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GENERALIZED n-TUPLED FIXED POINT THEOREMS IN PARTIALLY ORDERED METRIC SPACES INVOLVING AN ICS MAP

SHAMSHAD HUSAIN* AND HUMA SAHPER

Department of Applied Mathematics, Aligarh Muslim University, Aligarh 202002, India

Abstract. In this paper, we prove results on n-tupled fixed point (for even n) for mapping having the mixed monotone property in partially ordered metric spaces involving an ICS map. These results are the generalizations of the main results of Luong et al. (Bull. Math. Anal. Appl. 3, 129-140, 2011).

Keywords: Partially ordered set; metric space; mixed monotone property; ICS map; n-tupled fixed point.

2000 AMS Subject Classification: 54H10; 54H25

1. Introduction

Fixed point theory is one of the famous and traditional theories in mathematics and has large number of applications. The Banach contraction principle is one of the pivotal results of analysis. It is widely considered as the source of metric fixed point theory. Also, its significance lies in its vast applicability in a number of branches of mathematics. Generalization of the above principle has been a heavily investigated branch of mathematics. Existence of fixed points in partially ordered metric spaces was investigated in 2004 by

*Corresponding author

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Ran and Reurings [13] and then by Nieto and Lopez [14]. For some other results in partially ordered metric spaces, one can be referred to (cf. [1]-[19]) and the references cited therein.

In [1], Bhaskar and Lakshmikantham introduced the concept of a mixed monotone mapping and proved some coupled fixed point theorems for a mixed monotone mapping in partially ordered complete metric spaces. Later, various results on coupled fixed point have been obtained (see, for details, [1]-[11]).

Recently, Imdad et al. [12] introduced the concepts of n-tupled coincidence as well as n-tupled fixed point and utilize these two definitions to obtain n-tupled coincidence as well as n-tupled common fixed point theorems for nonlinear mappings satisfying ϕ-contraction condition in partially ordered complete metric spaces.

The purpose of this paper is to prove some n-tupled fixed point theorems for mapping having the mixed monotone property in partially ordered complete metric spaces involving an ICS map.

2. Preliminaries

We begin with the following definitions and results related to coupled fixed point in metric spaces.

Definition 2.1. [1] Let $(X, \preceq)$ be a partially ordered set equipped with a metric d such that (X, d) is a metric space. We endow the product space $X \times X$ with the following partial ordering:

$$\text{for } (x, y), (u, v) \in X \times X, \text{ define } (u, v) \preceq (x, y) \Leftrightarrow x \succeq u, y \preceq v.$$

Definition 2.2. Let $(X, \preceq)$ be a partially ordered set and $F : X \to X$ be a mapping. Then F is said to be nondecreasing if for all $x_1, x_2 \in X,\ x_1 \preceq x_2$ implies $F(x_1) \preceq F(x_2)$ and nonincreasing if for all $x_1, x_2 \in X,\ x_1 \preceq x_2$ implies $F(x_1) \succeq F(x_2)$.

Definition 2.3. [1] Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$ be a mapping. Then F is said to have mixed monotone property if for any $x, y \in X$, $F(x, y)$ is monotonically nondecreasing in first argument and monotonically nonincreasing in second argument, that is, for

$$x_1, x_2 \in X,\ x_1 \preceq x_2 \Rightarrow F(x_1, y) \preceq F(x_2, y),$$

$$y_1, y_2 \in X,\ y_1 \preceq y_2 \Rightarrow F(x, y_1) \succeq F(x, y_2).$$

Definition 2.4. [1] An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \to X$ if

$$F(x, y) = x \text{ and } F(y, x) = y.$$

Definition 2.5. [3] Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be ICS if T is injective, continuous and has the property: for every sequence $\{x_n\}$ in X, if $\{Tx_n\}$ is convergent then $\{x_n\}$ is also convergent.

Also Luong et al. [3] proved the following result:

Let Φ be the set of all functions $\phi : [0, \infty) \to [0, \infty)$ such that:

(i) $\phi(t) < t$ for all $t > 0$,

(ii) $\lim_{r \to t^+} \phi(r) < t$ for all $t > 0$.

Theorem 2.1. [3] *Let $(X, \preceq)$ be a partially ordered set and suppose there exists a metric d on X such that (X, d) is a complete metric space and $T : X \to X$ is an ICS mapping. Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X such that there exist two elements $x_0, y_0 \in X$ with $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$. Suppose there exists $\varphi \in \Phi$ such that*

$$d(TF(x, y), TF(u, v)) \leq \frac{1}{2}\varphi(d(Tx, Tu) + d(Ty, Tv))$$

for all $x, y, u, v \in X$ for which $x \succeq u$ and $y \preceq v$. Suppose either

(a) F is continuous or

(b) X has the following property:

(i) if a nondecreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all n;

(ii) if a nonincreasing sequence $\{y_n\} \to y$, then $y_n \succeq y$ for all n.

Then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$ i.e., F has a coupled fixed point in X.

Throughout the paper, we consider n to be an even positive integer.

For simplicity we will denote the cross product of $n \in \mathbb{N}$ copies of the space X by X^n.

Definition 2.6. [12] Let $(X, \preceq)$ be a partially ordered set and $F : X^n \to X$ be a mapping. The mapping F is said to have the mixed monotone property if F is nondecreasing in its odd position arguments and nonincreasing in its even position arguments, that is, if,

for all $x_1^1, x_2^1 \in X$, $x_1^1 \preceq x_2^1 \Rightarrow F(x_1^1, x^2, x^3, ..., x^n) \preceq F(x_2^1, x^2, x^3, ..., x^n)$

for all $x_1^2, x_2^2 \in X$, $x_1^2 \preceq x_2^2 \Rightarrow F(x^1, x_1^2, x^3, ..., x^n) \succeq F(x^1, x_2^2, x^3, ..., x^n)$

for all $x_1^3, x_2^3 \in X$, $x_1^3 \preceq x_2^3 \Rightarrow F(x^1, x^2, x_1^3, ..., x^n) \preceq F(x^1, x^2, x_2^3, ..., x^n)$

$$\vdots$$

for all $x_1^n, x_2^n \in X$, $x_1^n \preceq x_2^n \Rightarrow F(x^1, x^2, x^3, ..., x_1^n) \succeq F(x^1, x^2, x^3, ..., x_2^n)$.

Definition 2.7. [12] An element $(x^1, x^2, ..., x^n) \in X^n$ is called an n-tupled fixed point of the mapping $F : X^n \to X$ if

$$\begin{cases} F(x^1, x^2, x^3, ..., x^n) = x^1 \\ F(x^2, x^3, ..., x^n, x^1) = x^2 \\ F(x^3, ..., x^n, x^1, x^2) = x^3 \\ \vdots \\ F(x^n, x^1, x^2, ..., x^{n-1}) = x^n. \end{cases}$$

Example 2.1. Let (R, d) be a partially ordered metric space under natural setting and let $F : R^n \to R$ be a mapping defined by $F(x^1, x^2, x^3, ..., x^n) = \sin\,(x^1.x^2.x^3...x^n)$, for any $x^1, x^2, ..., x^n \in R$. Then $(0, 0, ..., 0)$ is an n-tupled fixed point of F.

Remark 2.1. Definition 2.7 with $n = 2$ yields the definition of coupled fixed point.

3. Main results

Now our main result is as follows:

Theorem 3.1. *Let $(X, \preceq, d)$ be a complete partially ordered metric space and $T : X \to X$ be an ICS mapping. Let $F : X^n \to X$ be a map enjoying the mixed monotone property on X. Suppose that there exists $\varphi \in \Phi$ such that*

$$d(TF(x^1, x^2, ..., x^n), TF(y^1, y^2, ..., y^n)) \leq \tfrac{1}{n}\varphi(d(Tx^1, Ty^1) + d(Tx^2, Ty^2) + ... + d(Tx^n, Ty^n)) \qquad (3.1)$$

for all $x^1, x^2, ..., x^n,\ y^1, y^2, ..., y^n \in X$ for which $x^1 \succeq y^1,\ x^2 \preceq y^2,\ x^3 \succeq y^3, ..., x^n \preceq y^n$. Suppose either

(a) F is continuous or

(b) X has the following property:

(i) if nondecreasing sequence $\{x_m\} \to x$, then $x_m \preceq x$ for all m;

(ii) if nonincreasing sequence $\{x_m\} \to x$, then $x_m \succeq x$ for all m.

If there exist $x_0^1, x_0^2, x_0^3, ..., x_0^n \in X$ such that

$$\begin{cases} x_0^1 \preceq F(x_0^1, x_0^2, x_0^3, ..., x_0^n) \\ x_0^2 \succeq F(x_0^2, x_0^3, ..., x_0^n, x_0^1) \\ x_0^3 \preceq F(x_0^3, ..., x_0^n, x_0^1, x_0^2) \\ \vdots \\ x_0^n \succeq F(x_0^n, x_0^1, x_0^2, ..., x_0^{n-1}). \end{cases}$$

Then there exist $x^1, x^2, ..., x^n \in X$ such that $x^1 = F(x^1, x^2, ..., x^n),\ x^2 = F(x^2, x^3, ..., x^1)$, ...,$x^n = F(x^n, x^1, ..., x^{n-1})$; that is, F has an n-tupled fixed point in X.

Proof. Let $x_0^1, x_0^2, x_0^3, ..., x_0^n \in X$ such that

$$\begin{cases} x_0^1 \preceq F(x_0^1, x_0^2, x_0^3, ..., x_0^n) \\ x_0^2 \succeq F(x_0^2, x_0^3, ..., x_0^n, x_0^1) \\ x_0^3 \preceq F(x_0^3, ..., x_0^n, x_0^1, x_0^2) \\ \vdots \\ x_0^n \succeq F(x_0^n, x_0^1, x_0^2, ..., x_0^{n-1}). \end{cases} \qquad (3.2)$$

Choose $x_1^1, x_1^2, x_1^3, ..., x_1^n \in X$ such that

$$\begin{cases} x_1^1 = F(x_0^1, x_0^2, x_0^3, ..., x_0^n) \\ x_1^2 = F(x_0^2, x_0^3, ..., x_0^n, x_0^1) \\ x_1^3 = F(x_0^3, ..., x_0^n, x_0^1, x_0^2) \\ \vdots \\ x_1^n = F(x_0^n, x_0^1, x_0^2, ..., x_0^{n-1}). \end{cases} \tag{3.3}$$

Continuing this process, we construct sequences $\{x_m^1\}, \{x_m^2\}, ..., \{x_m^n\}$, $(m \geq 0)$ such that

$$\begin{cases} x_{m+1}^1 = F(x_m^1, x_m^2, x_m^3, ..., x_m^n) \\ x_{m+1}^2 = F(x_m^2, x_m^3, ..., x_m^n, x_m^1) \\ \vdots \\ x_{m+1}^n = F(x_m^n, x_m^1, x_m^2, ..., x_m^{n-1}). \end{cases} \tag{3.4}$$

We are going to divide the proof into several steps in order to make it easily readable.

Step 1. We shall prove that for all $m \geq 0$,

$$x_m^1 \preceq x_{m+1}^1,\ x_m^2 \succeq x_{m+1}^2,\ x_m^3 \preceq x_{m+1}^3, ..., x_m^n \succeq x_{m+1}^n. \tag{3.5}$$

By using (3.2) and (3.3), we have

$$\begin{aligned} x_0^1 &\preceq F(x_0^1, x_0^2, x_0^3, ..., x_0^n) = x_1^1, \\ x_0^2 &\succeq F(x_0^2, x_0^3, ..., x_0^n, x_0^1) = x_1^2, \\ x_0^3 &\preceq F(x_0^3, ..., x_0^n, x_0^1, x_0^2) = x_1^3, \\ &\vdots \\ &\vdots \\ x_0^n &\succeq F(x_0^n, x_0^1, x_0^2, ..., x_0^{n-1}) = x_1^n. \end{aligned}$$

So, (3.5) holds for $m = 0$. Suppose that (3.5) holds for some $m > 0$. As F has the mixed monotone property, we have from (3.4) that

$$\begin{aligned} x_{m+1}^1 = F(x_m^1, x_m^2, x_m^3, ..., x_m^n) &\preceq F(x_{m+1}^1, x_m^2, x_m^3, ..., x_m^n) \\ &\preceq F(x_{m+1}^1, x_{m+1}^2, x_m^3, ..., x_m^n) \\ &\preceq F(x_{m+1}^1, x_{m+1}^2, x_{m+1}^3, ..., x_m^n) \end{aligned}$$

$$\begin{aligned}
&\preceq F(x^1_{m+1}, x^2_{m+1}, x^3_{m+1}, ..., x^n_{m+1}) \\
&= x^1_{m+2}, \\
x^2_{m+1} = F(x^2_m, x^3_m, ..., x^n_m, x^1_m) &\succeq F(x^2_{m+1}, x^3_m, ..., x^n_m, x^1_m) \\
&\succeq F(x^2_{m+1}, x^3_{m+1}, ..., x^n_m, x^1_m) \\
&\succeq F(x^2_{m+1}, x^3_{m+1}, ..., x^n_{m+1}, x^1_m) \\
&\succeq F(x^2_{m+1}, x^3_{m+1}, ..., x^n_{m+1}, x^1_{m+1}) \\
&= x^2_{m+2}.
\end{aligned}$$

Also for the same reason,

$$x^3_{m+1} = F(x^3_m, ..., x^n_m, x^1_m, x^2_m) \preceq F(x^3_{m+1}, ..., x^n_{m+1}, x^1_{m+1}, x^2_{m+1}) = x^3_{m+2},$$

$$\vdots$$

$$x^n_{m+1} = F(x^n_m, x^1_m, x^2_m, ..., x^{n-1}_m) \succeq F(x^n_{m+1}, x^1_{m+1}, x^2_{m+1}, ..., x^{n-1}_{m+1}) = x^n_{m+2}.$$

Thus by the mathematical induction we conclude that (3.5) holds for all $m \geq 0$. Therefore,

$$\begin{cases}
x^1_0 \preceq x^1_1 \preceq x^1_2 \preceq ... \preceq x^1_m \preceq x^1_{m+1} \preceq ... \\
x^2_0 \succeq x^2_1 \succeq x^2_2 \succeq ... \succeq x^2_m \succeq x^2_{m+1} \succeq ... \\
x^3_0 \preceq x^3_1 \preceq x^3_2 \preceq ... \preceq x^3_m \preceq x^3_{m+1} \preceq ... \\
\vdots \\
x^n_0 \succeq x^n_1 \succeq x^n_2 \succeq ... \succeq x^n_m \succeq x^n_{m+1} \succeq ....
\end{cases} \tag{3.6}$$

This completes the proof of our claim.

Step 2. We shall show that

$$\lim_{m\to\infty} (d(Tx^1_m, Tx^1_{m+1}) + d(Tx^2_m, Tx^2_{m+1}) + ... + d(Tx^n_m, Tx^n_{m+1})) = 0.$$

Due to (3.1) and (3.4), we have

$$d(Tx^1_m, Tx^1_{m+1}) = d(TF(x^1_{m-1}, x^2_{m-1}, x^3_{m-1}, ..., x^n_{m-1}), TF(x^1_m, x^2_m, x^3_m, ..., x^n_m))$$

$$\leq \frac{1}{n}\varphi(d(Tx^1_{m-1}, Tx^1_m) + d(Tx^2_{m-1}, Tx^2_m) + d(Tx^3_{m-1}, Tx^3_m) + ... + d(Tx^n_{m-1}, Tx^n_m)),$$

$$d(Tx^2_m, Tx^2_{m+1}) = d(TF(x^2_{m-1}, x^3_{m-1}, ..., x^n_{m-1}, x^1_{m-1}), TF(x^2_m, x^3_m, ..., x^n_m, x^1_m))$$

$$\leq \frac{1}{n}\varphi(d(Tx^2_{m-1}, Tx^2_m) + d(Tx^3_{m-1}, Tx^3_m) + ... + d(Tx^n_{m-1}, Tx^n_m) + d(Tx^1_{m-1}, Tx^1_m)).$$

Similarly, we can inductively write

$$d(Tx^n_m, Tx^n_{m+1}) = d(TF(x^n_{m-1}, x^1_{m-1}, x^2_{m-1}, ..., x^{n-1}_{m-1}), TF(x^n_m, x^1_m, x^2_m, ..., x^{n-1}_m))$$

$$\leq \frac{1}{n}\varphi(d(Tx^n_{m-1}, Tx^n_m) + d(Tx^1_{m-1}, Tx^1_m) + d(Tx^2_{m-1}, Tx^2_m) + ... + d(Tx^{n-1}_{m-1}, Tx^{n-1}_m)).$$

Adding the above inequalities, we obtain

$$d(Tx^1_m, Tx^1_{m+1}) + d(Tx^2_m, Tx^2_{m+1}) + ... + d(Tx^n_m, Tx^n_{m+1}) \leq \varphi(d(Tx^n_{m-1}, Tx^n_m)$$

$$+d(Tx^1_{m-1}, Tx^1_m) + d(Tx^2_{m-1}, Tx^2_m) + ... + d(Tx^{n-1}_{m-1}, Tx^{n-1}_m)). \tag{3.7}$$

Set

$$d_m := d(Tx^1_m, Tx^1_{m+1}) + d(Tx^2_m, Tx^2_{m+1}) + ... + d(Tx^n_m, Tx^n_{m+1}). \tag{3.8}$$

Using (3.7) we have,

$$d_m \leq \varphi(d_{m-1}). \tag{3.9}$$

Since $\varphi(t) < t$ for all $t > 0$, it follows from (3.9) that $\{d_m\}$ is a decreasing sequence of positive real numbers. Therefore, there exists some $d \geq 0$ such that

$$\lim_{m\to\infty}(d(Tx^1_m, Tx^1_{m+1}) + d(Tx^2_m, Tx^2_{m+1}) + ... + d(Tx^n_m, Tx^n_{m+1})) = \lim_{m\to\infty} d_m = d+ .$$

Assume that $d > 0$, taking $m \to \infty$ in both sides of (3.9) and using the property of φ, we have

$$d = \lim_{m\to\infty} d_m \leq \lim_{m\to\infty} \varphi(d_{m-1}) = \lim_{d_{m-1}\to d+} \varphi(d_{m-1}) < d$$

which is a contradiction. Thus $d = 0$, that is,

$$\lim_{m\to\infty}(d(Tx^1_m, Tx^1_{m+1}) + d(Tx^2_m, Tx^2_{m+1}) + ... + d(Tx^n_m, Tx^n_{m+1})) = \lim_{m\to\infty} d_m = 0. \tag{3.10}$$

This proves our claim.

Step 3. We shall show that $\{Tx^1_m\}, \{Tx^2_m\}, ..., \{Tx^n_m\}$ are Cauchy sequences.

Assume on contrary that atleast one of $\{Tx^1_m\}, \{Tx^2_m\}, ..., \{Tx^n_m\}$ is not a Cauchy sequence. Then there exists an $\varepsilon > 0$ for which we can find subsequences $\{Tx^1_{m(k)}\}, \{Tx^1_{t(k)}\}$

of $\{Tx^1_m\}$, $\{Tx^2_{m(k)}\},\{Tx^2_{t(k)}\}$ of $\{Tx^2_m\}$,...,$\{Tx^n_{m(k)}\},\{Tx^n_{t(k)}\}$ of $\{Tx^n_m\}$ with $m(k) > t(k) \geq k$ such that

$$d(Tx^1_{m(k)}, Tx^1_{t(k)}) + d(Tx^2_{m(k)}, Tx^2_{t(k)}) + ... + d(Tx^n_{m(k)}, Tx^n_{t(k)}) \geq \varepsilon. \tag{3.11}$$

Further, corresponding to $t(k)$, we can choose $m(k)$ in such a way that it is the smallest integer with $m(k) > t(k) \geq k$ satisfying (3.11). Then

$$d(Tx^1_{m(k)-1}, Tx^1_{t(k)}) + d(Tx^2_{m(k)-1}, Tx^2_{t(k)}) + ... + d(Tx^n_{m(k)-1}, Tx^n_{t(k)}) < \varepsilon. \tag{3.12}$$

Using (3.11),(3.12) and the triangle inequality, we have

$$\begin{aligned}\varepsilon \leq r_k &:= d(Tx^1_{m(k)}, Tx^1_{t(k)}) + d(Tx^2_{m(k)}, Tx^2_{t(k)}) + ... + d(Tx^n_{m(k)}, Tx^n_{t(k)})\\ &\leq d(Tx^1_{m(k)}, Tx^1_{m(k)-1}) + d(Tx^1_{m(k)-1}, Tx^1_{t(k)}) + d(Tx^2_{m(k)}, Tx^2_{m(k)-1})\\ &+d(Tx^2_{m(k)-1}, Tx^2_{t(k)})... + d(Tx^n_{m(k)}, Tx^n_{m(k)-1}) + d(Tx^n_{m(k)-1}, Tx^n_{t(k)})\\ &\leq d(Tx^1_{m(k)}, Tx^1_{m(k)-1}) + d(Tx^2_{m(k)}, Tx^2_{m(k)-1}) + ... + d(Tx^n_{m(k)}, Tx^n_{m(k)-1}) + \varepsilon.\end{aligned}$$

Letting $k \to \infty$ in above inequality, and using (3.10) we have

$$\lim_{k\to\infty} r_k = \lim_{k\to\infty}[d(Tx^1_{m(k)}, Tx^1_{t(k)}) + d(Tx^2_{m(k)}, Tx^2_{t(k)}) + ... + d(Tx^n_{m(k)}, Tx^n_{t(k)})] = \varepsilon+. \tag{3.13}$$

By the triangle inequality

$$\begin{aligned}r_k &= d(Tx^1_{m(k)}, Tx^1_{t(k)}) + d(Tx^2_{m(k)}, Tx^2_{t(k)}) + ... + d(Tx^n_{m(k)}, Tx^n_{t(k)})\\ &\leq d(Tx^1_{m(k)}, Tx^1_{m(k)+1}) + d(Tx^1_{m(k)+1}, Tx^1_{t(k)+1}) + d(Tx^1_{t(k)+1}, Tx^1_{t(k)})\\ &+d(Tx^2_{m(k)}, Tx^2_{m(k)+1}) + d(Tx^2_{m(k)+1}, Tx^2_{t(k)+1}) + d(Tx^2_{t(k)+1}, Tx^2_{t(k)})\\ &\vdots\\ &+d(Tx^n_{m(k)}, Tx^n_{m(k)+1}) + d(Tx^n_{m(k)+1}, Tx^n_{t(k)+1}) + d(Tx^n_{t(k)+1}, Tx^n_{t(k)})\\ &= d_m(k) + d_t(k) + d(Tx^1_{m(k)+1}, Tx^1_{t(k)+1}) + d(Tx^2_{m(k)+1}, Tx^2_{t(k)+1})\\ &\quad +... + d(Tx^n_{m(k)+1}, Tx^n_{t(k)+1}).\end{aligned} \tag{3.14}$$

Since $m(k) > t(k)$, we have

$$x^1_m(k) \succeq x^1_t(k), x^2_m(k) \preceq x^2_t(k), x^3_m(k) \succeq x^3_t(k), ..., x^n_m(k) \preceq x^n_t(k).$$

From (3.1) and (3.4), we have

$d(Tx^1_{m(k)+1}, Tx^1_{t(k)+1}) = d(TF(x^1_{m(k)}, x^2_{m(k)}, x^3_{m(k)}, ..., x^n_{m(k)}), TF(x^1_{t(k)}, x^2_{t(k)}, x^3_{t(k)}, ..., x^n_{t(k)}))$
$\leq \frac{1}{n}\varphi(d(Tx^1_{m(k)}, Tx^1_{t(k)}) + d(Tx^2_{m(k)}, Tx^2_{t(k)}) + d(Tx^3_{m(k)}, Tx^3_{t(k)}) + ... + d(Tx^n_{m(k)}, Tx^n_{t(k)}))$,
$d(Tx^2_{m(k)+1}, Tx^2_{t(k)+1}) = d(TF(x^2_{m(k)}, x^3_{m(k)}, ..., x^n_{m(k)}, x^1_{m(k)}), TF(x^2_{t(k)}, x^3_{t(k)}, ..., x^n_{t(k)}, x^1_{t(k)}))$
$\leq \frac{1}{n}\varphi(d(Tx^2_{m(k)}, Tx^2_{t(k)}) + d(Tx^3_{m(k)}, Tx^3_{t(k)}) + ... + d(Tx^n_{m(k)}, Tx^n_{t(k)}) + d(Tx^1_{m(k)}, Tx^1_{t(k)}))$.
Similarly, we can write
$d(Tx^n_{m(k)+1}, Tx^n_{t(k)+1}) = d(TF(x^n_{m(k)}, x^1_{m(k)}, x^2_{m(k)}, ...x^{n-1}_{m(k)}), TF(x^n_{t(k)}, x^1_{t(k)}, x^2_{t(k)}, ..., x^{n-1}_{t(k)}))$
$\leq \frac{1}{n}\varphi(d(Tx^n_{m(k)}, Tx^n_{t(k)}) + d(Tx^1_{m(k)}, Tx^1_{t(k)}) + d(Tx^2_{m(k)}, Tx^2_{t(k)}) + ... + d(Tx^{n-1}_{m(k)}, Tx^{n-1}_{t(k)}))$.

From the above inequalities, we have

$$\begin{aligned} r_k &\leq d_m(k) + d_t(k) + \varphi(d(T(x^1_{m(k)}, Tx^1_{t(k)}) + d(Tx^2_{m(k)}, Tx^2_{t(k)}) + \\ &\qquad d(Tx^3_{m(k)}, Tx^3_{t(k)}) + ... + d(Tx^n_{m(k)}, Tx^n_{t(k)})) \\ &= d_n(k) + d_m(k) + \varphi(r_k). \end{aligned}$$

Taking $k \to \infty$ in the above inequality, using (3.10),(3.13) and the property of φ, we have

$$\varepsilon = \lim_{k\to\infty} r_k \leq \lim_{k\to\infty}(d_n(k) + d_m(k) + \varphi(r_k)) = \lim_{r_k\to\varepsilon+} \varphi(r_k) < \varepsilon,$$

which is a contradiction. Therefore $\{Tx^1_m\}, \{Tx^2_m\}, ..., \{Tx^n_m\}$ are Cauchy sequences in X. Since X is a complete metric space, $\{Tx^1_m\}, \{Tx^2_m\}, ..., \{Tx^n_m\}$ are convergent sequences. Since T is an ICS mapping, there exist $x^1, x^2, ..., x^n \in X$ such that

$$\lim_{m\to\infty} x^1_m = x^1, \lim_{m\to\infty} x^2_m = x^2, ..., \lim_{m\to\infty} x^n_m = x^n. \tag{3.15}$$

Since T is continuous, we have

$$\lim_{m\to\infty} Tx^1_m = Tx^1, \lim_{m\to\infty} Tx^2_m = Tx^2, ..., \lim_{m\to\infty} Tx^n_m = Tx^n. \tag{3.16}$$

Suppose now the assumption (a) holds, that is, F is continuous. By (3.4), (3.15) and the continuity of F, we obtain

$$\begin{aligned} x^1 = \lim_{m\to\infty} x^1_{m+1} = \lim_{m\to\infty} F(x^1_m, x^2_m, ..., x^n_m) &= F(\lim_{m\to\infty} x^1_m, \lim_{m\to\infty} x^2_m, ..., \lim_{m\to\infty} x^n_m) \\ &= F(x^1, x^2, ..., x^n), \end{aligned}$$

$$
\begin{aligned}
x^2 = \lim_{m\to\infty} x^2_{m+1} = \lim_{m\to\infty} F(x^2_m, x^3_m, ..., x^1_m) &= F(\lim_{m\to\infty} x^2_m, \lim_{m\to\infty} x^3_m, ..., \lim_{m\to\infty} x^1_m,) \\
&= F(x^2, ..., x^n, x^1), \\
&\vdots \\
x^n = \lim_{m\to\infty} x^n_{m+1} = \lim_{m\to\infty} F(x^n_m, x^1_m, ..., x^{n-1}_m) &= F(\lim_{m\to\infty} x^n_m, \lim_{m\to\infty} x^1_m, ..., \lim_{m\to\infty} x^{n-1}_m) \\
&= F(x^n, x^1, ..., x^{n-1}).
\end{aligned}
$$

Thus $(x^1, x^2, ..., x^n) \in X^n$ is an n-tupled fixed point of F.

Next assume that the condition (b) holds. Since $\{x^i_m\}$ is non-decreasing or non-increasing according as i is odd or even and $x^i_m \longrightarrow x^i$ as $m \to \infty$. Then by assumption (b) we have for all m,

$$x^i_m \preceq x^i \text{ when i is odd,}$$

$$x^i_m \succeq x^i \text{ when i is even.}$$

Consider now,

$$
\begin{aligned}
d(Tx^1, TF(x^1, x^2, ..., x^n)) &\le d(Tx^1, Tx^1_{m+1}) + d(Tx^1_{m+1}, TF(x^1, x^2, ..., x^n)) \\
&= d(Tx^1, Tx^1_{m+1}) + d(TF(x^1_m, x^2_m, ..., x^n_m), TF(x^1, x^2, ..., x^n)) \\
&\le d(Tx^1, Tx^1_{m+1}) + \frac{1}{n}\varphi(d(Tx^1_m, Tx^1) + d(Tx^2_m, Tx^2) + ... + d(Tx^n_m, Tx^n)).
\end{aligned}
\tag{3.17}
$$

Taking $m \to \infty$ yields that $d(Tx^1, TF(x^1, x^2, ..., x^n)) \le 0$. Hence

$$d(Tx^1, TF(x^1, x^2, ..., x^n)) = 0.$$

Thus $Tx^1 = TF(x^1, x^2, ..., x^n)$ and since T is injective, we get that

$$x^1 = F(x^1, x^2, ..., x^n).$$

Analogously, we can show that

$$x^2 = F(x^2, x^3, ..., x^1), ..., x^n = F(x^n, x^1, ..., x^{n-1}).$$

Thus we proved that F has an n-tupled fixed point.

This completes the proof.

Corollary 3.1. *Let $(X, \preceq, d)$ be a complete partially ordered metric space. Let $F : X^n \to X$ be a map enjoying the mixed monotone property on X such that there exist $x_0^1, x_0^2, x_0^3, ..., x_0^n \in X$ with*

$$\begin{cases} x_0^1 \preceq F(x_0^1, x_0^2, x_0^3, ..., x_0^n) \\ x_0^2 \succeq F(x_0^2, x_0^3, ..., x_0^n, x_0^1) \\ x_0^3 \preceq F(x_0^3, ..., x_0^n, x_0^1, x_0^2) \\ \vdots \\ x_0^n \succeq F(x_0^n, x_0^1, x_0^2, ..., x_0^{n-1}). \end{cases}$$

Suppose that there exists $\varphi \in \Phi$ such that

$d(F(x^1, x^2, x^3 ..., x^n), F(y^1, y^2, y^3 ..., y^n)) \leq \frac{1}{n}\varphi(d(x^1, y^1) + d(x^2, y^2) + ... + d(x^n, y^n))$

for all $x^1, x^2, x^3 ..., x^n,\ y^1, y^2, y^3 ..., y^n \in X$ for which $x^1 \succeq y^1,\ x^2 \preceq y^2,\ x^3 \succeq y^3, ..., x^n \preceq y^n$. Suppose either

(a) F is continuous or

(b) X has the following property:

(i) if nondecreasing sequence $\{x_m\} \to x$, then $x_m \preceq x$ for all m;

(ii) if nonincreasing sequence $\{x_m\} \to x$, then $x_m \succeq x$ for all m.

Then there exist $x^1, x^2, ..., x^n \in X$ such that $x^1 = F(x^1, x^2, ..., x^n)$, $x^2 = F(x^2, x^3, ..., x^1)$, ...,$x^n = F(x^n, x^1, ..., x^{n-1})$; that is, F has an n-tupled fixed point in X.

Proof. It follows by taking $Tx = x$, for all $x \in X$, in Theorem 3.1.

Corollary 3.2. *Let $(X, \preceq, d)$ be a complete partially ordered metric space and $T : X \to X$ is an ICS mapping. Let $F : X^n \to X$ be a map enjoying the mixed monotone property on*

X such that there exist $x_0^1, x_0^2, x_0^3, ..., x_0^n \in X$ with

$$\begin{cases} x_0^1 \preceq F(x_0^1, x_0^2, x_0^3, ..., x_0^n) \\ x_0^2 \succeq F(x_0^2, x_0^3, ..., x_0^n, x_0^1) \\ x_0^3 \preceq F(x_0^3, ..., x_0^n, x_0^1, x_0^2) \\ \vdots \\ x_0^n \succeq F(x_0^n, x_0^1, x_0^2, ..., x_0^{n-1}). \end{cases}$$

Suppose that there exists $k \in [0, 1)$ such that

$$d(TF(x^1, x^2, x^3 ..., x^n), TF(y^1, y^2, y^3 ..., y^n)) \leq \frac{k}{n}(d(Tx^1, Ty^1) + d(Tx^2, Ty^2) + d(Tx^3, Ty^3) + \\ ... + d(Tx^n, Ty^n))$$

for all $x^1, x^2, x^3 ..., x^n,\ y^1, y^2, y^3 ..., y^n \in X$ for which $x^1 \succeq y^1,\ x^2 \preceq y^2,\ x^3 \succeq y^3, ..., x^n \preceq y^n$. Suppose either

(a) F is continuous or

(b) X has the following property:

(i) if nondecreasing sequence $\{x_m\} \to x$, then $x_m \preceq x$ for all m;

(ii) if nonincreasing sequence $\{x_m\} \to x$, then $x_m \succeq x$ for all m.

Then there exist $x^1, x^2, ..., x^n \in X$ such that $x^1 = F(x^1, x^2, ..., x^n)$, $x^2 = F(x^2, x^3, ..., x^1)$, ...,$x^n = F(x^n, x^1, ..., x^{n-1})$; that is, F has an n-tupled fixed point in X.

Proof. It follows by taking $\varphi(s) = ks$, for all $s \in [0, \infty)$, in Theorem 3.1.

Remark 3.1. Taking $n = 2$ in Theorem 3.1 and in Corollaries 3.1-3.2, we get Theorem 2.1 and Corollaries 2.2-2.3 of Luong et al. [3].

Now, we shall prove the uniqueness of an n-tupled fixed point. For a product X^n of a partially ordered set $(X, \preceq)$, we define a partial ordering in the following way: For $(x^1, x^2, x^3, ..., x^n),\ (y^1, y^2, y^3, ..., y^n) \in X^n$

$$(x^1, x^2, x^3, ..., x^n) \preceq (y^1, y^2, y^3, ..., y^n) \Leftrightarrow x^1 \preceq y^1,\ x^2 \succeq y^2, x^3 \preceq y^3 ..., x^n \succeq y^n.$$

Theorem 3.2. *In addition to the hypotheses of Theorem 3.1, suppose that for every $(x^1, x^2, x^3, ..., x^n),\ (y^1, y^2, y^3, ..., y^n) \in X^n$, there exists $(z^1, z^2, z^3, ..., z^n) \in X^n$ that is* *comparable to* $(x^1, x^2, x^3, ..., x^n)$ *and* $(y^1, y^2, y^3, ..., y^n)$. *Then* F *has a unique* n*-tupled fixed point* $(x^1, x^2, x^3, ..., x^n)$.

Proof. The set of n-tupled fixed points of F is non-empty due to Theorem 3.1. Assume now, $(x^1, x^2, x^3, ..., x^n)$ and $(y^1, y^2, y^3, ..., y^n)$ are two n-tupled fixed points of F, that is,

$$\begin{cases} x^1 = F(x^1, x^2, x^3, ..., x^n), \ y^1 = F(y^1, y^2, y^3, ..., y^n), \\ x^2 = F(x^2, x^3, ..., x^n, x^1), \ y^2 = F(y^2, y^3, ..., y^n, y^1), \\ \vdots \\ x^n = F(x^n, x^1, x^2, ..., x^{n-1}), \ y^n = F(y^n, y^1, y^2, ..., y^{n-1}). \end{cases} \tag{3.18}$$

We shall show that $(x^1, x^2, x^3, ..., x^n)$ and $(y^1, y^2, y^3, ..., y^n)$ are equal. By assumption, there exists $(z^1, z^2, z^3 ..., z^n) \in X^n$ that is comparable to $(x^1, x^2, x^3, ..., x^n)$, and $(y^1, y^2, y^3, ..., y^n)$. Define sequences $\{z_m^1\}, \{z_m^2\}, ..., \{z_m^n\}$ as follows:

$$z_0^1 = z^1, z_0^2 = z^2, ..., z_0^n = z^n,$$

$$\begin{cases} z_{m+1}^1 = F(z_m^1, z_m^2, z_m^3, ..., z_m^n) \\ z_{m+1}^2 = F(z_m^2, z_m^3, ..., z_m^n, z_m^1) \\ \vdots \\ z_{m+1}^n = F(z_m^n, z_m^1, z_m^3, ..., z_m^{n-1}), \ \text{for all } m. \end{cases} \tag{3.19}$$

Since $(z^1, z^2, z^3, ..., z^n)$ is comparable with $(x^1, x^2, x^3, ..., x^n)$, we may assume that

$$(x^1, x^2, x^3, ..., x^n) \succeq (z^1, z^2, z^3, ..., z^n) = (z_0^1, z_0^2, z_0^3, ...z_0^n).$$

Now we shall prove that

$$(x^1, x^2, x^3, ..., x^n) \succeq (z_m^1, z_m^2, z_m^3, ..., z_m^n), \ \text{for all } m. \tag{3.20}$$

Suppose that (3.20) holds for some $m \geq 0$. Then by the mixed monotone property of F, we have

$$\begin{cases} z^1_{m+1} = F(z^1_m, z^2_m, z^3_m, ..., z^n_m) \preceq F(x^1, x^2, x^3, ..., x^n) = x^1, \\ z^2_{m+1} = F(z^2_m, z^3_m, ..., z^n_m, z^1_m) \succeq F(x^2, x^3, ..., x^n, x^1) = x^2, \\ \vdots \\ z^n_{m+1} = F(z^n_m, z^1_m, z^2_m, ..., z^{n-1}_m) \succeq F(x^n, x^1, x^2, ..., x^{n-1}) = x^n. \end{cases}$$

Therefore, $(x^1, x^2, x^3, ..., x^n) \gtrsim (z^1_{m+1}, z^2_{m+1}, z^3_{m+1}, ..., z^n_{m+1})$ for all m. Hence (3.20) holds. From (3.18),(3.19) and (3.1), we have

$$\begin{aligned} d(Tx^1, Tz^1_m) &= d(TF(x^1, x^2, ..., x^n), TF(z^1_{m-1}, z^2_{m-1}, ..., z^n_{m-1})) \\ &\leq \tfrac{1}{n}\varphi(d(Tx^1, Tz^1_{m-1}) + d(Tx^2, Tz^2_{m-1}) + ... + d(Tx^n, Tz^n_{m-1})), \end{aligned}$$

$$\begin{aligned} d(Tx^2, Tz^2_m) &= d(TF(x^2, ..., x^n, x^1), TF(z^2_{m-1}, ..., z^n_{m-1}, z^1_{m-1})) \\ &\leq \tfrac{1}{n}\varphi(d(Tx^2, Tz^2_{m-1}) + ... + (Tx^n, Tz^n_{m-1}) + d(Tx^1, Tz^1_{m-1})), \end{aligned}$$

$$\vdots$$

$$\begin{aligned} d(Tx^n, Tz^n_m) &= d(TF(x^n, x^1, x^2, ..., x^{n-1}), TF(z^n_{m-1}, z^1_{m-1}, z^2_{m-1} ..., z^{n-1}_{m-1})) \\ &\leq \tfrac{1}{n}\varphi(d(Tx^n, Tz^n_{m-1}) + d(Tx^1, Tz^1_{m-1}) + d... + d(Tx^{n-1}, Tz^{n-1}_{m-1})). \end{aligned}$$

Adding the above inequalities we obtain

$$\begin{aligned} d(Tx^1, Tz^1_m) + d(Tx^2, Tz^2_m) + ... + d(Tx^n, Tz^n_m)) &\leq \varphi(d(Tx^1, Tz^1_{m-1}) \\ &+ d(Tx^2, Tz^2_{m-1}) + ... + d(Tx^n, Tz^n_{m-1})). \end{aligned} \tag{3.21}$$

Set $\delta_m = d(Tx^1, Tz^1_m) + d(Tx^2, Tz^2_m) + ... + d(Tx^n, Tz^n_m)$. It follows from (3.21) and the property of φ that $\{\delta_m\}$ is a monotone decreasing sequence of positive real numbers. Therefore there is some $\delta \geq 0$ such that

$$\lim_{m\to\infty} \delta_m = \lim_{m\to\infty} (d(Tx^1, Tz^1_m) + d(Tx^2, Tz^2_m) + ... + d(Tx^n, Tz^n_m)) = \delta + .$$

Assume that $\delta > 0$, taking $m \to \infty$ in both sides of (3.21), we have

$$\begin{aligned} \delta &= \lim_{m\to\infty} (d(Tx^1, Tz^1_m) + d(Tx^2, Tz^2_m) + ... + d(Tx^n, Tz^n_m)) \\ &\leq \lim_{m\to\infty} \varphi(d(Tx^1, Tz^1_{m-1}) + d(Tx^2, Tz^2_{m-1}) + ... + d(Tx^n, Tz^n_{m-1})) \end{aligned}$$

$$= \lim_{\delta_{m-1} \to \delta+} \varphi(\delta_{m-1}) < \delta,$$

which is a contradiction. Thus $\delta = 0$, that is,

$$\lim_{m\to\infty} (d(Tx^1, Tz_m^1) + d(Tx^2, Tz_m^2) + ... + d(Tx^n, Tz_m^n)) = 0.$$

This yields that

$$\lim_{m\to\infty} d(Tx^1, Tz_m^1) = 0, \ \lim_{m\to\infty} d(Tx^2, Tz_m^2) = 0, ..., \lim_{m\to\infty} d(Tx^n, Tz_m^n) = 0. \tag{3.22}$$

Analogously, we can show that

$$\lim_{m\to\infty} d(Ty^1, Tz_m^1) = 0, \ \lim_{m\to\infty} d(Ty^2, Tz_m^2) = 0, ..., \lim_{m\to\infty} d(Ty^n, Tz_m^n) = 0. \tag{3.23}$$

Combining (3.22) and (3.23) yields that $(Tx^1, Tx^2, ..., Tx^n)$ and $(Ty^1, Ty^2, ..., Ty^n)$ are equal. The fact that T is injective gives us $x^1 = y^1, x^2 = y^2, ..., x^n = y^n$.

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