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FIXED POINT THEOREMS UNDER CONDITIONAL SEMICOMPATIBILITY WITH CONTROL FUNCTION

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Abstract. The aim of the present paper is to generalize the elegant work of Pathak et al.[17] by using the new notion of "conditional semicompatibility". The new notion is proper generalization of semi compatibility and weak semi compatibility and can be applicable on commuting and compatible maps. We used compatible mappings, commuting mappings and absorbing mappings to prove theorems which also include (E.A.) property. In the last section we show that the new notion is a necessary condition for the existence of common fixed points.

Keywords: Common fixed point,fixed point theorems, compatible mapping, absorbing maps, commuting maps, occasionally weakly compatible mappings, expansion mappings.

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1. Introduction and new definitions -

It was the turning point in "fixed point arena" when the notion of weak commutativity was introduced by Sessa[20] as the sharper tool to obtain common fixed points of mappings. Now a day's most of the results either deal with commuting mappings or assume the notion of weak commutativity of mappings. It gives a new impetus to studying of common fixed points of mappings satisfying some contractive type conditions as well as expansive type conditions and numbers of interesting results have been found by various authors. In the same stream Pant,R.P.[16]introduced R-weak commuting mappings and further Pathak et al.[17]worked more with some new commutivity condition like R-weak commuting of type A_f and A_g . A

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bulk of results were produced and it was the centre of vigorous research activity in fixed point theory and other branches of mathematical sciences in last three decades .The major breakthrough was given by Jungck[12] when he introduced the notion of compatibility of mappings, also called asymptotic commutativity by Tiwari and Singh [24] in an independent formulation. Thereafter a flood of common fixed point theorems were produced by various researchers by using the improved notion of compatibility of mappings.

Definition1.1- Two self maps f & g of metric space (X,d) are called compatible [12] if $\lim d(fgx_n, gfx_n) = 0$,whenever $\{x_n\}$ is a sequence in X such that $\lim fx_n = \lim gx_n = t$ for some t in X .

In 1995 author [6] introduced the concept of semi compatibility and obtained the first result that established a situation in which a collection of mappings has a fixed point.

Definition1.2- Two self maps f & g of metric space (X,d) are called semi compatible if

(a) $fx = gx \Rightarrow fgx = gfx$ and

(b) $\lim fx_n = \lim gx_n = t$ for some $t \in X$ implies $\lim fgx_n = gt$ holds.

B.Singh and S.Jain [21] observe that (b) implies (a) . Hence they defined the semi compatibility by condition (b) only.

Let (X,d) be a metric space and let f and g be two maps from (X,d) into itself then f and g are called commuting maps if $fgx = gfx$ for all x in X .To generalize the notion of commuting maps, Sessa [20] introduced the concept of weakly commuting maps.

Definition1.3- Two self maps f & g of metric space (X,d) are called weak commuting if $d(fgx, gfx) \le d(fx, gx)$ for all $x \in X$.

In fact, every weak commuting pair of mappings is compatible but the converse is not true[12].

Definition1.4-Two self maps f & g of metric space (X,d) are called R -weak commuting[16]At point x in X if $d(fgx, gfx) \le Rd(fx, gx)$ for some real number $R > 0$.

Definition1.5- Two self maps f & g of metric space (X,d) are called R-weak commuting of type (A_f)[17]if there exist some real number $R>0$ such that $d(fgx, ggx) \le Rd(fx, gx)$ for all $x \in X$.

Definition1.6- Two self maps f & g of metric space (X,d) are called R-weak commuting of type (A_g) [17]if there exist some real number $R>0$ such that $d(gfx, ffx) \le Rd(fx, gx)$ for all $x \in X$.

Jungck et al.[10] made another generalization of weakly commuting maps by introducing the concept of compatible maps of type (A) .

Definition1.7- Two self maps f & g of metric space (X,d) are called compatible of type (A) If $\lim d(fgx_n, ggx_n) = 0$ and $\lim d(gfx_n, ffx_n) = 0$,whenever $\{x_n\}$ is a sequence in X such that $\lim fx_n = \lim gx_n = t$ for some t in X .

It is clear that weakly commuting maps are compatible of type (A), from [10] it follows that the implication is not reversible.

Definition1.8-Two self maps f & g of metric space (X,d) are called g-compatible([18]cited from[23])if $\lim d(gfx_n, ffx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim fx_n = \lim gx_n = t$ for some t in X .

Definition1.9-Two self maps f & g of metric space (X,d) are called f-compatible ([18]cited from[23])if $\lim d(fgx_n, ggx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim fx_n = \lim gx_n = t$ for some t in X .

Definition1.10- Let f and g $(f \ne g)$ be two self maps of metric space (X,d) then f will be called g-absorbing [8] if there exists a real number $R>0$ such that $d(gx, gfx) \le Rd(fx, gx)$ for all $x \in X$ .Similarly Let f and g $(f \ne g)$ be two self maps of metric space (X,d) then g will be called f-absorbing [8] if there exists a real number $R>0$ such that $d(fx, fgx) \le Rd(fx, gx)$ for all $x \in X$.

Definition1.11- Let f and g are two self mappings of metric space (X,d). The maps f and g satisfy the E.A. property [1] if there exists a sequence $\{x_n\}$ in X such that $\lim fx_n = \lim gx_n = t$ for some $t \in X$.

In a recent work Pant et al. [14] introduced the notion of weak reciprocal continuity as follows:

Definition1.12-Two self mappings f and g of metric space (X,d) are called weakly reciprocally continuous if $\lim fgx_n = ft$ or $\lim gfx_n = gt$, whenever $\{x_n\}$ is a sequence in X such that $\lim fx_n = \lim gx_n = t$ for some t in X.

More recently Pant and Bisht [15] generalized reciprocal continuity and introduced the notion of conditional reciprocal continuity (CRC) as follows,

Definition1.13-Two self mappings f and g of metric space (X,d) will be called conditional reciprocal continuous (CRC) if whenever the set of sequence $\{x_n\}$ satisfying $\lim fx_n = \lim gx_n$ is non empty, there exist a sequence $\{y_n\}$ satisfying $\lim fy_n = \lim gy_n = t$ (say) such that $\lim fgy_n = ft$ and $\lim gfy_n = gt$.

On the other hand Saluja et al. [19] introduced the notion weak semi compatibility as follows:

Definition1.14-Two self mappings f and g of a metric space (X,d) are called weak semi compatible mappings if $\lim_{n\to\infty} fgx_n = gt$ or $\lim_{n\to\infty} gfx_n = ft$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some t in X.

We now generalize the notion of semi compatibility and introduce the new notion “conditional semicompatibility” by unifying the approach of CRC. The new notion is a proper generalization of semi compatibility and weak semi compatibility.

Definition1.15-Two self mappings f and g of metric space (X,d) will be called conditional semi compatible mappings (CSC) if whenever the set of sequence $\{x_n\}$ satisfying $\lim fx_n = \lim gx_n$ is nonempty, then there exists at least a sequence $\{y_n\}$ satisfying $\lim fy_n = \lim gy_n = t$ (say) such that $\lim fgy_n = gt$ and $\lim gfy_n = ft$.

Example1.1- Let $X = [2,10]$ and d be the usual metric on X .Define $f,g : X \to X$ as follows

$fx = 2 \; if \;\; x = 2, \quad fx = \dfrac{x+8}{2} \; if \;\; 2 < x < 4, \quad fx = 4 \; if \;\; x \geq 4$

$gx = 2 \; if \;\; x = 2, \quad gx = x + 3 \; if \;\; 2 < x < 4, \quad fx = x \; if \;\; x \geq 4$

Let us consider the sequence $x_n = 2 + \dfrac{1}{n}$ then

$\lim fx_n = \lim f\left(2 + \frac{1}{n}\right) = 5 \; and \; \lim gx_n = \lim g\left(2 + \frac{1}{n}\right) = 5$

$\lim fx_n = \lim gx_n = 5$

$\lim fgx_n = \lim f\left(5 + \frac{1}{n}\right) = 4 \neq g(5) \; and \; \lim gfx_n = \lim g\left(5 + \frac{1}{2n}\right) = 5 \neq f(5)$.

If we take a sequence $y_n = 4 + \dfrac{1}{n}$ then

$\lim fy_n = \lim gy_n = 4$ and

$\lim fgy_n = \lim f\left(4 + \frac{1}{n}\right) = 4 = g(4) \; and \; \lim gfy_n = \lim g(4) = 4 = f(4)$.Also if we take a constant sequence $y_n = 2$ then $\lim fy_n = \lim gy_n = 2$ and $\lim fgy_n = 2 = g(2)$, $\lim gfy_n = 2 = f(2)$. Thus f and g are conditional semi compatible mappings.

Now we see more definitions which will help us to improve the results.

Definition1.16[11]- Let X be a set, f and g are self maps of X. A point x in X is called coincidence point of f and g *iff* $fx = gx$. We shall call $w = fx = gx$ a point of coincidence of f and g.

Definition1.17[13]- Two self maps f and g of a set X are occasionally weakly compatible (owc) *iff* there is a point x in X which is coincidence point of f and g at which f and g commute.

Lemma1.1 [11]-Let X be a set, f and g are owc self maps on X. If f and g have a unique point of coincidence, $w := fx = gx$, then w is the unique common fixed point of f and g.

Definition1.18-Let X be a set. A symmetric on X is a mapping $d : X \times X \to [0, \infty)$ such that

$d(x, y) = 0 \; iff \;\; x = y$, and $d(x, y) = d(y, x)$ for $x, y \in X$.

Lemma1.2[5]-If f and g are compatible of type (A) then they are owc ,but converse is not true in general.

Now we give the following lemma with the fact of above lemma of [5].

Lemma1.3- If f and g are either compatible or f-compatible or g-compatible then they are owc but converse is not true in general. We give the following examples to ensure it.

Example1.2-Let $X=[1,\infty)$ with d be the usual metric. Define $f,g:X\to X$ by,

$$fx=\begin{cases} 1/x & if\ x\in[1,2) \\ 1+x & if\ x\in[2,\infty) \end{cases}$$

$$gx=\begin{cases} 1/x^2 & if\ x\in[1,2) \\ 5-x & if\ x\in[2,\infty) \end{cases}$$

Here $f(1)=g(1)$ and $f(2)=g(2)$ also

$fg(1)=f(1)=1$ and $gf(1)=g(1)=1$ therefore $fg(1)=gf(1)$

But $fg(2)=f(3)=4$ and $gf(2)=g(3)=2$ and therefore $fg(2)\neq gf(2)$. Hence f and g are owc. Moreover if we take sequence $x_n=2+1/n$, for $n\in\{1,2,3...\}$.

$\lim fx_n=\lim gx_n=3$ and $\lim fgx_n=\lim fg\left(2+1/n\right)=\lim f\left(3-1/n\right)=4$ and

$\lim gfx_n=\lim gf\left(2+1/n\right)=\lim g\left(3+1/n\right)=2$ therefore $\lim fgx_n\neq\lim gfx_n$ and hence f and g are not compatible mappings.

Example1.3-Let $X=[0,\infty)$ with d be the usual metric. Define $f,g:X\to X$ by,

$$fx=\begin{cases} 3 & if\ x\in[0,1) \\ x & if\ x\in[1,\infty) \end{cases}$$

$$gx=\begin{cases} 2 & if\ x\in[0,1) \\ \frac{1}{x} & if\ x\in[1,\infty) \end{cases}$$

We have $f(1)=g(1)=1$ & $fg(1)=gf(1)=1$; that is f and g are owc. Now consider $x_n=1+1/n$ for , $n\in\{1,2,3...\}$ then $\lim fx_n=\lim f\left(1+1/n\right)=1$ and $\lim gx_n=\lim g\left(1+1/n\right)=1$.

But $\lim fgx_n=\lim fg\left(1+1/n\right)=3$ and $\lim ggx_n=\lim gg\left(1+1/n\right)=2$,

therefore $\lim d\left(fgx_n,ggx_n\right)\neq 0$ hence f and g are not f-compatible .

Example1.4-Let $X=[1,5]$ with d be the usual metric. Define $f,g:X\to X$ by,

$$fx=\begin{cases} 1 & if\ x=1 \\ 5 & if\ x\in(1,5] \end{cases} \quad \& \quad gx=\begin{cases} 1 & if\ x=1 \\ x+4 & if\ x\in(1,5] \end{cases}$$

We have $f(1)=g(1)=1$ & $fg(1)=gf(1)=1$ that is f and g are owc. Now consider

$x_n = 1+1/n$ for $n\in\{1,2,3...\}$ we have $\lim fx_n = \lim f(1+1/n) = 5$ and

$\lim gx_n = \lim g(1+1/n) = 5$. But $\lim gfx_n = \lim gf(1+1/n) = g(5) = 9$,

And $\lim ffx_n = \lim ff(1+1/n) = \lim f(5) = 5$,

Therefore, $\lim d(gfx_n, ffx_n) \neq 0$ and hence f and g are not g-compatible.

Theorem1.1- Let X be a set with a symmetric d. Suppose that f and g are owc self maps of X satisfying

(a) $d(gx, gy) \geq \phi d(fx, fy)$

Where ϕ is control function which is continuous, defined $\phi: R_+ \to R_+$ and $\phi(t) > t$ for all $t > 0$. Then f and g have a unique common fixed point.

Proof- Since the maps are owc, there exist a point x such that $fx = gx$ and $fgx = gfx$. We show now $ffx = fx$. If not then by (a)

$d(gx, gfx) \geq \phi d(fx, ffx)$. Thus

$d(fx, ffx) > d(fx, ffx)$, which is contradiction and hence $ffx = fx$ therefore fx is a fixed point of f. But, Since $gfx = fgx$, fx is also a fixed point of g.

Suppose that p and q are common fixed point of f and g. Now we show $p = q$. If not then by (a)

$d(gp, gq) \geq \phi d(fp, fq)$,this yields $d(p,q) > d(p,q)$,which is contradiction and hence $p = q$.

Theorem1.2[16]- Let (X,d) be a complete metric space and let f, g be R-weakly commuting self-mappings of X satisfying the condition:

$d(fx, fy) \leq \gamma(d(gx, gy))$

For all $x, y \in X$, where $\gamma: R_+ \to R_+$ is continuous function such that $\gamma(t) < t$ for each $t > 0$. If $f(X) \subseteq g(X)$ and if either f or g is continuous, then f and g have a unique common fixed point in X.

This result was generalized by Pathak et al. [17] with replacing R -weak commutativity by notion of R-weak commutivity of type (A_f)or (A_g). He has also given examples (1.1)and(1.2)(see [17]) which shows that theorem (1.2) do not hold if maps f and g to be discontinuous on X or the space X is not complete. He proved the following theorem.

Theorem1.3-Let (X,d)be metric space and let f,g be R-weak commuting self-mappings of type (A_f)or type $(A_g)$of X satisfying the condition

$d(fx, fy) \le \gamma(d(gx, gy))$

For all x, y in C, where $\gamma : R_+ \to R_+$ is continuous function such that $\gamma(t) < t$ for each $t > 0$, and C is the subset of X . If $f(C) \subseteq g(C)$, $f(C)$ is complete and if either f or g is continuous, then f and g have a unique common fixed point in X .

Simple statement and elegant proof of theorems (1.3) arise a natural question : " How theorem (1.3) can be improved ?" We give the answer. It seems that theorem (1.3) can be improved by two ways: either imposing certain restrictions on the space X or by replacing the notion of R-weak commutativity of type (A_f)or type (A_g).We used both of the way .In theorem (2.1)we take space X is complete, whereas in rest two theorems E.A. property is used. In these theorems we used generalized compatible maps, absorbing maps and generalized commuting maps respectively by using new notion of conditional semi compatibility (CSC) with Expansion mappings.

Now we prove our main theorem which is generalization of pathak et al.[17].

2. Main Results-

Theorem2.1-Let f and g are conditional semicompatible self mappings of a complete metric space (X,d) such that

(a) $f(X) \subseteq g(X)$

(b) $d(gx, gy) \ge \phi d(fx, fy)$

Where ϕ is control function which is continuous,defined $\phi : R_+ \to R_+$ and $\phi(t) > t$ for all $t > 0$.

(c) f and g are either compatible or f -compatible or g -compatible.

Then f and g have a common fixed point in X .

Proof- Let x_0 be any point in X . Since $f(X) \subseteq g(X)$, there exist $x_1 \in X$ such that $fx_0 = gx_1$. Similarly we can have a sequence $gx_{n+1} = fx_n$

Now by (b)

$$d(gx_{n+1}, gx_n) \geq \phi d(fx_{n+1}, fx_n)$$
$$d(fx_n, fx_{n-1}) \geq \phi d(fx_{n+1}, fx_n) \quad ...(1)$$
$$d(fx_n, fx_{n-1}) > d(fx_{n+1}, fx_n)$$

Therefore $d(fx_{n+1}, fx_n)$ is decreasing sequence. Let it tends to a non negative real number r, therefore $\lim d(fx_{n+1}, fx_n) = r$ where $r \geq 0$.

Now we claim that $r = 0$. Let we consider that $r > 0$. As $n \to \infty$ in (1), we have $r \geq \phi(r) > r$. Which is contradiction and hence $\lim d(fx_{n+1}, fx_n) = 0 \quad ...(2)$.

Now we show that $\{fx_n\}$ is Cauchy sequence. Let we assume contrary. Then there exist $\varepsilon > 0$, we choose integers m_i and n_i with $m_i < n_i < m_{i+1}$ such that

$d(fx_{m_i}, fx_{n_i}) \geq \varepsilon$ & $d(fx_{m_i}, fx_{n_{i-1}}) < \varepsilon$. It follows that

$$\varepsilon \leq d(fx_{m_i}, fx_{n_i}) \leq d(fx_{m_i}, fx_{n_{i-1}}) + d(fx_{n_{i-1}}, fx_{n_i})$$
$$< \varepsilon + d(fx_{n_{i-1}}, fx_{n_i})$$

On limiting $n \to \infty$ yields

$$\varepsilon \leq \lim d(fx_{m_i}, fx_{n_i}) < \varepsilon + \lim d(fx_{n_{i-1}}, fx_{n_i})$$
$$\varepsilon \leq \lim d(fx_{m_i}, fx_{n_i}) < \varepsilon$$

This yield $\lim d(fx_{m_i}, fx_{n_i}) = \varepsilon \quad ...(3)$

Now by (b),

$$d(gx_{m_i}, gx_{n_i}) \geq \phi d(fx_{m_i}, fx_{n_i})$$

$d(fx_{m_{i-1}}, fx_{n_{i-1}}) \geq \phi d(fx_{m_i}, fx_{n_i})$. Limiting $n \to \infty$ yields $\varepsilon \geq \phi(\varepsilon) > \varepsilon$. This is contradiction and hence $\{fx_n\}$ is Cauchy sequence. Since (X, d) is complete metric space therefore it will be converges to some t in X . Moreover $\lim fx_n = \lim gx_{n+1} = t \quad ...(4)$

Since f and g are conditional semi compatible mappings, then there exist at least a sequence $\{y_n\}$ in X such that $\lim fy_n = \lim gy_n = u$ then

$\lim fgy_n = gu$ and $\lim gfy_n = fu$ ...(5)

First we suppose that f & g are compatible, then $\lim d\left(fgy_n, gfy_n\right) = 0$ or $\lim fgy_n = \lim gfy_n$. With (5) this yields $fu = gu$. Since compatibility of f and g implies commutativity at their coincidence point. This yield $fgu = gfu$ or $fgu = gfu = ffu = ggu$.

Now we show $ffu = fu$. If not, then by (b), $d\left(gfu, gu\right) \geq \phi d\left(ffu, fu\right)$ or $d\left(ffu, fu\right) > d\left(ffu, fu\right)$. Which is contradiction and hence $ffu = fu$ or $ffu = gfu = fu$.Therefore fu is common fixed point of f and g.

Next we suppose that f & g are f - compatible, then $\lim d\left(fgy_n, ggy_n\right) = 0$ or $\lim fgy_n = \lim ggy_n$.

With (5) this yields $\lim ggy_n = gu$.Now we show $gu = u$. If not, then by (b) $d\left(ggy_n, gy_n\right) \geq \phi d\left(fgy_n, fy_n\right)$. Limiting $n \to \infty$ yields $d\left(gu, u\right) \geq \phi d\left(gu, u\right)$ or $d\left(gu, u\right) > d\left(gu, u\right)$. Which is contradiction and hence $gu = u$.

Since $\lim fgy_n = gu$ and $\lim fy_n = \lim gy_n$ these two together yields $\lim ffy_n = gu$.

Now we show $fu = u$. If not, then by (b),
$d\left(gfy_n, gu\right) \geq \phi d\left(ffy_n, fu\right)$. Limiting $n \to \infty$ yields $d\left(fu, gu\right) \geq \phi d\left(gu, fu\right)$ or $d\left(fu, u\right) > d\left(fu, u\right)$. Which is contradiction and hence $fu = u$. Therefore $fu = gu = u$ or u is a common fixed point of f and g.

Finally we suppose that f & g are g - compatible, then $\lim d\left(gfy_n, ffy_n\right) = 0$ or $\lim gfy_n = \lim ffy_n$.With (5) this yields $\lim ffy_n = fu$. Now we show that $fu = u$.If not, then by(b)
$d\left(gfy_n, gy_n\right) \geq \phi d\left(ffy_n, fy_n\right)$. Limiting $n \to \infty$ yields $d\left(fu, u\right) \geq \phi d\left(fu, u\right)$ or $d\left(fu, u\right) > d\left(fu, u\right)$. Which is contradiction and hence $fu = u$.Since $f\left(X\right) \subseteq g\left(X\right)$, then there exist a point v in X such that $fu = gv$. Now we show that $fv = u$. If not, then by (b) $d\left(gv, gy_n\right) \geq \phi d\left(fv, fy_n\right)$. Limiting $n \to \infty$ yields $d\left(u, u\right) \geq \phi d\left(fv, u\right)$ or $0 > d\left(fv, u\right)$. Which is contradiction and hence $fv = u$. Therefore $fv = gv$. Since f & g are g -compatible and g -compatibility of f and g implies commutivity at coincidence

point ,Therefore $fgv = gfv$ or $fgv = gfv = ffv = ggv$. Now we show $ffv = fv$. If not, then by (b)

$d(gfv, gv) \geq \phi d(ffv, fv)$ or $d(ffv, fv) > d(ffv, fv)$. Which is contradiction and hence $ffv = fv$. Since $fgv = gfv$ which yields $ffv = gfv = fv$. Or fv is a common fixed point of f and g.

Example-Let $x, y \in X\,(x \neq y)$ and $X = [2,5]$ and d be the usual metric on X. Define $f, g : X \to X$ as follows

$fx = \frac{x+2}{2}$ if $2 \leq x < 4$ and $fx = \frac{x+1}{5}$ if $x \geq 4$,

$gx = x$ if $2 \leq x < 4$,and $gx = x/4$ if $x \geq 4$.

when $x_n = 4 + \varepsilon_n$, where $\varepsilon_n \to 0$ as $n \to \infty$.

$\lim fx_n = \lim f(4+\varepsilon_n) = 1$ and $\lim gx_n = \lim g(4+\varepsilon_n) = 1$ therefore $\lim fx_n = \lim gx_n = 1$ (nonempty). Then we have a sequence as $y_n = 2 + \varepsilon_n$ where $\varepsilon_n \to 0$ as $n \to \infty$ for which

$\lim fy_n = \lim f(2+\varepsilon_n) = 2$ and $\lim gy_n = \lim g(2+\varepsilon_n) = 2$ therefore $\lim fy_n = \lim gy_n = 2$,

moreover $\lim fgy_n = \lim fg(2+\varepsilon_n) = \lim f(2+\varepsilon_n) = \lim\left(2 + \varepsilon_n/2\right) = 2 = g(2)$ and

$\lim gfy_n = \lim gf(2+\varepsilon_n) = \lim g\left(2 + \varepsilon_n/2\right) = \lim\left(2 + \varepsilon_n/2\right) = 2 = f(2)$. Therefore maps f and g are conditional semicompatible.

It is easy to see that with sequence $y_n = 2 + \varepsilon_n$ where $\varepsilon_n \to 0$ *as* $n \to \infty$,maps f and g are compatible that is $\lim fy_n = \lim gy_n = 2$ and $\lim d(fgy_n, gfy_n) = 0$.

Now for $x, y \in [2,4)$,

$$d(gx, gy) \geq \phi d(fx, fy)$$

$$|x-y| \geq \phi\left|\frac{x+2}{2} - \frac{y+2}{2}\right| \Rightarrow |x-y| > \left|\frac{x-y}{2}\right| (satisfied)$$

Also for $x, y \in [4,5]$,

$$d(gx, gy) \geq \phi d(fx, fy)$$

$$\left|\frac{x}{4} - \frac{y}{4}\right| \geq \phi\left|\frac{x+1}{5} - \frac{y+1}{5}\right| \Rightarrow \left|\frac{x-y}{4}\right| > \left|\frac{x-y}{5}\right| (satisfied)$$

And 2 is common fixed point of f and g.

Corollary2.1- Let f and g are conditional semicompatible self mappings of a complete metric space (X,d) such that

(a) $f(X) \subseteq g(X)$

(b) $d(gx, gy) \geq hd(fx, fy)$,Where $h > 1$

(c) f and g are either compatible or f -compatible or g -compatible.

Then f and g have a common fixed point in X .

Proof- For control function ϕ, if we define $\phi : R_+ \to R_+$ by $\phi(t) = ht$,where $h > 1$. Then proof of this corollary can be obtained from Theorem (2.1).

Corollary2.2- Let f and g are conditional semicompatible self mappings of a complete metric space (X,d) such that

(a) $f(X) \subseteq g(X)$

(b) $\phi d(gx, gy) \geq \phi(hd(fx, fy))$,Where $h > 1$

(c) f and g are either compatible or f -compatible or g -compatible.

Then f and g have a common fixed point in X .

Proof- For control function ϕ, if we define ϕ is monotone increasing and $\phi : R_+ \to R_+$ then by (b)

$d(gx, gy) \geq hd(fx, fy)$ .Now again we define ϕ as $\phi(t) = ht$. Then rest proof of this corollary can be obtained from theorem (2.1).

Remark-If g is an identity mappings then we get famous Banach fixed point theorem from corollary (2.1).

Corollary 2.3- Let X be a set, and d be the symmetric on X . Let maps f and g satisfy all the conditions of theorem (2.1). Since f and g are either compatible or f -compatible or g compatible , then by lemma (1.3) pair (f,g) will be owc and therefore the conclusion of theorem (2.1) follows from the theorem (1.1).

Theorem 2.2-Let f and g are conditional semi compatible self mappings of a metric space (X,d) such that

(a) $f(X) \subseteq g(X)$

(b) $d(gx, gy) \geq \phi d(fx, fy)$

Where ϕ is control function which is continuous, defined $\phi : R_+ \to R_+$ and $\phi(t) > t$ for all $t > 0$

(c) f is g -absorbing or g is f -absorbing.

If f and g satisfy E.A. property, then f and g have a common fixed point in X .

Proof -Since f and g satisfy E.A. property then there exist a sequence $\{x_n\}$ in X such that $\lim fx_n = \lim gx_n = t$ for some t in X . Again since f and g are conditional semicompatible mappings and $\lim fx_n = \lim gx_n = t$ (nonempty), then there exist at least a sequence $\{y_n\}$ in X such that $\lim fy_n = \lim gy_n = u$ such that $\lim fgy_n = gu$ and $\lim gfy_n = fu$.

First we suppose that f is g -absorbing , this yields $d(gy_n, gfy_n) \leq Rd(fy_n, gy_n)$. Now limiting $n \to \infty$ yields $\lim gfy_n = u$,and hence $fu = u$. Since $f(X) \subseteq g(X)$ then there exist a point v in X such that $fu = gv$. Now we show that $fv = u$. If not, then by (b) , $d(gv, gy_n) \geq \phi d(fv, fy_n)$. On limiting $n \to \infty$ yields $d(u,u) \geq \phi d(fv,u)$ or $0 > d(fv,u)$. Which is contradiction and hence $fv = u$ or $fv = gv$. Since f is g -absorbing yields $d(gv, gfv) \leq Rd(fv, gv)$. This implies $gfv = gv$ or $ggv = gv$. Now we show that $fgv = gv$. If not , then by (b), $d(ggv, gv) \geq \phi d(fgv, fv)$ or $d(gv, gv) > d(fgv, gv)$.Which is contradiction and hence $fgv = gv$. therefore $fgv = ggv = gv$ and gv is a common fixed point of f and g .

Finally we suppose that g is f -absorbing, this yields $d(fy_n, fgy_n) \leq Rd(fy_n, gy_n)$. Now limiting $n \to \infty$ yields $\lim fgy_n = u$ or $gu = u$. Now we show $fu = u$. If not, then by (b) $d(gu, gy_n) \geq \phi d(fu, fy_n)$. On limiting $n \to \infty$ yields $d(u,u) \geq \phi d(fu,u)$ or $0 > d(fu,u)$. Which is contradiction and hence $fu = u$. Therefore $fu = gu = u$ Or u is a common fixed point of f and g .

Example-Let $x, y \in X\, (x \neq y)$ and $X = [2,7]$ and d be the usual metric on X . Define $f, g : X \to X$ as follows

$fx = \dfrac{x+8}{2}$ if $2 \leq x < 5$ and $fx = \dfrac{x+5}{2}$ if $x \geq 5$,

$gx = x + 3$ if $2 \leq x < 5$,and $gx = x$ if $x \geq 5$.

when $x_n = 2+\varepsilon_n$, where $\varepsilon_n \to 0$ as $n \to \infty$.

$\lim fx_n = 5$ and $\lim gx_n = 5$ therefore $\lim fx_n = \lim gx_n = 5$ (nonempty) .Then we have a sequence as $y_n = 5+\varepsilon_n$ where $\varepsilon_n \to 0$ as $n \to \infty$ for which

$\lim fy_n = \lim f(5+\varepsilon_n) = 5$ and $\lim gy_n = \lim g(5+\varepsilon_n) = 5$ therefore $\lim fy_n = \lim gy_n = 5$, moreover $\lim fgy_n = \lim fg(5+\varepsilon_n) = \lim f(5+\varepsilon_n) = \lim\left(5+\varepsilon_n/2\right) = 5 = g(5)$ and

$\lim gfy_n = \lim gf(5+\varepsilon_n) = \lim g\left(5+\varepsilon_n/2\right) = \lim\left(5+\varepsilon_n/2\right) = 5 = f(5)$. Therefore maps f and g are conditional semicompatible. It is easy to see that f and g satisfy E.A. property. By taking sequences $\{5+\varepsilon_n\}$ *or* $\{2+\varepsilon_n\}$ where $\varepsilon_n \to 0$ as $n \to \infty$,one can verify it.

For $2 \le x < 5$, $gfx = g\left(\frac{x+8}{2}\right) = \frac{x+8}{2}$, then $d(gx, gfx) = \left|(x+3) - \frac{x+8}{2}\right| = \frac{x-2}{2}$ and

$d(gx, fx) = \left|(x+3) - \frac{x+8}{2}\right| = \frac{x-2}{2}$ .Therefore f and g Satisfy $d(gx, gfx) \le Rd(fx, gx)$ with

$R = 1$ .Also for $x \ge 5$, $gfx = \frac{x+5}{2}$, then $d(gx, gfx) = \left|\frac{x-5}{2}\right|$ and $d(gx, fx) = \left|\frac{x-5}{2}\right|$. Therefore f and g Satisfy $d(gx, gfx) \le Rd(fx, gx)$ with $R = 1$ .Or f is g -absorbing with $R = 1$.

Now for $x, y \in [2,5)$,

$$d(gx, gy) \ge \phi d(fx, fy)$$

$$|(x+3)-(y+3)| \ge \phi\left|\frac{x+8}{2} - \frac{y+8}{2}\right| \Rightarrow |x-y| > \left|\frac{x-y}{2}\right| (satisfied)$$

Also for $x, y \in [5,7]$,

$$d(gx, gy) \ge \phi d(fx, fy)$$

$$|x-y| \ge \phi\left|\frac{x+5}{2} - \frac{y+5}{2}\right| \Rightarrow |x-y| > \left|\frac{x-y}{2}\right| (satisfied)$$

And 5 is common fixed point of f and g .

Theorem2.3-Let f and g are conditional semicompatible self mappings of a metric space (X,d) such that

(a) $f(X) \subseteq g(X)$

(b) $d(gx, gy) \ge \phi d(fx, fy)$

Where ϕ is control function which is continuous, defined $\phi : R_+ \to R_+$ and $\phi(t) > t$ for all $t > 0$

(c) f and g are either R-weak commuting type of A_f or A_g.

If f and g satisfy E.A. property, then f and g have a common fixed point in X.

Proof **-**Since f and g satisfy E.A. property then there exist a sequence $\{x_n\}$ in X such that $\lim fx_n = \lim gx_n = t$ for some t in X. Again since f and g are conditional semicompatible mappings and $\lim fx_n = \lim gx_n = t$ (nonempty), then there exist at least a sequence $\{y_n\}$ in X such that $\lim fy_n = \lim gy_n = u$ such that $\lim fgy_n = gu$ and $\lim gfy_n = fu$.

First we suppose that f and g are R -weak commuting type of A_f , Then $d(fgy_n, ggy_n) \le Rd(fy_n, gy_n)$. On limiting $n \to \infty$ yields $\lim fgy_n = \lim ggy_n$ or $\lim ggy_n = gu$.

Now we show $fu = gu$. If not, then by (b) $d(ggy_n, gu) \ge \phi d(fgy_n, fu)$. On limiting $n \to \infty$ yields $d(gu, gu) \ge \phi d(gu, fu)$ or $0 > d(gu, fu)$. Which is contradiction and hence $fu = gu$.

Since f and g are R-weak commuting type of A_f, Then $d(fgu, ggu) \le Rd(fu, gu)$. This yield $fgu = ggu$. Now we show that $fgu = gu$. If not, then by (b), $d(ggu, gu) \ge \phi d(fgu, fu)$ or $d(fgu, gu) > d(fgu, gu)$. Which is contradiction and hence $fgu = gu$ or $fgu = ggu = gu$ and gu is a common fixed point of f and g.

Finally we suppose that f and g are R -weak commuting type of A_g , Then $d(gfy_n, ffy_n) \le Rd(fy_n, gy_n)$. On limiting $n \to \infty$ yields $\lim gfy_n = \lim ffy_n$ or $\lim ffy_n = fu$.

Now we show $fu = u$. If not, then by (b) , $d(gfy_n, gy_n) \ge \phi d(ffy_n, fy_n)$. On limiting $n \to \infty$ yields $d(fu, u) > d(fu, u)$. Which is contradiction and hence $fu = u$ .Since $f(x) \subseteq g(X)$,then there exist a point v in X such that $fu = gv$. Now we show that $fv = u$. If not, then by (b), $d(gv, gy_n) \ge \phi d(fv, fy_n)$. On limiting $n \to \infty$ yields $d(fu, u) \ge \phi d(fv, u)$ or $d(u, u) > d(fv, u)$.Which is contradiction and hence $fv = u$ or $fv = gv$. Since f and g are R-weak commuting type of A_g, Then $d(gfv, ffv) \le Rd(fv, gv)$. This yield $ffv = gfv$.Now we show $gfv = fv$. If not, then by (b), $d(gfv, gv) \ge \phi d(ffv, fv)$ or

$d(gfv, fv) > d(gfv, fv)$.Which is contradiction and hence $gfv = fv$ or $ffv = gfv = fv$ and fv is a common fixed point of f and g.

Example-Let $x, y \in X(x \neq y)$ and $X = [2,7]$ and d be the usual metric on X. Define $f, g : X \to X$ as follows

$fx = \frac{x+2}{2}$ if $2 \le x \le 5$ and $fx = \frac{x+20}{5}$ if $x > 5$,

$gx = x$ if $2 \le x \le 5$,and $gx = \frac{x+15}{4}$ if $x > 5$.

when $x_n = 5 + \varepsilon_n$, where $\varepsilon_n \to 0$ as $n \to \infty$.

$\lim fx_n = \lim f(5+\varepsilon_n) = 5$ and $\lim gx_n = \lim g(5+\varepsilon_n) = 5$ therefore $\lim fx_n = \lim gx_n = 5$ (nonempty) .Then we have a sequence as $y_n = 2 + \varepsilon_n$ where $\varepsilon_n \to 0$ as $n \to \infty$ for which $\lim fy_n = \lim f(2+\varepsilon_n) = 2$ and $\lim gy_n = \lim g(2+\varepsilon_n) = 2$ therefore $\lim fy_n = \lim gy_n = 2$, moreover $\lim fgy_n = \lim fg(2+\varepsilon_n) = \lim f(2+\varepsilon_n) = \lim\left(2 + \varepsilon_n/2\right) = 2 = g(2)$ and

$\lim gfy_n = \lim gf(2+\varepsilon_n) = \lim g\left(2 + \varepsilon_n/2\right) = \lim\left(2 + \varepsilon_n/2\right) = 2 = f(2)$. Therefore maps f and g are conditional semi compatible. It is easy to see that f and g satisfy E.A. property. By taking sequences $\{5+\varepsilon_n\}$ *or* $\{2+\varepsilon_n\}$ where $\varepsilon_n \to 0$ as $n \to \infty$,one can verify it.

For, $2 \le x \le 5$, $fgx = \frac{x+2}{2}$ and $ggx = x$ then $d(fgx, ggx) = \left|\frac{x-2}{2}\right|$ and $d(fx, gx) = \left|\frac{x-2}{2}\right|$. Therefore f and g Satisfy $d(fgx, ggx) \le Rd(fx, gx)$ with $R = 1$ .Now for $x > 5$, $fgx = \frac{x+95}{20}$ and $ggx = \frac{x+75}{16}$ then $d(fgx, ggx) = \left|\frac{x-5}{80}\right|$ and $d(fx, gx) = \left|\frac{x-5}{20}\right|$. Therefore f and g Satisfy $d(fgx, ggx) \le Rd(fx, gx)$ with $R = 1$ .And f and g are R-weak commuting type of A_f for $R = 1$.

Now for $x, y \in [2,5]$,

$$d(gx, gy) \ge \phi d(fx, fy)$$

$$|x-y| \ge \phi\left|\frac{x+2}{2} - \frac{y+2}{2}\right| \Rightarrow |x-y| > \left|\frac{x-y}{2}\right| (\textit{satisfied})$$

Also for $x, y \in (5,7]$,

$d(gx,gy)\geq \phi d(fx,fy)$

$\left|\frac{x+15}{4}-\frac{y+15}{4}\right|\geq \phi\left|\frac{x+20}{5}-\frac{y+20}{5}\right| \Rightarrow \left|\frac{x-y}{4}\right|>\left|\frac{x-y}{5}\right|(satisfied)$

And 2 is common fixed point of f and g.

Now we show that the new notion required necessary condition for the existence of common fixed points.

Suppose f & g are self mappings of metric space (X,d). Let v be the fixed point of f & g. Therefore $fv=gv=v$ also $fgv=gfv=v$. If we take constant sequence $\{x_n\}=v$ then $\lim fx_n=\lim gx_n=v$. Also $\lim fgx_n=fgv=v=gv$ and $\lim gfx_n=gfv=v=fv$, therefore f & g are conditional semi compatible mappings. This shows that when f & g have common fixed point they will necessarily be conditional semi compatible, i.e. conditional semi compatibility is necessary condition for the existence of common fixed point of given mappings f & g. Whereas the conditional semi compatible mappings is not sufficient condition for existence of common fixed point. To see this we do following example

Example-

Let $X=[1,30]$ and d be the usual metric on X. Let $x,y\in X\,(x\neq y)$.

We define $f,g:X\to X$ such that

$fx=x+1 \ \ if \ \ 1\leq x<5$, $fx=\frac{x+1}{6} \ \ if \ \ x\geq 5$

$gx=2x \ \ if \ \ 1\leq x<5$, $gx=x/5 \ \ if \ \ x\geq 5$

It is easy to observe that $fX\subseteq gX$. If we take sequence $x_n=5+1/n$, $\lim fx_n=\lim f(5+1/n)=1$ and $\lim gx_n=\lim g(5+1/n)=1$ therefore $\lim fx_n=\lim gx_n=1$

Moreover $\lim fgx_n=\lim fg(5+1/n)=\lim f(1+1/5n)=2=g(1)$ and $\lim gfx_n=\lim gf(5+1/n)=\lim g\left(1+\frac{1}{6n}\right)=2=f(1)$. This shows that f & g are conditional semi compatible. Also f & g satisfy $d(gx,gy)\geq \phi d(fx,fy)$ for $fx=x+1$ & $gx=2x$ when $x,y\in[1,5)$. Again f & g satisfy same inequality for $fx=\frac{x+1}{6}$ & $gx=x/5$ when $x,y\geq 5$.

According to present working it can be said that f & g satisfy a necessary condition which is conditional semicompatibility, yet they do not have any common fixed point.

Conflict of Interests

The author declares that there is no conflict of interests.

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