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CONE METRIC SPACES AND FIXED POINT RESULTS OF GENERALIZED CONTRACTIVE MAPPINGS

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Abstract: The object of this note is to establish some fixed point theorems for generalized contractive mappings on cone metric spaces.

Key Words: Cone metric spaces,generalized contractive mappings,fixed point.

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1. Introduction and Preliminaries

In 2007 Huang and Zhang [8] have generalized the concept of a metric space, replacing the set of real numbers by an ordered Banach space and obtained some fixed point theorems for mapping satisfying different contractive conditions. Subsequently, many authors have established and extended different types of contractive mappings in cone metric spaces, see for instance [3],[4],[6],[7],[11] and [12]. The author [3] proved fixed point theorems for mappings satisfying generalized contractive condition in cone metric spaces.

The purpose of this paper is to extend and improve the fixed point theorems of [3,8,12].

We recall some definitions of cone metric spaces and some of their properties.

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Definition 1.1 : Let (E,τ) be a topological vector space and $P \subset E$. Then P is called a cone whenever

(a) P is closed, non-empty and $P \neq \{0\}$;

(b) $ax + by \in$ P for all $x, y \in P$ and non-negative real numbers a, b;

(c) $x \in P$ and $-x \in P \Rightarrow x = 0 \Leftrightarrow$ P $\cap$ (−P) = {0}.

Given a cone $P \subset E$, a partial ordering is defined as $\leq$ with respect to P, by $x \leq y$ if and only if $y - x \in P$. We write $x < y$ to indicate that $x \leq y$ but $x \neq y$.

For $x, y \in P, x \ll$ y stand for $y - x \in int\ P$, where $int\ P$ is the interior of P.

Definition 1.2 (See[8])**:** Let X be a non-empty set, a mapping $d: X \times X \to E$ is called cone metric on X if the following conditions are satisfied:

(d1) $0 \leq d(x,y)$ for all $x, y \in X$, and $d(x,y) = 0$ if and only if $x = y$;

(d2) $d(x,y) = d(y,x)$ for all $x, y \in X$;

(d3) $d(x,y) \leq d(x,z) + d(z,y)$ for all $x, y, z \in X$.

From now on, we assume that E is a normed space, P is a cone in E with $int\ (P) \neq \emptyset$ and $\leq$ is a partial ordering with respect to P, and (X, d) is called cone metric space.

Definition 1.3 (See[8]): Let (X, d) be a cone metric space and $\{x_n\}$ be a sequence of points of X. Then

(i) $\{x_n\}$ converges to $x \in X$ and denoted by $\lim_{n\to\infty} x_n = x\ or\ x_n \to x$, if for any $c \in int\ (P)$, there exists N such that for all $n > N, d(x_n, x) \ll c$.

(ii) $\{x_n\}$is called Cauchy if for every $c \in int\,(P)$, there exists N such that for all $n, m > N, d(x_n, x_m) \ll c$.

(iii) (X, d)iscomplete if every Cauchy sequence in X is convergent.

Definition 1.4 (See[3])**:** A function$F: P \to P$ is called $\ll$- increasing if, for each $x, y \in P; x \ll y$ if and only if $f(x) \ll f(y)$.

Let $F: P \to P$ be a function such that

(F1) $F(t) = 0$ if and only if $t = 0$;

(F2) F is $\ll -$ increasing;

(F3) F is surjective.

We denote by $\Upsilon(P, P)$ the family of functions satisfying $(F1), (F2)\ and\ (F3)$.

Lemma 1.1(See[4])**:** Let E be a topological vector space. If $c_n \in E$ and $c_n \to 0$, then for each $c \in\ int\,(P)$ there exists N such that $c_n \ll c$ for all n>N.

2.Fixed Point Theorems

Theorem 2.1: Let (X,d) be a complete cone metric space. Suppose that a mapping$T: X \to X$ satisfies

$F(d(Tx, Ty)) \le k\{F\big(d(Tx, x) + d(x, y) + d(Ty, y)\big)\}$ ……………..(2.1)

For all $x, y \in X$; where $k \in \left[0, \frac{1}{2}\right)\ and\ F \in \Upsilon(\mathrm{P}, \mathrm{P})$ such that

(1) F is sub-additive;

(2) If, for $\{c_n\} \subset P, \lim_{n\to\infty} F(c_n) = 0\ then\ \lim_{n\to\infty} c_n = 0$.

Then T has a unique fixed common point in X. For each$x \in X$, the iterativesequence$\{T^n x\}$is convergent to the fixed point.

Proof: Let each $x_o \in X$befixed. Let $x_1 = Tx_o$and` let$x_{n+1} = Tx_n = T^{n+1}x_0$ for all $n \in \mathbb{N}$.

From (2.1) with $x = x_n$ and $y = x_{n-1}$, we have

$$F\big(d(x_{n+1}, x_n)\big) = F(d(Tx_n, Tx_{n-1}))$$

$$\leq k\{F(d(Tx_n, x_n) + d(x_n, x_{n-1}) + d(Tx_{n-1}, x_{n-1}))\}$$

$$= k\{F(d(x_{n+1}, x_n) + d(x_n, x_{n-1}) + d(x_n, x_{n-1}))\},$$

Which implies

$F(d(x_{n+1}, x_n)) \leq rF\big(d(x_n, x_{n-1})\big)$for all $n \in \mathbb{N}$

Where $r = \frac{2k}{1-k}$.

Hence

$$F(d(x_{n+1}, x_n)) \leq rF(d(x_n, x_{n-1})) \leq r^2F(d(x_{n-1}, x_{n-2}))$$

$$\leq\text{- - - - - -}\leq r^nF(d(x_1, x_o)).$$

We now show that $\{x_n\}$ is a Cauchy sequence in X.

For $m > n$, we have

$$F(d(x_n, x_m)) \leq F(d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ----$$

$$--- + d(x_{m-1}, x_m))$$

$$\leq F\big(d(x_n\ , x_{n+1})\big) + F\big(d(x_{n+1}, x_{n+2})\big) + ----$$

$$---- + F(d(x_{m-1}, x_m))$$

$$\leq r^nF\big(d(x_1, x_o)\big) + r^{n+1}F\big(d(x_1, x_0)\big) + -----$$

$$---- + r^{m-1}F(d(x_1, x_0))$$

$$\leq \frac{r^m}{1-r}F(d(x_1, x_0)) \to 0.$$

Hence $\lim_{n,m\to\infty} d(x_{n,}x_m) = 0$ by (ii). Applying Lemma 1.1, $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $z \in X$ such that $\lim_{n\to\infty} x_n = z$.

Let $c \in int\ (P)$ be given. We can choose $N \in \mathbb{N}$ such that

$d(x_{n+1}, x_n) \ll F^{-1}\left(\frac{c(1-k)}{3k}\right)$and

$d\left(x_{n,}, z\right) \ll F^{-1}\left(\frac{c(1-k)}{3}\right)$for all $n > N$.

By (F2) and (F3),

$F(d(x_{n+1}, x_n)) \ll \frac{c(1-k)}{3k}$and

$F(d(x_n, z)) \ll \frac{c(1-k)}{3}$for all $n > N$.

Then we have

$$F(d(Tz,z)) \leq F(d(Tz,Tx_{n-1}) + d(Tx_{n-1},z))$$

$$\leq k\{F\left(d(Tx_{n-1},x_{n-1}) + d(x_{n-1},z) + d(Tz,z)\right)\}$$

$$+F(d(x_n,z))$$

$$= k\{F\left(d(x_n,x_{n-1}) + d(x_{n-1},z) + d(Tz,z)\right)\}$$

$$+F(d(x_n,z))$$

Hence wehave

$$F(d(Tz,z)) \leq \frac{k}{1-k}F(d(x_{n-1},x_n)) + \frac{k}{1-k}F(d(x_{n-1},z)) + \frac{1}{1-k}F(d(x_n,z))$$

$$\ll \frac{c}{3} + \frac{c}{3} + \frac{c}{3} = c.$$

Thus, $F(d(Tz,z)) \ll \frac{c}{n}$ for all $n \in \mathbb{N}$, and so $\frac{c}{n} - F(d(Tz,z)) \in P$. Since$\frac{c}{n} \to 0$and P is closed,$-F(d(Tz,z)) \in P$. Hence $F(d(Tz,z)) = 0$.

By $(F1), d(Tz,z) = 0$ and so $z = Tz$.

Assume that u is another fixed point of T.

Then from (2.1) ,we have

$$F\big(d(z,u)\big) = F(d(Tz,Tu))$$

$$\leq k\{F(d(Tz,z) + d(z,u) + d(Tu,u))\}$$

$$= k\{F\big(d(z,z) + d(z,u) + d(u,u)\big)\}$$

$$F(d(z,u)) \leq kF(d(z,u))$$

Which implies$z = u$.

Therefore, T has a unique fixed point in X.

Theorem 2.2: Let (X,d) be a complete cone metric space. Suppose that a mapping$T: X \to X$ satisfies

$F(d(Tx,Ty)) \leq k\{F\big(d(Tx,y) + d(x,y) + d(x,Ty)\big)\}$ ……………..(2.2)

For all $x, y \in X$; where $k \in \left[0, \frac{1}{2}\right)$ and $F \in \Upsilon(\mathrm{P},\mathrm{P})$ such that

(1) F is sub-additive;
(2) If, for $\{c_n\} \subset P, \lim_{n\to\infty} F(c_n) = 0$ then $\lim_{n\to\infty} c_n = 0$.

Then T has a unique fixed point in X. For each $x \in X$, the iterative sequence$\{T^n x\}$is convergent to the fixed point.

Proof: Let $each\ x_o \in X$ be fixed. Let $x_1 = Tx_0$and let $x_{n+1} = Tx_n = T^{n+1}x_0$ for all$n \in \mathbb{N}$.

From (2.2) with $x = x_n$and $y = x_{n-1}$, we have

$$F(d(x_{n+1,}x_n)) = F(d(Tx_n, Tx_{n-1}))$$

$$\leq k\{F(d(Tx_n , x_{n-1}) + d(x_n , x_{n-1}) + d(x_n , Tx_{n-1}))\}$$

$$= k\{F(d(x_{n+1} , x_{n-1}) + d(x_n , x_{n-1}) + d(x_n , x_n))\}$$

$$\leq k\{F\big(d(x_{n+1} , x_{n-1})\big)\} + k\{F(d(x_n , x_{n-1})\big)\}$$

Which implies

$F(d(x_{n+1}, x_n)) \leq rF(d(x_n, x_{n-1}))$for all $n \in \mathbb{N}$,

Where $= \frac{2k}{1-k}$.

Hence,

$$F(d(x_{n+1}, x_n)) \leq rF(d(x_n, x_{n-1})) \leq r^2 F(d(x_{n-1}, x_{n-2}))$$

$$\leq \text{- - - - - -} \leq r^n F(d(x_1, x_o)).$$

We now show that $\{x_n\}$ is a Cauchy sequence in X.

For $m > n$, we have

$$F\big(d(x_n, x_m)\big) \leq F(d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + - - -$$

$$\text{- - - -} + d(x_{m-1}, x_m)).$$

$$\leq F\big(d(x_n, x_{n+1})\big) + F\big(d(x_{n+1}, x_{n+2})\big) + - - -$$

$$\text{- - - - -} + F(d(x_{m-1} , x_m)).$$

$$\leq r^n F\big(d(x_1, x_0)\big) + r^{n+1} F(d(x_1, x_0)) + - - -$$
$$\text{- - - - -} + r^{m-1} F(d(x_1, x_0))$$

$$\leq \frac{r^m}{1-r} F(d(x_1, x_0)) \to 0.$$

Hence $\lim_{n,m\to\infty} d(x_{n,}x_m) = 0$ by (ii). Applying Lemma 1.1, $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $z \in X$ such that $\lim_{n\to\infty} x_n = z$.

Let $c \in int\ (P)$ be given. We can choose $N \in \mathbb{N}$ such that

$d(x_n, z) \ll F^{-1}\left(\frac{c(1-k)}{4}\right)$for all n>N

By (F2) and (F3),

$F(d(x_n, z)) \ll \frac{c(1-k)}{4}$for all $n > N$.

Then we have

$$F(d(Tz, z)) \leq F(d(Tz, Tx_{n-1}) + d(Tx_{n-1}, z))$$

$$\leq k\{F\big(d(Tx_{n-1}, z) + d(x_{n-1}, z) + d(x_{n-1}, Tz)\big)\}$$

$$+F(d(x_n, z))$$

$$= k\{F\big(d(x_n, z) + d(x_{n-1}, z) + d(x_{n-1}, Tz)\big)\}$$

$$+F(d(x_n, z))$$

$$\leq k\{F(d(x_n, z) + d(x_{n-1}, z) + d(x_{n-1}, z) + d(Tz, z))\}$$

$$+F(d(x_n, z))$$

Hence we have,

$$(1-k)F(d(Tz, z)) \leq 2kF(d(x_{n-1}, z)) + kF(d(x_n, z))$$

$$+F(d(x_n, z))$$

$$F(d(Tz, z)) \leq \frac{2k}{1-k}\, F(d(x_{n-1}, z)) + \frac{k}{1-k} F(d(x_n, z)) + \frac{1}{1-k} F\big(d(x_n, z)\big)$$

$\ll \frac{2c}{4} + \frac{c}{4} + \frac{c}{4} = c.$

Thus, $F(d(Tz, z)) \ll \frac{c}{n}$ for all $n \in \mathbb{N}$, and so$\frac{c}{n} - F(d(Tz, z)) \in P$. Since $\frac{c}{n} \to 0$ and P is closed, $-F(d(Tz, z)) \in P$. Hence $F(d(Tz, z)) = 0$.

By $(F1), d(Tz, z) = 0$ and so $z = Tz$.

Assume thatu is another fixed point of T.

Then from (2.2) ,we have

$$F\big(d(z, u)\big) = F(d(Tz, Tu))$$

$$\leq k\{F(d(Tz, u) + d(z, u) + d(z, Tu))\}$$

$$\leq k\{F(d(z, u) + d(z, u) + d(z, u))\}$$

$$\leq 3kF(d(z, u))$$

Hence $d(z, u) = 0$and so$z = u$.

Therefore,the fixed pointof T is unique.

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