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SYMPLECTIC CYCLIC GROUP ACTIONS ON HOMOTOPY E(N) SURFACES

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Abstract. Let $G = Z_p$ be a symplectic cyclic group action of prime order p on the homotopy $E(n)$ surface X. We study the existence of homologically trivial, pseudofree actions Z_{17} and Z_{19} on X. If the actions exist, we give the concrete structure of the fixed-point sets and realize the fixed-point data by locally linear, pseudofree actions on X.

Keywords: Elliptic surfaces, Cyclic group actions, Locally linear, Fixed-point set.

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1. Introduction

Let $E(1) = \mathbf{C}P^2 \sharp 9\overline{\mathbf{C}P^2}$ with an elliptic fibration. An elliptic surface $E(n)$ is defined as the n-fold fiber sum of copies of $E(1)$. Obviously, the elliptic surface defined as above is a simply connected 4-manifold. The intersection form of $E(n)$ is $nE8 \oplus (2n-1)H$ which is an even symmetric bilinear form, where $E8$ is a negative definite and unitary module form of rank 8 and H is a hyperbolic form. As to elliptic surfaces, We can refer to [4] for details. Note that $sign(E(n)) = -8n$ and $\chi(E(n)) = 12n$. As a Kähler manifold, $E(n)$ is

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obviously a symplectic manifold. In this paper, we always assume $E(n)$ to be a minimal homotopy elliptic surface.

Let G be a symplectic cyclic group action preserving the symplectic structure. In the study of symplectic group actions on 4-manifolds, a central problem is describing the structure of fixed-point set and action around it, where the later one can be realized by the local representation of fixed points. For example, by using the G-equivariant Seiberg-Witten Taubes theory [8, 9], W. Chen and S. Kwasik [2] give a complete description of the fixed-point set structure of the symplectic cyclic action of prime order on a minimal symplectic 4-manifold with $c_1^2 = 0$. Besides, [7, 10] study the symplectic cyclic group actions on elliptic surfaces $E(n)$ with small p. In this paper, we further study the pseudofree, homologically trivial, symplectic cyclic actions on $E(n)$ with order $p = 17, 19$. We give the concrete structure of the fixed-point set. Besides, we can realize the fixed-point set by the locally linear pseudofree actions on homotopy $E(n)$. Our results are as follows.

Theorem 1.1. *Let $X = E(n)$ be a homotopy elliptic surface and $G = Z_p$ the symplectic cyclic group action of order 17 on X. Suppose the action of G on X is nontrivial while the induced action on the rational cohomology is trivial. Then the fixed-point set of G is consisted of $12n$ isolated points. $4n$ with local representation $(z_1, z_2) \mapsto (\mu_{17}^k z_1, \mu_{17}^{2k} z_2)$ and the other $8n$ with local representation $(z_1, z_2) \mapsto (\mu_{17}^{-k} z_1, \mu_{17}^{4k} z_2)$, where $\mu_{17} = exp(\frac{2\pi i}{17})$ and $k \neq 0 \mod p$.*

Theorem 1.2. *Let $X = E(n)$ be a homotopy elliptic surface and $G = Z_p$ the symplectic cyclic group action of order 19 on X. Suppose the action of G on X is nontrivial while the induced action on the rational cohomology is trivial. Then the fixed-point set of G is composed of type (2) and (4). Besides, $a_2 + 2a_4 = 6n$, where a_2, a_4 denote the number of fixed-points of type (2) and (4) respectively.*

We arrange the following content as below: In section 2, we introduce some preliminaries used in our study. In section 3, we give the proof of Theorem 1.1 and Theorem 1.2.

2. Preliminaries

The study of the structure of fixed-point set can usually be connected with the induced representation of G on rational cohomology of X. In this section, we introduce Lefschetz fixed point formula and G-signature theorem and some relative results of fixed-point set of symplectic cyclic group actions in W.Chen and S.Kwasik [2].

Let $\tau : X \to X$ be a generator of the group action G on a closed simply connected 4-manifold X. The Lefschetz fixed point formula is $\chi(F) = \Lambda(\tau) = 2 + \text{Trace}[\tau_* : H_2(X) \to H_2(X)]$. For details, we can refer to [1].

Let X be a closed, oriented smooth 4-manifold and $G = Z_p$ be the orientation-preserving cyclic group action with prime order. If the fixed-point set is nonempty, it must be composed of isolated fixed points and 2-dim submanifolds according to [5]. Meanwhile, there is a complex representation $(z_1, z_2) \to (\mu_p^k z_1, \mu_p^{qk} z_2))$ near every fixed point, where $k, q \neq 0$ mod p, $\mu_p = exp(\frac{2\pi i}{p})$. Besides, q is uniquely determined for each fixed point and k is determined up to sign.

Theorem 2.1.[6](G-signature Theorem)

$$|G| \cdot sign(X/G) = sign(X) + \sum_{m\in F} def_m + \sum_{Y\in F} def_Y,$$

where m stands for an isolated fixed point, and Y stands for a 2-dimensional component of M^G. The terms def_m and def_Y are called signature defects and defined by

$$def_m = \sum_{k=1}^{p-1} \frac{(1+\mu_p^k)(1+\mu_p^{kq})}{(1-\mu_p^k)(1-\mu_p^{kq})},\ def_Y = \frac{P^2-1}{3} \cdot (Y \cdot Y),$$

where $Y \cdot Y$ is the self-intersection of Y and the local representation at m is $(z_1, z_2) \mapsto (\mu_p^k z_1, \mu_p^{kq} z_2)$.

Theorem 2.2. [2] *Let M be a minimal symplectic 4-manifold with $c_1^2 = 0$ and $b_2^+ \geq 2$, which admits a nontrivial, pseudofree action of $G \equiv \mathbf{Z}_p$, where p is prime, such that the symplectic structure is preserved under the action and the induced action on $H^2(M;Q)$ is trivial. Then the set of fixed points of G can be divided into groups each of which belongs to one of the following five possible types.*

(1) *One fixed point with local representation $(z_1, z_2) \mapsto (\mu_p^k z_1, \mu_p^{-k} z_2)$ for some $k \neq 0$ mod p, i.e., with local representation contained in $SL_2(\mathbf{C})$.*

(2) *Two fixed points with local representation* $(z_1, z_2) \mapsto (\mu_p^{2k} z_1, \mu_p^{3k} z_2)$, $(z_1, z_2) \mapsto (\mu_p^{-k} z_1, \mu_p^{6k} z_2)$ *for some* $k \neq 0$ *mod* p *respectively. Fixed points of this type occur only when* $p > 5$.

(3) *Three fixed points, one with local representation* $(z_1, z_2) \mapsto (\mu_p^{k} z_1, \mu_p^{2k} z_2)$ *and the other two with local representation* $(z_1, z_2) \mapsto (\mu_p^{-k} z_1, \mu_p^{4k} z_2)$ *for some* $k \neq 0$ *mod* p. *Fixed points of this type occur only when* $p > 3$.

(4) *Four fixed points, one with local representation* $(z_1, z_2) \mapsto (\mu_p^{k} z_1, \mu_p^{k} z_2)$ *and the other three with local representation* $(z_1, z_2) \mapsto (\mu_p^{-k} z_1, \mu_p^{3k} z_2)$ *for some* $k \neq 0$ *mod* p. *Fixed points of this type occur only when* $p > 3$.

(5) *Three fixed points, each with local representation* $(z_1, z_2) \mapsto (\mu_p^{k} z_1, \mu_p^{k} z_2)$ *for some* $k \neq 0$ *mod* p. *Fixed points of this type occur only when* $p = 3$.

Theorem 2.3. [2] *Let* M *be a minimal symplectic 4-manifold with* $c_1^2 = 0$ *and* $b_2^+ \geq 2$, *which admits a homologically trivial (over* $\mathbf{Q}$ *coefficients), pseudofree, symplectic* $\mathbf{Z}_p$-*action for a prime* $p > 1$. *Then the following conclusions hold.*

(1) *The action is trivial if* $p \neq 1$ *mod* 4, $p \neq 1$ *mod* 6, *and the signature of* M *is nonzero. In particular, if the signature of* M *is nonzero, then for infinitely many primes* p *the manifold* M *does not admit any such nontrivial* $\mathbf{Z}_p$-*actions.*

(2) *The action is trivial as long as there is a fixed point of type (1) in Theorem 2.3.*

Theorem 2.4. [2] *Let* $def_{(k)}$ *be the total signature defect contributed by one group of fixed points of type* (k) *in the Theorem 2.2, where* $k = 2, 3, 4$. *Then we have*

(1) $def_{(2)} = -8r$ *if* $p = 6r + 1$, $def_{(2)} = 8r + 8$ *if* $p = 6r + 5$.

(2) $def_{(3)} = -8r$ *if* $p = 4r + 1$, $def_{(3)} = 2$ *if* $p = 4r + 3$.

(3) $def_{(4)} = -8r$ *if* $p = 3r + 1$, $def_{(4)} = -4r$ *if* $p = 3r + 2$.

3. Proof of Main Theorems

In this section, we study the homologically trivial, pseudofree symplectic Z_{17}, Z_{19} action on homotopy $E(n)$ surface X. We give the concrete structure of fixed-point set. In the sense of topology, we also realize the fixed-point set by locally linear pseudofree action.

At first, the Z_{17}, Z_{19} actions on X can not be determined to be trivial by Theorem 2.2.(1). If the action is trivial, there is nothing to be discussed. Thus we suppose both actions are nontrivial.

Proof of Theorem 1.1. *Since the action of G on X is homologically trivial,*

$$sign(X/G) = sign(X) = -8n,\ \chi(X/G) = \chi(X) = 12n.$$

In addition, since the action is supposed to be pseudofree and nontrivial, the fixed-point set of G is composed of type (2), (3) and (4) according to Theorem 2.2 and Theorem 2.3. Denote the number of fixed points of type (2), (3) and (4) by a_2, a_3, a_4 respectively. By Theorem 2.3, $def_{(2)} = 6, def_{(3)} = -32, def_{(4)} = -20$ for $G = Z_{17}$. Then from the Lefschetz fixed point formula and G-Signature theorem, we have the following equations.

$$(17-1)(-8n) = 6a_2 - 32a_3 - 20a_4$$

$$12n = 2a_2 + 3a_3 + 4a_4$$

Since $a_2, a_3, a_4 \geq 0$, the solution of the above equations is $a_2 = 0, a_3 = 4n, a_4 = 0$. Thus the fixed-point set is composed of $4n$ groups of type (3) fixed points. Then Theorem 1.1 is proved.

In fact, the fixed-point set in Theorem 1.1 can be realized by locally linear pseudofree group actions on X. In order to realize it, we suppose $n = 4m, m \in \mathrm{Z}^+$. Then the above action $G = Z_{17}$ has $48m$ fixed points. Now we divide the fixed-point set into m groups. Each group has 48 fixed points with local representation

$$(1)\quad (z_1, z_2) \mapsto (\mu_{17}^{k} z_1, \mu_{17}^{2k} z_2),\ (z_1, z_2) \mapsto (\mu_{17}^{-k} z_1, \mu_{17}^{4k} z_2),\ (z_1, z_2) \mapsto (\mu_{17}^{-k} z_1, \mu_{17}^{4k} z_2),$$

where $\mu_{17} = exp(\frac{2\pi i}{17}), k = 1, 2, \cdots, 16$.

In order to realize the above fixed point data, we need to verify the REP, GSF and TOR conditions in realization theorem of Edmonds and Ewings [3]. However, since the group action is homologically trivial, the REP condition is naturally satisfied. Besides, since the order of G is 17, the TOR condition is contained in GSF according to [3]. Therefore we only need to verify the GSF condition. In this case, the GSF condition is

$$mdef_{(3)} = sign(g, X).$$

On one hand, since the G action is supposed to be homologically trivial, $sign(g, X) = sign(X) = -8n$. On the other hand, by Theorem 2.3 $def_{(3)} = -32$. Thus the GSF condition is obviously satisfied. That is to say there exists a locally linear pseudofree topological group action on homotopy elliptic surface X with fixed point data (1).

Proof of Theorem 1.2. *At first, from Theorem 2.2, 2.3 and the assumption of Theorem 1.1, the fixed points of* $G = \mathrm{Z}_{19}$ *is composed of type (2), (3) and (4). Denote the number of fixed points of type (2), (3) and (4) by* a_2, a_3, a_4 *respectively. For* $G = Z_{19}$, $def(2) = -24, def(3) = 2, def(4) = -48$ *according to Theorem 2.3. Besides, we also have*

$$sign(X/G) = sign(X) = -8n, \ \chi(X/G) = \chi(X) = 12n$$

and the the following equations

$$(19-1)(-8n) = -24a_2 + 2a_3 - 48a_4$$

$$12n = 2a_2 + 3a_3 + 4a_4. \tag{2}$$

Since $a_2, a_3, a_4 \geq 0$, *the solution of the above equations is* $a_3 = 0, a_2 + 2a_4 = 6n$. *These complete the proof of Theorem 1.2.*

Next we realize the fixed-point set in Theorem 1.2 by locally linear pseudofree actions on homotopy elliptic surfaces X.

Obviously, the solution of equations (2) is determined by the value of n. For convenience, we choose one solution $a_2 = 6n, a_3 = a_4 = 0$ of (2). That is to say, there are only $6n$ type (2) fixed points of G. In addition, we suppose $n = 3m, m \in \mathrm{Z}^+$. Then the above action $G = Z_{19}$ has $36m$ fixed points. Now we depart the fixed-point set into m groups. Each group has 36 fixed points with local representation

$$(z_1, z_2) \mapsto (\mu_{19}^{2k} z_1, \mu_{19}^{3k} z_2), \ (z_1, z_2) \mapsto (\mu_{19}^{-k} z_1, \mu_{19}^{6k} z_2), \tag{3}$$

where $\mu_{19} = exp(\frac{2\pi i}{17}), k = 1, 2, \cdots, 18$.

When $p = 19$, we only need to verify the GSF conditions by realization theorem of Edmonds and Ewings [3]. In this case, the GSF condition is

$$mdef_{(2)} = sign(g, X).$$

On one hand, since the G action is supposed to be homologically trivial, $sign(g, X) = sign(X) = -8n$. On the other hand, $def_{(2)} = -24$ by Theorem 2.3. Thus the GSF condition is obviously satisfied. Hence the fixed point data (3) can be realized by locally linear pseudofree topological action on homotopy elliptic surface X.

As to the other solutions of equations (2), we can use the same method to construct the fixed-point set of G and verify the realization theorem.

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