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GENERALIZATION OF SUZUKI TYPE FIXED POINT THEOREMS

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Abstract. The aim of this paper is to examine the existence of fixed point in complete metric space for Suzuki-type contraction mapping. The generalization of Suzuki's result[1] is exemplified. Also the results which are given by Kadelburg et al.[3] are proved in complete cone metric space.

Keywords: Fixed point, Contraction mapping, complete metric space, cone metric space.

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1. Introduction

In 2009, Suzuki introduced a new contractive definition[1] which is weaker than Edelstein's one and also a generalized version of Edeltein's result has been obtained. The generalized version of Edelstein's theorem follows as

Theorem 1.1. *Let f be a self-mapping defined on compact metric space (X, d). If the mapping f satisfies*

$$\frac{1}{2}d(x, fx) < d(x, y) \text{ implies } d(fx, fy) < d(x, y) \tag{1}$$

for all $x, y \in X$, then f has a unique fixed point.

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Although there are thousands of fixed point theorems, Suzuki's result is the new one which belongs to $(T2)$. The fixed point theorems are categorized into four types as below

: (T1) Leader-type[2]- f has a unique fixed point and $\{f^n x\}$ converges to the fixed point for all $x \in X$. Such a mapping is called a Picard operator in [4].

: (T2) Suzuki-type[1]- f has a unique fixed point and $\{f^n x\}$ does not necessarily converge to the fixed point.

: (T3) Subrahmanyam-type[5]- f may have more than one fixed point and $\{f^n x\}$ converges to a fixed point for all $x \in X$. Such a mapping is callled a weakly Picard operator in [6, 7].

: (T4) Caristi-type[8, 9]- f may have more than one fixed point and $\{f^n x\}$ does not necessarily converge to a fixed point.

We know that Edelstein's theorem belongs to $(T1)$ but its generalization obtained by Suzuki belongs to $(T2)$.

Kadelburg et al.[3], generalized Suzuki's result and also extended to abstract cone metric space.

In this paper, we prove some fixed point theorems in complete metric space which are the generalization of Suzuki's result and Kadelburg et al., results. An example is presented for backing. And also we extend it to complete cone metric space.

2. Preliminaries

Let P be a cone in a real Banach space E with θ as the zero element. For the given cone P, a partial ordering $\preceq$ with respect to P is introduced in the following way: $x \preceq y$ if and only if $y - x \in P$. We write $x \prec y$ to indicate that $x \preceq y$, but $x \neq y$. If $y - x \in intP$, interior of P, we write $x \ll y$. If $intP \neq \emptyset$, then the cone P is called solid.

The cone P in E is called normal if there exists $K > 0$ such that for all $x, y \in E$, $\theta \preceq x \preceq y$ implies that $||x|| \leq K||y||$ (infimum of such K is called normal constant of P). It is known that there exists a norm $||.||_1$ on E, equivalent to the given one, which is monotone, i.e., such that $\theta \preceq x \preceq y$ implies that $||x||_1 \leq ||y||_1$.

Consider a abstract cone metric space (X, d) on a real Banach space E. If the cone P

is solid and normal, then we can always suppose that the normal constant of P is $K = 1$ and that the given norm in E is monotone. In particular, we can define a metric in X as $D : X \times X \to \mathbb{R}$ by $D(x, y) = ||d(x, y)||$. Since it follows from [10, 11] that this metric D and the cone metric d give the same topologies on X, i.e., these spaces have the same collections of open, closed, bounded and compact sets, etc.
In particular, for all $x, y' \in E$ and $\alpha > 0$, where $y' = \alpha y$:

$$\theta \preceq x \ll y' \Rightarrow ||x|| < \alpha||y||. \tag{2}$$

To prove Cauchy property of a sequence, we follow Jungck's lemma[12]

Lemma 2.1. *Let $\{y_n\}$ be a sequence in a complete metric space (X, d). If there exists $\alpha \in (0, 1)$ such that $d(y_{n+1}, y_n) \leq \alpha d(y_n, y_{n-1})$ for all n, then $\{y_n\}$ converges to a point in X.*

3. Main results

Our first result follows as

Theorem 3.1. *Let (X, d) be a complete metric space. If a self-mapping $f : X \to X$ satisfies*

$$\frac{1}{2}d(x, fx) < d(x, y) \text{ implies } d(fx, fy) \leq rd(x, y), \; r \in (0, 1) \tag{3}$$

then f has a unique fixed point in X.

Proof: Let $x_0 \in X$. If this point x_0 is not a fixed point, then $d(x_0, fx_0) \neq 0$. Now we have by (3)

$$\frac{1}{2}d(x_0, fx_0) < d(x_0, fx_0) \;\Rightarrow\; d(fx_0, f(fx_0)) \leq rd(x_0, fx_0).$$

Let it be $x_n = fx_{n-1}$. If for some $n \in N$, $fx_n = x_n$ holds, then nothing needs to prove. We assume that for all $n \in N$, $x_n \neq fx_n$. Then by (3)

$$\frac{1}{2}d(x_n, fx_n) < d(x_n, fx_n) \;\Rightarrow\; d(fx_n, f(fx_n)) \leq rd(x_n, fx_n).$$

holds for all $n \in N$. That is

$$d(x_{n+2}, x_{n+1}) \leq r d(x_{n+1}, x_n) \text{ holds for all } n \in N.$$

By Jungck's Lemma, $\{x_n\}$ is a cauchy sequence in X. The completeness of X assures the existence of a point p in X such that $\{x_n\}$ converges to p. We claim that the point p is the fixed point of f. Suppose

$$\begin{aligned} \frac{1}{2}d(x_n, fx_n) &\geq d(x_n, p) \text{ and} \\ \frac{1}{2}d(fx_n, f(fx_n)) &\geq d(fx_n, p) \text{ holds for all } n \in N. \\ d(x_n, fx_n) &\leq d(x_n, p) + d(p, fx_n) \\ &\leq \frac{1}{2}d(x_n, fx_n) + \frac{1}{2}d(fx_n, f(fx_n)) \end{aligned} \tag{4}$$

Because that $x_n \neq fx_n$ for all n,

$$\frac{1}{2}d(x_n, fx_n) < d(x_n, fx_n) \Rightarrow d(fx_n, f(fx_n)) \leq r d(x_n, fx_n),$$

Equation (4) becomes,

$$\begin{aligned} d(x_n, fx_n) &\leq \frac{1}{2}d(x_n, fx_n) + \frac{1}{2}r d(x_n, fx_n) \\ d(x_n, fx_n) &< \frac{1}{2}d(x_n, fx_n) + \frac{1}{2}d(x_n, fx_n) = d(x_n, fx_n) \end{aligned}$$

which is absurd. Therefore for all $n \in N$, either

$$\frac{1}{2}d(x_n, fx_n) < d(x_n, p) \text{ or } \frac{1}{2}d(fx_n, f(fx_n)) < d(fx_n, p)$$

holds. In the first case, we have

$$d(p, fp) = \lim_{n\to\infty} d(fx_n, fp) \leq \lim_{n\to\infty} d(x_n, p) = 0$$

which implies $fp = p$. In the second case, we have,

$$d(p, fp) = \lim_{n\to\infty} d(f(fx_n), fp) \leq \lim_{n\to\infty} d(fx_n, p) = 0$$

which implies $fp = p$. Hence by both the cases, we have a fixed point for f. Suppose there exists a point $q \in X$ distinct from p such that $fq = q$. By the equation (3),

$$\begin{aligned} 0 = \frac{1}{2}d(p, fp) < d(p, q) \;\Rightarrow\; d(fp, fq) &\leq r \quad d(p, q). \\ d(p, q) &< \quad d(p, q) \end{aligned}$$

which is absurd. Therefore the fixed point of f is unique.

Now we present an example as follows

Example 3.1. *Let $X = (-\infty, -\frac{1}{2}] \cup \{0\} \cup [\frac{1}{2}, \infty)$ be the complete metric space with usual metric $d(x, y) = |x - y|$. Define $f : X \to X$ by*

$$f(x) = \begin{cases} -\frac{x}{32} & \text{if } x \in (-\infty, -2) \cup (2, \infty) \\ 0 & \text{if } x \in [-2, -\frac{1}{2}] \cup \{0\} \cup [\frac{1}{2}, 2] \end{cases} \tag{5}$$

Proof:

Case (i): $x, y \in (-\infty, -2) \cup (2, \infty)$

$$d(fx, fy) = |\frac{-x}{32} + \frac{y}{32}| = \frac{1}{32}|x - y| = \frac{1}{32}d(x, y)$$

Case (ii): $x, y \in [-2, -\frac{1}{2}] \cup \{0\} \cup [\frac{1}{2}, 2]$

$$d(fx, fy) = 0 = \alpha d(x, y)$$

for all $\alpha \in (0, 1)$.

Case (iii):

If $x \in (-\infty, -2)$ *and* $y \in [-2, -\frac{1}{2}]$, *then*

$$\begin{aligned} d(x, fx) &= |x - (-\frac{x}{32})| = |x + \frac{x}{32}| = \frac{33}{32}|x|. \\ d(fx, fy) &= |-\frac{x}{32}| = \frac{1}{32}|x|. \\ \text{If } \frac{1}{2}d(x, fx) &< d(x, y) \\ \text{i.e., } \frac{33}{64}|x| &< |x - y| \\ \Rightarrow \frac{1}{32}|x| &< \frac{2}{33}|x - y| \\ \text{Thus } \frac{1}{2}d(x, fx) &< d(x, y) \Rightarrow d(fx, fy) < \frac{2}{32}d(x, y) \end{aligned}$$

If $x \in [-2, \frac{1}{2}]$ *and* $y \in (-\infty, -2)$, *then*

$$\begin{aligned} d(x, fx) &= |x|; \;\; d(fx, fy) = \frac{1}{32}|y| \\ \text{If } \frac{1}{2}d(x, fx) &< d(x, y) \\ i.e., \frac{1}{2}|x| &< |x-y| \\ \text{Since } \frac{1}{2} \leq |x| \leq 2 &\Rightarrow \frac{1}{4} \leq \frac{|x|}{2} \leq 1 \\ \text{i.e., } \frac{1}{4} \leq \frac{|x|}{2} &< |x-y|, \end{aligned}$$

We claim that $\frac{|y|}{32} \leq \frac{1}{4}|x-y|$. *Suppose there exist* x *and* y *such that* $\frac{|y|}{32} > \frac{1}{4}|x-y|$.

$$\begin{aligned} \frac{1}{4}|x-y| < \frac{2}{32} \leq \frac{|y|}{32} \\ \Rightarrow |x-y| < \frac{1}{4} \end{aligned}$$

which gives contradiction to $\frac{1}{4} \leq \frac{|x|}{2} < |x-y|$. *Therefore*

$$\frac{1}{2}d(x, fx) < d(x, y) \Rightarrow d(fx, fy) \leq \frac{1}{4}d(x, y) \text{ holds.}$$

Hence for all $x, y \in X$, *the mapping* f *satisfies the equation (3) in Theorem* (3.1) *for* $r = \frac{1}{4} = \max\{\frac{1}{4}, \frac{1}{32}, \frac{2}{32}\}$. *Moreover* $f^n(x)$ *converges for all* $x \in X$.

Remark 3.1. *The presented result which belongs to Leader type, not* $(T2)$ *type.*

The following corollaries due to [3] are proved in the version of complete metric space.

Corollary 3.1. *Let* (X, d) *be a complete metric space and let* $f : X \to X$. *Assume that*

$$\begin{aligned} \frac{1}{2}d(x, fx) &< d(x, y) \\ \Rightarrow \quad & d(fx, fy) \leq r[a_1 d(x, y) + a_2 d(x, fx) + a_3 d(y, fy)] \end{aligned} \tag{6}$$

holds for all $x, y \in X$, *where* $r \in (0, 1)$ *and* a_1, a_2 *and* a_3 *are non-negative real numbers with* $r(a_1 + a_2) + a_3 < 1$. *Then* f *has a unique fixed point in* X.

Corollary 3.2. *Let $f : X \to X$ be a self-mapping defined on a complete metric space (X, d) such that*

$$\frac{1}{2}d(x, fx) < d(x, y) \text{ implies } d(fx, fy) \leq r[a_1 d(x, fx) + a_2 d(y, fy)]$$

for all $x, y \in X$ and a_1, a_2 are non-negative real numbers such that $ra_1 + a_2 < 1$. Then f has a unique fixed point.

Theorem 3.2. *Let (X, d) be a complete cone metric over a normal, solid cone P and $f : X \to X$. If for all $x, y \in X$ with $x \neq y$*

$$\frac{1}{2}d(x, fx) - d(x, y) \notin intP \quad \Rightarrow \quad d(fx, fy) \ll \alpha d(x, y) \tag{7}$$

holds true for some $\alpha \in (0, 1)$, then f has a unique fixed point.

Proof: By the normality of the cone P, we can have a norm which is monotone, i.e., $\theta \preceq c \ll d$ implies $||c|| < ||d||$. That is for $\alpha \geq 0$, $\theta \preceq c \ll \alpha d$ implies $||c|| < \alpha||d||$. The proof of our result is as follows 3.7 in [3].

Corollary 3.3. *Let (X, d) be a complete cone metric over a normal, solid cone P and $f : X \to X$. If for all $x, y \in X$ with $x \neq y$*

$$\frac{1}{2}d(x, fx) - d(x, y) \notin intP$$
$$\Rightarrow d(fx, fy) \ll r[a_1 d(x, y) + a_2 d(x, fx) + a_3 d(y, fy)]$$

where $r \in (0, 1)$ and a_1, a_2 and a_3 are non-negative real numbers with $r(a_1 + a_2) + a_3 < 1$, then f has a unique fixed point in X.

Corollary 3.4. *Let (X, d) be a complete cone metric over a normal, solid cone P and $T : X \to X$. If for all $x, y \in X$ with $x \neq y$*

$$\frac{1}{2}d(x, fx) - d(x, y) \notin intP \text{ implies } d(fx, fy) \ll r[a_1 d(x, fx) + a_2 d(y, fy)]$$

a_1, a_2 are non-negative real numbers such that $ra_1 + a_2 < 1$, then f has a unique fixed point.

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