_2(z^*)-\eta\{T_2(z_n,x_n,y_n)-T_2(z^*,x^*,y^*)\}\| \\
&\le (1-\beta_n)\|y_n-y^*\|+\beta_n\|y_n-y^*-\{g_2(y_n)-g_2(y^*)\}\| \\
&\quad +\beta_n\|z_n-z^*-\{h_2(z_n)-h_2(z^*)\}\| \\
&\quad +\beta_n\|z_n-z^*-\eta\{T_2(z_n,x_n,y_n)-T_2(z^*,x^*,y^*)\}\|\,. \qquad (14)
\end{aligned}$$

By the assumption that T_2 is relaxed (ϕ_2,ψ_2)-cocoercive and μ_2-Lipschitz continuous in the first variable, we obtain

$$\|z_n-z^*-\eta\{T_2(z_n,x_n,y_n)-T_2(z^*,x^*,y^*)\}\| \le \theta_4\|z_n-z^*\|\,, \qquad (15)$$

where $\theta_4=\sqrt{1-2\eta\psi_2+2\eta\phi_2\mu_2^2+\eta^2\mu_2^2}$.

Since h_2 is relaxed (δ_2,λ_2)-cocoercive and ν_2-Lipschitz continuous, we have

$$\|z_n-z^*-\{h_2(z_n)-h_2(z^*)\}\| \le \theta_5\|z_n-z^*\|\,, \qquad (16)$$

where $\theta_5=\sqrt{1-2\lambda_2+2\delta_2\nu_2^2+\nu_2^2}$.

Similarly, since g_2 is relaxed (ζ_2,ω_2)-cocoercive and τ_2-Lipschitz continuous, we have

$$\|y_n-y^*-\{g_2(y_n)-g_2(y^*)\}\| \le \theta_6\|y_n-y^*\|\,, \qquad (17)$$

where $\theta_6=\sqrt{1-2\omega_2+2\zeta_2\tau_2^2+\tau_2^2}$.

Substituting (15), (16) and (17) into (14), we have

$$\|y_{n+1}-y^*\| \le (1-\beta_n(1-\theta_6))\,\|y_n-y^*\|+\beta_n(\theta_4+\theta_5)\,\|z_n-z^*\|\,. \tag{18}$$

Again, using (6), and the similar argument as above, we get that

$$\|z_{n+1}-z^*\| \le (1-\gamma_n(1-\theta_9))\,\|z_n-z^*\|+\gamma_n(\theta_7+\theta_8)\,\|x_n-x^*\|\,, \tag{19}$$

where

$$\theta_7=\sqrt{1-2\sigma\psi_3+2\sigma\phi_3\mu_3^2+\sigma^2\mu_3^2}\,,$$
$$\theta_8=\sqrt{1-2\lambda_3+2\delta_3\nu_3^2+\nu_3^2}\,,$$
$$\theta_9=\sqrt{1-2\omega_3+2\zeta_3\tau_3^2+\tau_3^2}\,.$$

From (13), (18) and (19), it follows that

$$\begin{aligned}
&\|x_{n+1}-x^*\|+\|y_{n+1}-y^*\|+\|z_{n+1}-z^*\|\\
&\le [1-\{\alpha_n(1-\theta_3)-\gamma_n(\theta_7+\theta_8)\}]\,\|x_n-x^*\|\\
&\quad+[1-\{\beta_n(1-\theta_6)-\alpha_n(\theta_1+\theta_2)\}]\,\|y_n-y^*\|\\
&\quad+[1-\{\gamma_n(1-\theta_9)-\beta_n(\theta_4+\theta_5)\}]\,\|z_n-z^*\|\\
&\le \max\{(1-\Theta_{1n}),(1-\Theta_{2n}),(1-\Theta_{3n})\}\,[\|x_n-x^*\|+\|y_n-y^*\|+\|z_n-z^*\|]\,, \qquad (20)
\end{aligned}$$

where

$$\Theta_{1n}=\{\alpha_n(1-\theta_3)-\gamma_n(\theta_7+\theta_8)\}\,,$$
$$\Theta_{2n}=\{\beta_n(1-\theta_6)-\alpha_n(\theta_1+\theta_2)\}\,,$$
$$\Theta_{3n}=\{\gamma_n(1-\theta_9)-\beta_n(\theta_4+\theta_5)\}\,.$$

Now, define the norm $\|\cdot\|_1$ on $H\times H\times H$ by

$$\|(x,y,z)\|_1=\|x\|+\|y\|+\|z\|\,,\quad \forall\,(x,y,z)\in H\times H\times H\,.$$

Then $(H\times H\times H,\|\cdot\|_1)$ is a Banach space. Hence, (20) implies that

$$\begin{aligned}&\|(x_{n+1},y_{n+1},z_{n+1})-(x^*,y^*,z^*)\|_1\\ &\leq \max\{(1-\Theta_{1n}),(1-\Theta_{2n}),(1-\Theta_{3n})\}\|(x_n,y_n,z_n)-(x^*,y^*,z^*)\|_1 .\end{aligned} \tag{21}$$

By assumption $0\leq\alpha_n,\beta_n,\gamma_n\leq 1$ and $\Theta_1,\Theta_2,\Theta_3\geq 0$ such that $\sum_{n=1}^{\infty}\Theta_{1n}=\infty$, $\sum_{n=1}^{\infty}\Theta_{2n}=\infty$ and $\sum_{n=1}^{\infty}\Theta_{3n}=\infty$.

By Lemma 3, we get

$$\lim_{n\to\infty}\|(x_n,y_n,z_n)-(x^*,y^*,z^*)\|_1=0 .$$

Therefore, sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ converges to x^*, y^* and z^* respectively.

This completes the proof. □

From Theorem 4, we get the following results immediately :

Corollary 5. *Let K be a closed convex subset of a real Hilbert space H. Let $T_i:H\times H\times H\to H$ be a relaxed (ϕ_i,ψ_i)-cocoercive and μ_i-Lipschitz continuous in the first variable, $g_i:K\to H$ be a relaxed (ζ_i,ω_i)-cocoercive and τ_i-Lipschitz continuous mapping. Suppose that $x^*,y^*,z^*\in H$ are solutions of the problem (2) and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in $[0,1]$. Assume that the following conditions are satisfies:*

(i) $\theta_3,\ \theta_6,\ \theta_9<1$,

(ii) $\Theta_{1n}=\alpha_n(1-\theta_3)-\gamma_n\theta_7\geq 0$, $\Theta_{2n}=\beta_n(1-\theta_6)-\alpha_n\theta_1\geq 0$, *and* $\Theta_{3n}=\gamma_n(1-\theta_9)-\beta_n\theta_4\geq 0$,

(iii) $\sum_{n=0}^{\infty}\Theta_{1n}=\infty$, $\sum_{n=0}^{\infty}\Theta_{2n}=\infty$ *and* $\sum_{n=0}^{\infty}\Theta_{3n}=\infty$,

where

$$\begin{aligned}\theta_1&=\sqrt{1-2\rho\psi_1+2\rho\phi_1\mu_1^2+\rho^2\mu_1^2}\,, &\theta_3&=\sqrt{1-2\omega_1+2\zeta_1\tau_1^2+\tau_1^2}\,,\\ \theta_4&=\sqrt{1-2\eta\psi_2+2\eta\phi_2\mu_2^2+\eta^2\mu_2^2}\,, &\theta_6&=\sqrt{1-2\omega_2+2\zeta_2\tau_2^2+\tau_2^2}\,,\\ \theta_7&=\sqrt{1-2\sigma\psi_3+2\sigma\phi_3\mu_3^2+\sigma^2\mu_3^2}\,, &\theta_9&=\sqrt{1-2\omega_3+2\zeta_3\tau_3^2+\tau_3^2}\,.\end{aligned}$$

Then the sequence $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ generated by Algorithm 2, converges strongly to x^*,y^* and z^*, respectively.

Corollary 6. *Let K be a closed convex subset of a real Hilbert space H. Let $T_i : H \times H \times H \to H$ be a relaxed (ϕ_i, ψ_i)-cocoercive and μ_i-Lipschitz continuous in the first variable. Suppose that $x^*, y^*, z^* \in H$ are solutions of the problem* (3) *and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in $[0,1]$. Assume that the following conditions are satisfies:*

(i) $\Theta_{1n} = \alpha_n - \gamma_n\theta_7 \geq 0$, $\Theta_{2n} = \beta_n - \alpha_n\theta_1 \geq 0$, *and* $\Theta_{3n} = \gamma_n - \beta_n\theta_4 \geq 0$,

(ii) $\sum_{n=0}^{\infty} \Theta_{1n} = \infty$, $\sum_{n=0}^{\infty} \Theta_{2n} = \infty$ *and* $\sum_{n=0}^{\infty} \Theta_{3n} = \infty$,

where

$$\theta_1 = \sqrt{1 - 2\rho\psi_1 + 2\rho\phi_1\mu_1^2 + \rho^2\mu_1^2}\,, \qquad \theta_4 = \sqrt{1 - 2\eta\psi_2 + 2\eta\phi_2\mu_2^2 + \eta^2\mu_2^2}\,,$$
$$\theta_7 = \sqrt{1 - 2\sigma\psi_3 + 2\sigma\phi_3\mu_3^2 + \sigma^2\mu_3^2}\,.$$

Then the sequence $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ generated by Algorithm 3, converges strongly to x^*, y^* and z^*, respectively.

5. Algorithms for common elements

Now, we consider, based on the projection method, the approximation solvability of a system of generalized extended nonlinear variational inequality problem with nine different mappings which is also a common fixed point of three nonexpansive mappings in the framework of Hilbert spaces.

We propose a general three-step model for the projection methods, which can be applied to the convergence analysis of the approximation solvability of the SGENVI problem (1) and common fixed point of three nonexpansive mappings.

Algorithm 4. *For any $x_0, y_0, z_0 \in K$, compute the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by the iterative process*

$$\begin{aligned} z_{n+1} &= (1-\gamma_n)z_n + \gamma_n S_3\left\{z_n - g_3(z_n) + P_K\left[h_3(x_n) - \sigma\left(T_3(x_n, y_n, z_n)\right)\right]\right\}, \\ y_{n+1} &= (1-\beta_n)y_n + \beta_n S_2\left\{y_n - g_2(y_n) + P_K\left[h_2(z_n) - \eta\left(T_2(z_n, x_n, y_n)\right)\right]\right\}, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n S_1\left\{x_n - g_1(x_n) + P_K\left[h_1(y_n) - \rho\left(T_1(y_n, z_n, x_n)\right)\right]\right\}, \end{aligned} \tag{22}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ *are sequences in* $[0,1]$ *for all* $n \geq 0$ *and* S_1, S_2, S_3 *are nonexpansive mappings.*

If $g_i = h_i$ for $i = 1,2,3$ then the Algorithm 4 reduces to the following

Algorithm 5. *For any* $x_0, y_0, z_0 \in H$*, compute the sequences* $\{x_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *by the iterative process*

$$
\begin{aligned}
z_{n+1} &= (1-\gamma_n)z_n + \gamma_n S_3 \left\{z_n - g_3(z_n) + P_K\left[g_3(x_n) - \sigma\left(T_3(x_n,y_n,z_n)\right)\right]\right\}, \\
y_{n+1} &= (1-\beta_n)y_n + \beta_n S_2 \left\{y_n - g_2(y_n) + P_K\left[g_2(z_n) - \eta\left(T_2(z_n,x_n,y_n)\right)\right]\right\}, \\
x_{n+1} &= (1-\alpha_n)x_n + \alpha_n S_1 \left\{x_n - g_1(x_n) + P_K\left[g_1(y_n) - \rho\left(T_1(y_n,z_n,x_n)\right)\right]\right\},
\end{aligned}
\tag{23}
$$

where $\{\alpha_n\}, \{\beta_n\}$, $\{\gamma_n\}$ *are sequences in* $[0,1]$ *for all* $n \geq 0$ *and* S_1, S_2, S_3 *are nonexpansive mappings.*

If $g_i = h_i =$identity operator, for $i = 1,2,3$ then the Algorithm 4 reduces to the following

Algorithm 6. *For any* $x_0, y_0, z_0 \in H$*, compute the sequences* $\{x_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *by the iterative process*

$$
\begin{aligned}
z_{n+1} &= (1-\gamma_n)z_n + \gamma_n S_3 P_K\left[x_n - \sigma\left(T_3(x_n,y_n,z_n)\right)\right], \\
y_{n+1} &= (1-\beta_n)y_n + \beta_n S_2 P_K\left[z_n - \eta\left(T_2(z_n,x_n,y_n)\right)\right], \\
x_{n+1} &= (1-\alpha_n)x_n + \alpha_n S_1 P_K\left[y_n - \rho\left(T_1(y_n,z_n,x_n)\right)\right],
\end{aligned}
\tag{24}
$$

where $\{\alpha_n\}, \{\beta_n\}$, $\{\gamma_n\}$ *are sequences in* $[0,1]$ *for all* $n \geq 0$ *and* S_1, S_2, S_3 *are nonexpansive mappings.*

Theorem 7. *Let* K *be a closed convex subset of a real Hilbert space* H*. For* $i = 1,2,3$*, let* T_i*,* g_i*,* h_i *be as in Theorem 4 and* $S_i : K \to K$ *be a nonexpansive mapping. Suppose that* $x^*, y^*, z^* \in H$ *are solutions of the SGENVI problem* (1) *also* $x^*, y^*, z^* \in \cap_{i=1}^{3} F(S_i)$ *and* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ *are sequences in* $[0,1]$*. Assume that the conditions (i), (ii) and (iii) of Theorem 4 are satisfied. Then the sequence* $\{x_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *generated by Algorithm 4, converges strongly to* x^*, y^* *and* z^**, respectively.*

Proof. Since x^*, y^* and z^* are common elements of the set of solution to the SGENVI problem (1) and the set of common fixed points of S_1, S_2 and S_3, we have from (5) that

$$z^* = S_3\{z^* - g_3(z^*) + P_K[h_3(x^*) - \sigma(T_3(x^*, y^*, z^*))]\},$$

$$y^* = S_2\{y^* - g_2(y^*) + P_K[h_2(z^*) - \eta(T_2(z^*, x^*, y^*))]\},$$

$$x^* = S_3\{x^* - g_1(x^*) + P_K[h_1(y^*) - \rho(T_1(y^*, z^*, x^*))]\}.$$

Therefore,

$$z^* = (1-\gamma_n)z^* + \gamma_n S_3\{z^* - g_3(z^*) + P_K[h_3(x^*) - \sigma(T_3(x^*, y^*, z^*))]\},$$

$$y^* = (1-\beta_n)y^* + \beta_n S_2\{y^* - g_2(y^*) + P_K[h_2(z^*) - \eta(T_2(z^*, x^*, y^*))]\},$$

$$x^* = (1-\alpha_n)x^* + \alpha_n S_1\{x^* - g_1(x^*) + P_K[h_1(y^*) - \rho(T_1(y^*, z^*, x^*))]\}.$$

Using (22), we obtain

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|(1-\alpha_n)x_n + \alpha_n S_1\{x_n - g_1(x_n) + P_K[h_1(y_n) - \rho(T_1(y_n, z_n, x_n))]\} - x^*\| \\
&= \|(1-\alpha_n)x_n + \alpha_n\{x_n - g_1(x_n) + P_K[h_1(y_n) - \rho(T_1(y_n, z_n, x_n))]\} \\
&\qquad - [(1-\alpha_n)x^* + \alpha_n S_1\{x^* - g_1(x^*) + P_K[h_1(y^*) - \rho(T_1(y^*, z^*, x^*))]\}]\| \\
&\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|S_1\{x_n - g_1(x_n) + P_K[h_1(y_n) - \rho(T_1(y_n, z_n, x_n))]\} \\
&\qquad + S_1\{x^* - g_1(x^*) + P_K[h_1(y^*) - \rho(T_1(y^*, z^*, x^*))]\}\| \\
&\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|\{x_n - g_1(x_n) + P_K[h_1(y_n) - \rho(T_1(y_n, z_n, x_n))]\} \\
&\qquad + \{x^* - g_1(x^*) + P_K[h_1(y^*) - \rho(T_1(y^*, z^*, x^*))]\}\| \\
&\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|x_n - x^* - \{g_1(x_n) - g_1(x^*)\}\| \\
&\quad + \alpha_n\|P_K[h_1(y_n) - \rho(T_1(y_n, z_n, x_n))] - P_K[h_1(y^*) - \rho(T_1(y^*, z^*, x^*))]\| \\
&\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|x_n - x^* - \{g_1(x_n) - g_1(x^*)\}\| \\
&\quad + \alpha_n\|h_1(y_n) - h_1(y^*) - \rho\{T_1(y_n, z_n, x_n) - T_1(y^*, z^*, x^*)\}\| \\
&\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|x_n - x^* - \{g_1(x_n) - g_1(x^*)\}\| \\
&\quad + \alpha_n\|y_n - y^* - \{h_1(y_n) - h_1(y^*)\}\| \\
&\quad + \alpha_n\|y_n - y^* - \rho\{T_1(y_n, z_n, x_n) - T_1(y^*, z^*, x^*)\}\|. \qquad (25)
\end{aligned}$$

Using (10), (11) and (12), we get from (25), that

$$\|x_{n+1}-x^*\| \leq (1-\alpha_n(1-\theta_3))\,\|x_n-x^*\| + \alpha_n(\theta_1+\theta_2)\,\|y_n-y^*\|\,, \tag{26}$$

In a similar way, we get

$$\|y_{n+1}-y^*\| \leq (1-\beta_n(1-\theta_6))\,\|y_n-y^*\| + \beta_n(\theta_4+\theta_5)\,\|z_n-z^*\|\,, \tag{27}$$

and

$$\|z_{n+1}-z^*\| \leq (1-\gamma_n(1-\theta_9))\,\|z_n-z^*\| + \gamma_n(\theta_7+\theta_8)\,\|x_n-x^*\|\,, \tag{28}$$

where $\theta_i,\ i=1,2,\ldots,9$ are as in Theorem 4.

Using the arguments as in the proof of Theorem 4, we get that, sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ converges to x^*, y^* and z^*. respectively.

□

We immediately obtain following results from the Theorem 7.

Corollary 8. *Let K be a closed convex subset of a real Hilbert space H. For $i=1,2,3$, let T_i, g_i be as in Theorem 4 and $S_i : K \to K$ be a nonexpansive mapping. Suppose that $x^*, y^*, z^* \in H$ are solutions of the problem* (2) *also $x^*, y^*, z^* \in \cap_{i=1}^{3} F(S_i)$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in* $[0,1]$. *Assume that the conditions (i), (ii) and (iii) of Corollary 5 are satisfied. Then the sequence $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ generated by Algorithm 5, converges strongly to x^*, y^* and z^*, respectively.*

Corollary 9. *Let K be a closed convex subset of a real Hilbert space H. For $i=1,2,3$, let T_i be as in Theorem 4 and $S_i : K \to K$ be a nonexpansive mapping. Suppose that $x^*, y^*, z^* \in H$ are solutions of the problem* (3) *also $x^*, y^*, z^* \in \cap_{i=1}^{3} F(S_i)$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in* $[0,1]$. *Assume that the conditions (i) and (ii) of Corollary 6 are satisfied. Then the sequence $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ generated by Algorithm 6, converges strongly to x^*, y^* and z^*, respectively.*

6. Conclusion

In this paper, we have introduced and considered a new system of generalized extended nonlinear variational inequalities involving three different trivariate operators and six univariate operators. We have established the equivalence between generalized extended nonlinear variational inequality and fixed point problem using projection mapping. Using this equivalence, we suggest and analyze some iterative methods for approximate solvability of the system of generalized extended nonlinear variational inequalities. Iterative methods to find common element of fixed point set of three different nonexpansive mapping and solution set of the system of generalized extended nonlinear variational inequalities has also been studied and analyzed. Several special cases are as well discussed.

Conflict of Interests

The authors declare that there is no conflict of interests.

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