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A NEW ITERATIVE METHOD FOR SOLVING A SYSTEM OF GENERALIZED EQUILIBRIUM PROBLEMS, GENERALIZED MIXED EQUILIBRIUM PROBLEMS AND COMMON FIXED POINT PROBLEMS IN HILBERT SPACES

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Abstract. In this paper, we introduce an iterative method for finding a common element of the set of solutions of a generalized mixed equilibrium problem (GMEP), the solutions of a general system of equilibrium problem and the set of common fixed points of a finite family of nonexpansive mappings in a real Hilbert space. Then, we prove that the sequence converges strongly to a common element of the above three sets. Furthermore, we apply our result to prove four new strong convergence theorems in fixed point problems, mixed equilibrium problems, generalized equilibrium problems , equilibrium problems and variational inequality.

Keywords: Mixed equilibrium problems; Equilibrium problems; Nonexpansive mappings; Fixed point, inverse-strongly monotone mapping; variational inequality.

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1. INTRODUCTION

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H. Let $\varphi : C \to \mathbb{R}$ be a real value function, $A : C \to H$ a nonlinear mapping and let $\Phi : C \times C \to \mathbb{R}$ be a bifunction, i.e., $\Phi(x,x) = 0$ for each $x \in C$.

Peng and Yao [11] considered the generalized mixed equilibrium problem of finding $x^* \in C$ such that

$$(GMEP): \qquad \Phi(x^*,y) + \varphi(y) - \varphi(x^*) + \langle Ax^*, y - x^*\rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of solutions for problem (1.1) is denoted by Ω, i.e.,

$$\Omega = \{x^* \in C : \Phi(x^*,y) + \varphi(y) - \varphi(x^*) + \langle Ax^*, y - x^*\rangle \geq 0, \quad \forall y \in C\}. \tag{1.2}$$

If $A \equiv 0$ in (1.1), then (GMEP) (1.1) reduces to the classical mixed equilibrum problem (for short, MEP) and Ω is denoted by $MEP(\Phi,\varphi)$, that is,

$$MEP(\Phi,\varphi) = \{x^* \in C : \Phi(x^*,y) + \varphi(y) - \varphi(x^*) \geq 0, \quad \forall y \in C\}, \tag{1.3}$$

which was considered by Ceng and Yao [3].

If $\varphi \equiv 0$ in (1.1), then (GMEP) (1.1) reduces to the generalized equilibrium problem (for short, GEP) and Ω is denoted by EP, that is,

$$EP = \{x^* \in C : \Phi(x^*,y) + \langle Ax^*, y - x^*\rangle \geq 0, \quad \forall y \in C\}. \tag{1.4}$$

which was studied by Takahashi and Takahashi [16] and many other for instance, [8,15-17].

If $\varphi \equiv 0$ and $A \equiv 0$ in (1.1), then (GMEP) (1.1) reduces to the classical equilibrium problem (for short, EP) and Ω is denoted by $EP(\Phi)$, that is,

$$EP(\Phi) = \{x^* \in C : \Phi(x^*,y) \geq 0, \quad \forall y \in C\}. \tag{1.5}$$

If $\Phi \equiv 0$ and $\varphi \equiv 0$ in (1.1), then (GMEP) (1.1) reduces to the classical variational inequality and Ω is denoted by $VI(A,C)$, that is,

$$VI(A,C) = \{x^* \in C : \langle Ax^*, y - x^*\rangle \geq 0, \quad \forall y \in C\}. \tag{1.6}$$

In 2005, Combettes and Hirstoaga [4] introduced an iterative scheme of finding the best approximation to the initial data when $EP(\Phi) \neq \emptyset$ and proved a strong convergence theorem.

In 2006, Takahashi and Takahashi [17] introduced an iterative scheme by the viscosity approximation method for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of nonexpansive mapping in a Hilbert space and proved a strong convergence theorem.

In 2007, Tada and Takahashi [15] introduced two iterative schemes for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space and obtained both strong convergence theorem and weak convergence theorem. In 2008, Takahashi and Takahashi [16] introduced an iterative method for finding a common element of the set of solutions of a generalized equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space and then obtain that the sequence converges strongly to a common element of two sets. Moreover they proved three new strong convergence theorems in fixed point problems, variational inequalities and equilibrium problems.

In 2008, Ceng and Yao [3] introduced a hybrid iterative scheme for finding a common element of the set of solutions of mixed equilibrium problem (1.3) and the set of common fixed points of finitely many nonexpansive mappings and they proved that the sequences generated by the hybrid iterative scheme converge strongly to a common element of the set of solutions of mixed equilibrium problem and the set of common fixed points of finitely many nonexpansive mappings.

The generalized mixed equilibrium problems includes, optimization problems, variational inequalities, the Nash equilibrium problem in noncooperative games and others; see, for example [1, 3, 16]. Peng and Yao [11] obtained some strong convergence theorems for iterative schemes based on the hybrid method and the extragradient method for finding a common element of the set of solutions of problem (1.1), the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality.

Very recently, Jeong [7] consedered the generalized equilibrium problem $(\bar{x},\bar{y}) \in C\times C$ such that

$$\text{(1.7)} \qquad \begin{cases} G_1(\bar{x},x)+\langle F_1\bar{y},x-\bar{x}\rangle+\frac{1}{\mu_1}\langle \bar{x}-\bar{y},x-\bar{x}\rangle \geq 0, & \forall x\in C, \\ G_2(\bar{y},y)+\langle F_2\bar{x},y-\bar{y}\rangle+\frac{1}{\mu_2}\langle \bar{y}-\bar{x},y-\bar{y}\rangle \geq 0, & \forall y\in C, \end{cases}$$

where $G_1,G_2 : C\times C \to \mathbb{R}$ are two bifunctions, $F_1,F_2 : C \to H$ are two nonlinear and $\mu_1 > 0$ and $\mu_2 > 0$ are two constants.

In this paper, we will introduced an iterative scheme by the general iterative method (3.1) for finding an element of the set of solutions of the generalized mixed equilibrium problem (1.1), the set of solutions of the generalized equilibrium problem (1.7) and the set of common fixed points of finitely many nonexpansive mappings in real Hilbert space, where$A,F_1,F_2 : C \to H$ be $\eta-$inverse strongly monotone, ζ_1-inverse strongly monotone and ζ_2-inverse strongly monotone, respectively, and then obtain a strong convergence theorem. Moreover we using this theorem to the problem for finding a common elements of $\cap_{i=1}^N F(T_i)\cap MEP(\Phi,\varphi)\cap O$, $\cap_{i=1}^N F(T_i)\cap EP\cap O$, $\cap_{i=1}^N F(T_i)\cap EP(\Phi)\cap O$ and $\cap_{i=1}^N F(T_i)\cap VI(A,C)\cap O$, respectively, where O is the set of solutions of the generalized equilibrium problem (1.7).

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let symbols $\rightharpoonup$ and $\to$ denote weak and strong convergence, respectively. Let C be a nonempty closed convex subset of H. Then, for any $x\in H$, there exists a unique nearest point in C, denoted by $P_C(x)$ such that $\|x-P_C(x)\| \leq \|x-y\|, \ \forall y\in C$. The mapping $P_C : x \to P_C(x)$ is called the *metric projection* of H onto C. We know that P_C is nonexpansive.

The following characterizes the projection P_C.

Lemma 2.1. *(See* [14]*) Given $x\in H$ and $y\in C$. Then $P_C(x)=y$ if and only if there holds the inequality*

$$\langle x-y,y-z\rangle \geq 0, \ \ \forall z\in C.$$

Recall that the following definitions.

(1) A mapping $T: C \to C$ is called **nonexpansive** if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. Next, we denote by $F(T)$ the set of fixed points of T, i.e., $F(T) = \{x \in C : Tx = x\}$.

(2) A mapping $f: H \to H$ is said to be a **contraction** if there exists a constant $\rho \in (0,1)$ such that $\|f(x) - f(y)\| \le \rho \|x - y\|$ for all $x, y \in H$.

(3) A mapping $A: C \to H$ is called **monotone** if $\langle Ax - Ay, x - y\rangle \ge 0$ for all $x, y \in C$ and it is called $\alpha-$**inverse strongly monotone** if there exists a positive real number α such that $\langle x - y, Ax - Ay\rangle \ge \alpha \|Ax - Ay\|^2, \ \forall x, y \in C$. We can see that if A is $\alpha-$**inverse strongly monotone**, then A is monotone mapping.

The following lemmas will be useful for proving our main results.

Lemma 2.2. *(See* [14]*) For all $x, y \in H$, there holds the inequality*

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle.$$

Lemma 2.3. *(See* [14]*) In a strictly convex Banach space E, if*

$$\|x\| = \|y\| = \|\lambda x + (1 - \lambda) y\|,$$

for all $x, y \in E$ and $\lambda \in (0,1)$, then $x = y$.

Lemma 2.4. *(See* [19]*) Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying $a_{n+1} = (1 - \alpha_n) a_n + \alpha_n \beta_n, \ \forall n \ge 0$ where $\{\alpha_n\}, \{\beta_n\}$ satisfy the conditions*

(i) $\{\alpha_n\} \subset [0,1], \quad \sum_{n=1}^{\infty} \alpha_n = \infty;$

(ii) $\limsup_{n\to\infty} \beta_n \le 0.$

Then $\lim_{n\to\infty} a_n = 0.$

Lemma 2.5. *(See* [13]*) Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$. Suppose*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) y_n,$$

for all integer $n \ge 0$ and

$$\limsup_{n\to\infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0.$$

Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0.$

Lemma 2.6. *(See [3]) Let C be a nonempty closed convex subset of H , $\varphi : C \to \mathbb{R}$ be a lower semicontinuous and convex function and let Φ be a bifunction of $C \times C$ in to $\mathbb{R}$ satisfy*

(A1) $\Phi(x,x) = 0$ *for all* $x \in C$;

(A2) Φ *is monotone, i.e.,* $\Phi(x,y) + \Phi(y,x) \leq 0, \quad \forall x,y \in C$;

(A3) *for all* $x,y,z \in C, \quad \lim_{t\to 0} \Phi(tz+(1-t)x,y) \leq \Phi(x,y)$;

(A4) *for all* $x \in C, \quad y \mapsto \Phi(x,y)$ *is convex and lower semicontinuous;*

(B1) *for each* $x \in H$ *and* $r > 0$, *there exists a bounded subset* $D_x \subset C$ *and* $y_x \in C$ *such that for any* $z \in C \backslash D_x$,

$$\Phi(z,y_x) + \varphi(y_x) + \frac{1}{r}\langle y_x - z, z - x\rangle < \varphi(z).$$

(B2) C *is bounded set.*

Assume that either (B1) or (B2) holds. For $x \in C$ and $r > 0$, define a mapping $T_r^{(\Phi,\varphi)} : H \to C$ as follows.

$$T_r^{(\Phi,\varphi)}(x) := \{z \in C : \Phi(z,y) + \varphi(y) + \frac{1}{r}\langle y - z, z - x\rangle \geq \varphi(z), \ \ \forall y \in C\}$$

for all $x \in H$. Then , the following conditions hold:

(i) *For each* $x \in H, T_r^{(\Phi,\varphi)}(x) \neq \emptyset$;

(ii) $T_r^{(\Phi,\varphi)}$ *is single-valued;*

(iii) $T_r^{(\Phi,\varphi)}$ *is firmly nonexpansive, i.e.,*

$$\|T_r^{(\Phi,\varphi)}x - T_r^{(\Phi,\varphi)}y\|^2 \leq \langle T_r^{(\Phi,\varphi)}x - T_r^{(\Phi,\varphi)}y, x - y\rangle, \ \ \forall x,y \in H;$$

(iv) $F(T_r^{(\Phi,\varphi)}) = MEP(\Phi,\varphi)$;

(v) $MEP(\Phi,\varphi)$ *is closed and convex.*

Remark 2.7. If $\varphi \equiv 0$ then $T_r^{(\Phi,\varphi)}$ is rewritten as T_r^{Φ}.

Lemma 2.8. *(see [7]) Let C be a nonempty closed convex subset of H. let $G_1, G_2 : C \times C \longrightarrow \mathbb{R}$ be two a bifunctions satisfying conditions (A1)-(A4) and let the mapping $F_1, F_2 : C \longrightarrow H$ be ζ_1- inverse strongly monotone and ζ_2- inverse strongly monotone, respectively. Then, for* *given* $\overline{x}, \overline{y} \in C$, $(\overline{x}, \overline{y})$ *is a solution* (1.7) *if and only if* $\overline{x}$ *is a fixed point of the mapping* $\Gamma : C \longrightarrow C$ *defined by*

$$\Gamma(x) = T_{\mu_1}^{G_1}(T_{\mu_2}^{G_2}(x - \mu_2 F_2 x) - \mu_1 F_1 T_{\mu_2}^{G_2}(x - \mu_2 F_2 x)), \quad \forall x \in C,$$

where $\overline{y} = T_{\mu_2}^{G_2}(\overline{x} - \mu_2 F_2 \overline{x})$.

The set of fixed points of the mapping Γ is denoted by O.

Proposition 2.9. *(see* [16]*) Let* C, H, Φ, φ *and* $T_r^{(\Phi,\varphi)}$ *be as in Lemma 2.6. Then the following holds:*

$$\|T_s^{(\Phi,\varphi)}x - T_t^{(\Phi,\varphi)}x\|^2 \leq \frac{s-t}{s}\langle T_s^{(\Phi,\varphi)}x - T_t^{(\Phi,\varphi)}x, T_s^{(\Phi,\varphi)}x - x\rangle$$

for all $s, t > 0$ *and* $x \in H$.

Lemma 2.10. *(see* [5]*) Assume that* T *is a nonexpansive self-mapping of a nonempty closed convex subset* C *of* H*. If* T *has a fixed point, then* $I - T$ *is demi-closed, that is, when* $\{x_n\}$ *is a sequence in* C *converging weakly to some* $x \in C$ *and the sequence* $\{(I-T)x_n\}$ *converges strongly to some y, it follows that* $(I-T)x = y$.

Let X be a real Hilbert space and C a nonempty closed convex subset of X. For a finite family of nonexpansive mappings $T_1, T_2, \ldots, T_N$ and sequence $\{\lambda_{n,i}\}_{i=1}^N$ in $[0,1]$, Kangtunyakarn and Suantai [8] defined the mapping $K_n : C \to C$ as follows:

$$U_{n,1} = \lambda_{n,1}T_1 + (1 - \lambda_{n,1})I,$$
$$U_{n,2} = \lambda_{n,2}T_2U_{n,1} + (1 - \lambda_{n,2})U_{n,1},$$
$$U_{n,3} = \lambda_{n,2}T_3U_{n,2} + (1 - \lambda_{n,3})U_{n,2},$$
$$\vdots$$
$$U_{n,N-1} = \lambda_{n,N-1}T_{N-1}U_{n,N-2} + (1 - \lambda_{n,N-1})U_{n,N-2},$$
$$K_n = U_{n,N} = \lambda_{n,N}T_NU_{n,N-1} + (1 - \lambda_{n,N})U_{n,N-1} \tag{2.1}$$

Such a mapping K_n is called the $K-$mapping generated by $T_1, T_2, \ldots, T_N$ and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$.

Definition 2.11. (See [8]) Let C be a nonempty convex subset of a real Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mapping of C into itself, and let $\lambda_1, \ldots, \lambda_N$ be real numbers such that $0 \leq \lambda_i \leq 1$ for every $i = 1, \ldots, N$. They define a mapping $K : C \to C$ as follows:

$$
\begin{aligned}
U_1 &= \lambda_1 T_1 + (1-\lambda_1)I,\\
U_2 &= \lambda_2 T_2 U_1 + (1-\lambda_2)U_1,\\
U_3 &= \lambda_3 T_3 U_2 + (1-\lambda_3)U_2,\\
&\vdots\\
U_{N-1} &= \lambda_{N-1} T_{N-1} U_{N-2} + (1-\lambda_{N-1})U_{N-2},\\
K = U_N &= \lambda_N T_N U_{N-1} + (1-\lambda_N)U_{N-1}.
\end{aligned}
$$

Such a mapping K is called the $K-$*mapping* generated by $T_1, \ldots, T_N$ and $\lambda_1, \ldots, \lambda_N$.

Lemma 2.12. *(See* [8]*) Let C be a nonempty closed convex subset of a strictly convex Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself with $\cap_{i=1}^N F(T_i) \neq \emptyset$ and let $\lambda_1, \ldots, \lambda_N$ be real numbers such that $0 < \lambda_i < 1$ for every $i = 1, \ldots, N-1$ and $0 < \lambda_N \leq 1$. Let K be the $K-$mapping generated by $T_1, \ldots, T_N$ and $\lambda_1, \ldots, \lambda_N$. Then $F(K) = \cap_{i=1}^N F(T_i)$.*

Lemma 2.13. *(See* [8]*) Let C be a nonempty closed convex subset of a Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself and $\{\lambda_{n,i}\}_{i=1}^N$ sequences in $[0,1]$ such that $\lambda_{n,i} \to \lambda_i$, as $n \to \infty$ $(i = 1, 2, \ldots, N)$. Moreover, for every $n \in \mathbb{N}$, let K and K_n be the $K-$mappings generated by $T_1, T_2, \ldots, T_N$ and $\lambda_1, \lambda_2, \ldots, \lambda_N$ and $T_1, T_2, \ldots, T_N$ and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$, respectively. Then, for every $x \in C$,*

$$\lim_{n\to\infty} \|K_n x - Kx\| = 0.$$

Lemma 2.14. *(see* [12]*) Let $\{x_n\}$ be a bounded sequence in a Hilbert space H. Then there exits $L > 0$ such that*

$$\|K_{n+1}x_{n+1} - K_n x_n\| \leq \|x_{n+1} - x_n\| + L\sum_{i=1}^{N} |\lambda_{n+1,i} - \lambda_{n,i}|, \ \ \forall n \geq 0. \tag{2.2}$$

3. MAIN RESULTS

We are now in a position to prove the main result of this paper.

Theorem 3.1. *Let H be a real Hilbert space, C a closed convex nonempty subset of H. Let $\Phi, G_1, G_2 : C \times C \to \mathbb{R}$ be three bifunctions which satisfying (A1)-(A4) and $\varphi : C \to \mathbb{R}$ a lower semicontinuous and convex functional. Let $A, F_1, F_2 : C \to H$ be $\eta-$inverse strongly monotone, ζ_1-inverse strongly monotone and ζ_2-inverse strongly monotone, respectively. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself such that $\Delta = \cap_{i=1}^N F(T_i) \cap \Omega \cap O \neq \emptyset$ and f a $\rho-$contraction of C into itself. Assume that either (B1) or (B2) holds. Let $x_1 \in C$ and let $\{x_n\}$ be a sequence defined by*

$$(3.1) \qquad \begin{cases} u_n = T_{r_n}^{(\Phi,\varphi)}(x_n - r_n A x_n), \\ y_n = T_{\mu_1}^{G_1}[T_{\mu_2}^{G_2}(u_n - \mu_2 F_2 u_n) - \mu_1 F_1 T_{\mu_2}^{G_2}(u_n - \mu_2 F_2 u_n)], \\ x_{n+1} = \alpha_n f(K_n x_n) + \beta_n x_n + \gamma_n K_n y_n, \forall n \geq 1, \end{cases}$$

where K_n is a $K-$mapping generated by $T_1, T_2, \ldots, T_N$ and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$ and $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in $(0,1)$ with $\alpha_n + \beta_n + \gamma_n = 1$, $\{\lambda_{n,i}\}_{i=1}^N$ a sequence in $[a,b]$ with $0 < a \leq b < 1$, $\{r_n\}$ a sequence in $[0, 2\eta]$ for all $n \in \mathbb{N}$, $\mu_1 \in (0, 2\zeta_1)$ and $\mu_2 \in (0, 2\zeta_2)$ satisfy the following conditions:

(i) *the sequence $\{r_n\}$ satisfies*

(C1) $0 < c \leq r_n \leq d < 2\eta$*; and*

(C2) $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$*;*

(ii) *the sequence $\{\alpha_n\}$ satisfies*

(D1) $\lim_{n\to\infty} \alpha_n = 0$*; and*

(D2) $\sum_{n=0}^{\infty} \alpha_n = \infty$*;*

(iii) *the sequence $\{\beta_n\}$ satisfies*

(E1) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$*;*

(iv) *the finite family of sequences $\{\lambda_{n,i}\}_{i=1}^N$ satisfies*

(F1) $\lim_{n\to\infty} |\lambda_{n+1,i} - \lambda_{n,i}| = 0$ *for every $i \in \{1, 2, \ldots, N\}$.*

Then $\{x_n\}$ *converges strongly to* $x^* = P_\Delta f(x^*)$ *where* $\Delta = \cap_{i=1}^N F(T_i) \cap \Omega \cap O$ *and* (x^*, y^*) *is a solution of problem* (1.7) *where* $y^* = T_{\mu_2}^{G_2}(x^* - \mu_2 F_2 x^*)$.

Proof. Let $x, y \in C$. Since A is $\eta-$inverse strongly monotone and $r_n \in (0, 2\eta)$, $\forall n \in \mathbb{N}$, we have

$$\begin{aligned}
\|(I - r_nA)x - (I - r_nA)y\|^2 &= \|x - y - r_n(Ax - Ay)\|^2 \\
&= \|x - y\|^2 - 2r_n\langle x - y, Ax - Ay\rangle + r_n^2\|Ax - Ay\|^2 \\
&\leq \|x - y\|^2 - 2r_n\eta\|Ax - Ay\|^2 + r_n^2\|Ax - Ay\|^2 \\
&= \|x - y\|^2 + r_n(r_n - 2\eta)\|Ax - Ay\|^2 \\
&\leq \|x - y\|^2,
\end{aligned}$$

then the mapping $I - r_nA$ is a nonexpansive mapping, and so are $I - \mu_1 F_1$ and $I - \mu_2 F_2$, provided $\mu_1 \in (0, 2\zeta_1)$ and $\mu_2 \in (0, 2\zeta_2)$, respectively.

We shall divide the proof into several steps.

step 1. We shall show that the sequences $\{x_n\}$ is bounded.

Let $p \in \Delta = \cap_{i=1}^N F(T_i) \cap \Omega \cap O$. Since $p = T_{r_n}^{(\Phi,\varphi)}(p - r_nAp)$ and $T_{r_n}^{(\Phi,\varphi)}$ and $(I - r_nA)$ are nonexpansive, we obtain that for any $n \geq 1$,

$$\begin{aligned}
\|u_n - p\| &= \|T_{r_n}^{(\Phi,\varphi)}(x_n - r_nAx_n) - T_{r_n}^{(\Phi,\varphi)}(p - r_nAp)\| \\
&\leq \|(x_n - r_nAx_n) - (p - r_nAp)\| \\
&\leq \|x_n - p\|. \qquad (3.2)
\end{aligned}$$

Putting $z_n = T_{\mu_2}^{G_2}(u_n - \mu_2 F_2 u_n)$ and $z = T_{\mu_2}^{G_2}(p - \mu_2 F_2 p)$, we have

$$\begin{aligned}
\|z_n - z\| &= \|T_{\mu_2}^{G_2}(u_n - \mu_2 F_2 u_n) - T_{\mu_2}^{G_2}(p - \mu_2 F_2 p)\| \\
&\leq \|(u_n - \mu_2 F_2 u_n) - (p - \mu_2 F_2 p)\| \\
&\leq \|u_n - p\|. \qquad (3.3)
\end{aligned}$$

And since $p = T_{\mu_1}^{G_1}(z - \mu_1 F_1 z)$, we know that for any $n \geq 1$,

$$\begin{aligned} \|y_n - p\| &= \|T_{\mu_1}^{G_1}(z_n - \mu_1 F_1 z_n) - T_{\mu_1}^{G_1}(z - \mu_1 F_1 z)\| \\ &\leq \|(z_n - \mu_1 F_1 z_n) - (z - \mu_1 F_1 z)\| \\ &\leq \|z_n - z\| \\ &\leq \|u_n - p\|. \end{aligned} \tag{3.4}$$

Furthermore, from (3.1), (3.2) and (3.4) we have

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n f(K_n x_n) + \beta_n x_n + \gamma_n K_n y_n - p\| \\ &= \|\alpha_n f(K_n x_n) + \beta_n x_n + \gamma_n K_n y_n - (\alpha_n + \beta_n + \gamma_n) p\| \\ &\leq \alpha_n \|f(K_n x_n) - p\| + \beta_n \|x_n - p\| + \gamma_n \|K_n y_n - p\| \\ &\leq \alpha_n (\|f(K_n x_n) - f(p)\| + \|f(p) - p\|) + \beta_n \|x_n - p\| + \gamma_n \|y_n - p\| \\ &\leq \alpha_n (\rho \|x_n - p\| + \|f(p) - p\|) + \beta_n \|x_n - p\| + \gamma_n \|x_n - p\| \\ &= \alpha_n \rho \|x_n - p\| + \beta_n \|x_n - p\| + \gamma_n \|x_n - p\| + \alpha_n \|f(p) - p\| \\ &= \alpha_n \rho \|x_n - p\| + (1 - \alpha_n) \|x_n - p\| + \alpha_n \|f(p) - p\| \\ &= (1 - \alpha_n(1 - \rho)) \|x_n - p\| + \alpha_n \|f(p) - p\| \\ &= (1 - \alpha_n(1 - \rho)) \|x_n - p\| + \alpha_n (1 - \rho) \cdot \frac{1}{1 - \rho} \|f(p) - p\|. \end{aligned} \tag{3.5}$$

It follows from (3.5) induction that

$$\|x_n - p\| \leq M, \quad \forall n \geq 1$$

where $M = \max\{\|x_0 - p\|, \frac{1}{1-\rho}\|f(p) - p\|\}$. Hence $\{x_n\}$ is bounded, and so are $\{u_n\}, \{y_n\}, \{Ax_n\}, \{f(K_n x_n)\}$ and $\{K_n y_n\}$.

Step 2. We claim that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

Define

$$w_n = \frac{\alpha_n}{1 - \beta_n} f(K_n x_n) + \frac{\gamma_n}{1 - \beta_n} K_n y_n, \tag{3.6}$$

we have

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) w_n, \quad \forall n \geq 0.$$

Notice that

$$
\begin{aligned}
\|w_{n+1}-w_n\| &= \|\frac{\alpha_{n+1}}{1-\beta_{n+1}}f(K_{n+1}x_{n+1})+\frac{\gamma_{n+1}}{1-\beta_{n+1}}K_{n+1}y_{n+1}-\frac{\alpha_n}{1-\beta_n}f(K_nx_n)-\frac{\gamma_n}{1-\beta_n}K_ny_n\| \\
&\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(K_{n+1}x_{n+1})-f(K_nx_n)\|+|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}|\|f(K_nx_n)\| \\
&\quad +\frac{\gamma_{n+1}}{1-\beta_{n+1}}\|K_{n+1}y_{n+1}-K_ny_n\|+|\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n}|\|K_ny_n\| \\
&\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\rho\|K_{n+1}x_{n+1}-K_nx_n\|+|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}|(\|f(K_nx_n)\|+\|K_ny_n\|) \\
&\quad +\frac{\gamma_{n+1}}{1-\beta_{n+1}}\|K_{n+1}y_{n+1}-K_ny_n\|. \qquad (3.7)
\end{aligned}
$$

From Lemma 2.14, there exist $L_1 > 0$ and $L_2 > 0$ such that

$$
\begin{aligned}
\|w_{n+1}-w_n\| \leq & \frac{\alpha_{n+1}}{1-\beta_{n+1}}\rho(\|x_{n+1}-x_n\|+L_1\sum_{i=1}^{N}|\lambda_{n+1,i}-\lambda_{n,i}|) \\
&+|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}|(\|f(K_nx_n)\|+\|K_ny_n\|) \\
&+\frac{\gamma_{n+1}}{1-\beta_{n+1}}(\|y_{n+1}-y_n\|+L_2\sum_{i=1}^{N}|\lambda_{n+1,i}-\lambda_{n,i}|). \qquad (3.8)
\end{aligned}
$$

Notice that

$$
\begin{aligned}
\|y_{n+1}-y_n\|^2 &= \|T_{\mu_1}^{G_1}(z_{n+1}-\mu_1F_1z_{n+1})-T_{\mu_1}^{G_1}(z_n-\mu_1F_1z_n)\|^2 \\
&\leq \|(z_{n+1}-z_n)-\mu_1(F_1z_{n+1}-F_1z_n)\|^2 \\
&= \|z_{n+1}-z_n\|^2-2\mu_1\langle z_{n+1}-z_n, F_1z_{n+1}-F_1z_n\rangle+\mu_1^2\|F_1z_{n+1}-F_1z_n\|^2 \\
&\leq \|z_{n+1}-z_n\|^2-2\mu_1\zeta_1\|F_1z_{n+1}-F_1z_n\|^2+\mu_1^2\|F_1z_{n+1}-F_1z_n\|^2 \\
&= \|z_{n+1}-z_n\|^2+\mu_1(\mu_1-2\zeta_1)\|F_1z_{n+1}-F_1z_n\|^2
\end{aligned}
$$

$$\begin{aligned}
\|y_{n+1}-y_n\|^2 &\le \|z_{n+1}-z_n\|^2 \\
&= \|T_{\mu_2}^{G_2}(u_{n+1}-\mu_2 F_2 u_{n+1}) - T_{\mu_2}^{G_2}(u_n-\mu_2 F_2 u_n)\|^2 \\
&\le \|(u_{n+1}-u_n)-\mu_2(F_2u_{n+1}-F_2u_n)\|^2 \\
&= \|u_{n+1}-u_n\|^2 - 2\mu_2\langle u_{n+1}-u_n, F_2u_{n+1}-F_2u_n\rangle + \mu_2^2\|F_2u_{n+1}-F_2u_n\|^2 \\
&\le \|u_{n+1}-u_n\|^2 - 2\mu_2\zeta_2\|F_2u_{n+1}-F_2u_n\|^2 + \mu_2^2\|F_2u_{n+1}-F_2u_n\|^2 \\
&= \|u_{n+1}-u_n\|^2 + \mu_2(\mu_2-2\zeta_2)\|F_2u_{n+1}-F_2u_n\|^2 \\
&\le \|u_{n+1}-u_n\|^2. \qquad (3.9)
\end{aligned}$$

And

$$\begin{aligned}
\|u_{n+1}-u_n\| &= \|T_{r_{n+1}}^{(\Phi,\varphi)}(x_{n+1}-r_{n+1}Ax_{n+1}) - T_{r_n}^{(\Phi,\varphi)}(x_n-r_nAx_n)\| \\
&\le \|T_{r_{n+1}}^{(\Phi,\varphi)}(x_{n+1}-r_{n+1}Ax_{n+1}) - T_{r_{n+1}}^{(\Phi,\varphi)}(x_n-r_nAx_n)\| \\
&\quad + \|T_{r_{n+1}}^{(\Phi,\varphi)}(x_n-r_nAx_n) - T_{r_n}^{(\Phi,\varphi)}(x_n-r_nAx_n)\| \\
&\le \|(x_{n+1}-r_{n+1}Ax_{n+1})-(x_n-r_nAx_n)\| \\
&\quad + \|T_{r_{n+1}}^{(\Phi,\varphi)}(x_n-r_nAx_n) - T_{r_n}^{(\Phi,\varphi)}(x_n-r_nAx_n)\| \\
&= \|x_{n+1}-x_n-r_{n+1}Ax_{n+1}+r_{n+1}Ax_n-r_{n+1}Ax_n+r_nAx_n\| \\
&\quad + \|T_{r_{n+1}}^{(\Phi,\varphi)}(x_n-r_nAx_n) - T_{r_n}^{(\Phi,\varphi)}(x_n-r_nAx_n)\|
\end{aligned}$$

$$\begin{aligned}
&\le \|(x_{n+1}-r_{n+1}Ax_{n+1})-(x_n-r_{n+1}Ax_n)\| + \|r_nAx_n-r_{n+1}Ax_n\| \\
&\quad + \|T_{r_{n+1}}^{(\Phi,\varphi)}(x_n-r_nAx_n) - T_{r_n}^{(\Phi,\varphi)}(x_n-r_nAx_n)\| \\
&\le \|x_{n+1}-x_n\| + |r_{n+1}-r_n|\|Ax_n\| \\
&\quad + \|T_{r_{n+1}}^{(\Phi,\varphi)}(x_n-r_nAx_n) - T_{r_n}^{(\Phi,\varphi)}(x_n-r_nAx_n)\|. \qquad (3.10)
\end{aligned}$$

It follows from (3.9) and (3.10) that

$$\begin{aligned}\|y_{n+1}-y_n\| &\le \|u_{n+1}-u_n\| \\ &\le \|x_{n+1}-x_n\|+|r_{n+1}-r_n|\|Ax_n\| \\ &\quad +\|T_{r_{n+1}}^{(\Phi,\varphi)}(x_n-r_nAx_n)-T_{r_n}^{(\Phi,\varphi)}(x_n-r_nAx_n)\|. \end{aligned} \tag{3.11}$$

Without loss of generality, let us assume that there exists a real number k such that $r_n > k > 0$ for all n. Utilizing Proposition 2.9, we have

$$\begin{aligned}&\|T_{r_{n+1}}^{(\Phi,\varphi)}(x_n-r_nAx_n)-T_{r_n}^{(\Phi,\varphi)}(x_n-r_nAx_n)\| \\ &\le \frac{|r_{n+1}-r_n|}{r_{n+1}}\|T_{r_{n+1}}^{(\Phi,\varphi)}(I-r_nA)x_n\| \\ &\le \frac{|r_{n+1}-r_n|}{k}\|T_{r_{n+1}}^{(\Phi,\varphi)}(I-r_nA)x_n\|. \end{aligned} \tag{3.12}$$

By (3.11) and (3.12) , we have

$$\|y_{n+1}-y_n\| \le \|x_{n+1}-x_n\|+|r_{n+1}-r_n|\|Ax_n\|+\frac{|r_{n+1}-r_n|}{k}\|T_{r_{n+1}}^{(\Phi,\varphi)}(I-r_nA)x_n\|. \tag{3.13}$$

Combining (3.8) and (3.13), we deduce

$$\begin{aligned}\|w_{n+1}-w_n\| &\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\rho(\|x_{n+1}-x_n\|+L_1\sum_{i=1}^{N}|\lambda_{n+1,i}-\lambda_{n,i}|) \\ &\quad +|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}|(\|f(K_nx_n)\|+\|K_ny_n\|) \\ &\quad +\frac{\gamma_{n+1}}{1-\beta_{n+1}}(\|x_{n+1}-x_n\|+|r_{n+1}-r_n|\|Ax_n\|+\frac{|r_{n+1}-r_n|}{k}\|T_{r_{n+1}}^{(\Phi,\varphi)}(I-r_nA)x_n\| \\ &\quad +L_2\sum_{i=1}^{N}|\lambda_{n+1,i}-\lambda_{n,i}|) \\ &\le \|x_{n+1}-x_n\|+|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}|(\|f(K_nx_n)\|+\|K_ny_n\|) \\ &\quad +\frac{\gamma_{n+1}}{1-\beta_{n+1}}|r_{n+1}-r_n|\|Ax_n\|+\frac{\gamma_{n+1}}{1-\beta_{n+1}}\cdot\frac{1}{k}|r_{n+1}-r_n|\|T_{r_{n+1}}^{(\Phi,\varphi)}(I-r_nA)x_n\| \\ &\quad +\frac{\gamma_{n+1}}{1-\beta_{n+1}}L_2\sum_{i=1}^{N}|\lambda_{n+1,i}-\lambda_{n,i}|+\frac{\alpha_{n+1}}{1-\beta_{n+1}}L_1\sum_{i=1}^{N}|\lambda_{n+1,i}-\lambda_{n,i}|.\end{aligned}$$

Therefore

$$\|w_{n+1}-w_n\|-\|x_{n+1}-x_n\| \le \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right|(\|f(K_nx_n)\|+\|K_ny_n\|)$$
$$+\frac{\gamma_{n+1}}{1-\beta_{n+1}}|r_{n+1}-r_n|\|Ax_n\|+\frac{\gamma_{n+1}}{1-\beta_{n+1}}\cdot\frac{1}{k}|r_{n+1}-r_n|\|T_{r_{n+1}}^{(\Phi,\varphi)}(I-r_nA)x_n\|$$
$$+\frac{\gamma_{n+1}}{1-\beta_{n+1}}L_2\sum_{i=1}^{N}|\lambda_{n+1,i}-\lambda_{n,i}|+\frac{\alpha_{n+1}}{1-\beta_{n+1}}L_1\sum_{i=1}^{N}|\lambda_{n+1,i}-\lambda_{n,i}|. \tag{3.14}$$

Applying the conditions (C2), (D1), (E1) and (F1) and taking the superior limit as $n\to\infty$ to (3.14), we have

$$\limsup_{n\to\infty}(\|w_{n+1}-w_n\|-\|x_{n+1}-x_n\|)=0.$$

Hence, by Lemma 2.5, we get $\lim_{n\to\infty}\|w_n-x_n\|=0$. This implies that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\beta_n)\|w_n-x_n\|=0.$$

Step 3. We shall show that $\lim_{n\to\infty}\|x_n-u_n\|=0$, $\lim_{n\to\infty}\|u_n-y_n\|=0$, $\lim_{n\to\infty}\|x_n-y_n\|=0$ and $\lim_{n\to\infty}\|K_ny_n-y_n\|=0$.
Since $x_{n+1}=\alpha_nf(K_nx_n)+\beta_nx_n+\gamma_nK_ny_n$, we obtain

$$\begin{aligned}\|x_n-K_ny_n\| &\le \|x_n-x_{n+1}\|+\|x_{n+1}-K_ny_n\|\\ &=\|x_n-x_{n+1}\|+\|\alpha_nf(K_nx_n)+\beta_nx_n+\gamma_nK_ny_n-K_ny_n\|\\ &=\|x_n-x_{n+1}\|+\|\alpha_nf(K_nx_n)+\beta_nx_n-(1-\gamma_n)K_ny_n\|\\ &=\|x_n-x_{n+1}\|+\|\alpha_nf(K_nx_n)+\beta_nx_n-(\alpha_n+\beta_n)K_ny_n\|\\ &\le \|x_n-x_{n+1}\|+\alpha_n\|f(K_nx_n)-K_ny_n\|+\beta_n\|x_n-K_ny_n\|\end{aligned}$$

and hence

$$\|x_n-K_ny_n\|\le\frac{1}{1-\beta_n}\|x_n-x_{n+1}\|+\frac{\alpha_n}{1-\beta_n}\|f(K_nx_n)-K_ny_n\|. \tag{3.15}$$

Since $\alpha_n\to 0$ and $\|x_n-x_{n+1}\|\to 0$ as $n\to\infty$, (3.15) implies that

$$\lim_{n\to\infty}\|x_n-K_ny_n\|=0. \tag{3.16}$$

Since A, F_1 and F_2 are $\eta-$inverse strongly monotone, ζ_1-inverse strongly monotone and ζ_2-inverse strongly monotone, respectively and $p \in \Delta$, we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|\alpha_n f(K_nx_n)+\beta_nx_n+\gamma_nK_ny_n-p\|^2 \\
&\leq \alpha_n\|f(K_nx_n)-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n\|K_ny_n-p\|^2 \\
&\leq \alpha_n\|f(K_nx_n)-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n\|y_n-p\|^2 \\
&= \alpha_n\|f(K_nx_n)-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n\|T_{\mu_1}^{G_1}(z_n-\mu_1F_1z_n)-T_{\mu_1}^{G_1}(z-\mu_1F_1z)\|^2 \\
&\leq \alpha_n\|f(K_nx_n)-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n\|(z_n-z)-\mu_1(F_1z_n-F_1z)\|^2 \\
&\leq \alpha_n\|f(K_nx_n)-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n[\|z_n-z\|^2-2\mu_1\langle z_n-z, F_1z_n-F_1z\rangle \\
&\quad +\mu_1^2\|F_1z_n-F_1z\|^2] \\
&\leq \alpha_n\|f(K_nx_n)-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n[\|z_n-z\|^2-2\mu_1\zeta_1\|F_1z_n-F_1z\|^2 \\
&\quad +\mu_1^2\|F_1z_n-F_1z\|^2] \\
&= \alpha_n\|f(K_nx_n)-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n[\|z_n-z\|^2+\mu_1(\mu_1-2\zeta_1)\|F_1z_n-F_1z\|^2]
\end{aligned}$$

$$\begin{aligned}
&= \alpha_n\|f(K_nx_n)-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n[\|T_{\mu_2}^{G_2}(u_n-\mu_2F_2u_n)-T_{\mu_2}^{G_2}(p-\mu_2F_2p)\|^2 \\
&\quad +\mu_1(\mu_1-2\zeta_1)\|F_1z_n-F_1z\|^2] \\
&\leq \alpha_n\|f(K_nx_n)-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n[\|u_n-p\|^2+\mu_2(\mu_2-2\zeta_2)\|F_2u_n-F_2p\|^2 \\
&\quad +\mu_1(\mu_1-2\zeta_1)\|F_1z_n-F_1z\|^2] \\
&= \alpha_n\|f(K_nx_n)-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n[\|T_{r_n}^{(\Phi,\varphi)}(x_n-r_nAx_n)-T_{r_n}^{(\Phi,\varphi)}(p-r_nAp)\|^2 \\
&\quad +\mu_2(\mu_2-2\zeta_2)\|F_2u_n-F_2p\|^2+\mu_1(\mu_1-2\zeta_1)\|F_1z_n-F_1z\|^2] \\
&\leq \alpha_n\|f(K_nx_n)-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n[\|x_n-p\|^2+r_n(r_n-2\eta)\|Ax_n-Ap\|^2 \\
&\quad +\mu_2(\mu_2-2\zeta_2)\|F_2u_n-F_2p\|^2+\mu_1(\mu_1-2\zeta_1)\|F_1z_n-F_1z\|^2]. \qquad (3.17)
\end{aligned}$$

It follows that

$$
\begin{aligned}
&\gamma_n r_n(2\eta - r_n)\|Ax_n - Ap\|^2 + \gamma_n\mu_2(2\zeta_2 - \mu_2)\|F_2u_n - F_2p\|^2 + \gamma_n\mu_1(2\zeta_1 - \mu_1)\|F_1z_n - F_1z\|^2 \\
&\qquad \le \alpha_n\|f(K_nx_n) - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\qquad = \alpha_n\|f(K_nx_n) - p\|^2 + (1 - \alpha_n)\|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\qquad = \alpha_n\|f(K_nx_n) - p\|^2 - \alpha_n\|x_n - p\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\qquad \le \alpha_n\|f(K_nx_n) - p\|^2 - \alpha_n\|x_n - p\|^2 + \|x_n - x_{n+1}\| \times (\|x_n - p\| + \|x_{n+1} - p\|).
\end{aligned}
$$

Since $0 < c \le r_n \le d < 2\eta$, we have

$$
\begin{aligned}
&\gamma_n c(2\eta - d)\|Ax_n - Ap\|^2 + \gamma_n\mu_2(2\zeta_2 - \mu_2)\|F_2u_n - F_2p\|^2 + \gamma_n\mu_1(2\zeta_1 - \mu_1)\|F_1z_n - F_1z\|^2 \\
&\qquad \le \alpha_n\|f(K_nx_n) - p\|^2 - \alpha_n\|x_n - p\|^2 + \|x_n - x_{n+1}\| \times (\|x_n - p\| + \|x_{n+1} - p\|).
\end{aligned} \tag{3.18}
$$

From $\alpha_n \to 0$, $\|x_n - x_{n+1}\| \to 0$ as $n \to \infty$ and the boundedness of $\{x_n\}$ and $\{f(K_nx_n)\}$, we have

$$
\lim_{n\to\infty}\|Ax_n - Ap\| = 0, \lim_{n\to\infty}\|F_1z_n - F_1z\| = 0 \quad \text{and} \lim_{n\to\infty}\|F_2u_n - F_2p\| = 0.
$$

Indeed, from (3.2), (3.3) and Lemma 2.6, we have

$$
\begin{aligned}
\|z_n - z\|^2 &= \|T_{\mu_2}^{G_2}(u_n - \mu_2F_2u_n) - T_{\mu_2}^{G_2}(p - \mu_2F_2p)\|^2 \\
&\le \langle T_{\mu_2}^{G_2}(u_n - \mu_2F_2u_n) - T_{\mu_2}^{G_2}(p - \mu_2F_2p), (u_n - \mu_2F_2u_n) - (p - \mu_2F_2p)\rangle \\
&= \langle (u_n - \mu_2F_2u_n) - (p - \mu_2F_2p), z_n - z\rangle \\
&= \frac{1}{2}(\|(u_n - \mu_2F_2u_n) - (p - \mu_2F_2p)\|^2 + \|z_n - z\|^2 - \|(u_n - \mu_2F_2u_n) - (p - \mu_2F_2p) - (z_n - z)\|^2) \\
&\le \frac{1}{2}(\|u_n - p\|^2 + \|z_n - z\|^2 - \|(u_n - z_n) - \mu_2(F_2u_n - F_2p) - (p - z)\|^2) \\
&\le \frac{1}{2}(\|x_n - p\|^2 + \|z_n - z\|^2 - \|(u_n - z_n) - (p - z)\|^2 \\
&\quad + 2\mu_2\langle (u_n - z_n) - (p - z), F_2u_n - F_2p\rangle - \mu_2^2\|F_2u_n - F_2p\|^2),
\end{aligned}
$$

and

$$
\begin{aligned}
\|y_n - p\|^2 &= \|T_{\mu_1}^{G_1}(z_n - \mu_1 F_1 z_n) - T_{\mu_1}^{G_1}(z - \mu_1 F_1 z)\|^2 \\
&\leq \langle T_{\mu_1}^{G_1}(z_n - \mu_1 F_1 z_n) - T_{\mu_1}^{G_1}(z - \mu_1 F_1 z), (z_n - \mu_1 F_1 z_n) - (z - \mu_1 F_1 z)\rangle \\
&= \langle (z_n - \mu_1 F_1 z_n) - (z - \mu_1 F_1 z), y_n - p\rangle \\
&= \frac{1}{2}(\|(z_n - \mu_1 F_1 z_n) - (z - \mu_1 F_1 z)\|^2 + \|y_n - p\|^2 \\
&\quad - \|(z_n - \mu_1 F_1 z_n) - (z - \mu_1 F_1 z) - (y_n - p)\|^2) \\
&\leq \frac{1}{2}(\|z_n - z\|^2 + \|y_n - p\|^2 - \|(z_n - y_n) + (p - z)\|^2 \\
&\quad + 2\mu_1 \langle (z_n - y_n) + (p - z), F_1 z_n - F_1 z\rangle - \mu_1^2 \|F_1 z_n - F_1 z\|^2) \\
&\leq \frac{1}{2}(\|x_n - p\|^2 + \|y_n - p\|^2 - \|(z_n - y_n) + (p - z)\|^2 \\
&\quad + 2\mu_1 \langle (z_n - y_n) + (p - z), F_1 z_n - F_1 z\rangle
\end{aligned}
$$

which implies that

$$
\|z_n - z\|^2 \leq \|x_n - p\|^2 - \|(u_n - z_n) - (p - z)\|^2 + 2\mu_2 \|(u_n - z_n) - (p - z)\| \|F_2 u_n - F_2 p\| \tag{3.19}
$$

and

$$
\|y_n - p\|^2 \leq \|x_n - p\|^2 - \|(z_n - y_n) + (p - z)\|^2 + 2\mu_1 \|F_1 z_n - F_1 z\| \|(z_n - y_n) + (p - z)\|. \tag{3.20}
$$

It follows from (3.20) that

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \alpha_n \|f(K_n x_n) - p\|^2 + \beta_n \|x_n - p\|^2 + \gamma_n \|y_n - p\|^2 \\
&\leq \alpha_n \|f(K_n x_n) - p\|^2 + \beta_n \|x_n - p\|^2 + \gamma_n (\|x_n - p\|^2 - \|(z_n - y_n) + (p - z)\|^2 \\
&\quad + 2\mu_1 \|F_1 z_n - F_1 z\| \|(z_n - y_n) + (p - z)\|),
\end{aligned}
$$

which finds that

$$
\begin{aligned}
\gamma_n\|(z_n-y_n)+(p-z)\|^2 \le\, & \alpha_n\|f(K_nx_n)-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n\|x_n-p\|^2-\|x_{n+1}-p\|^2 \\
& +2\mu_1\gamma_n\|F_1z_n-F_1z\|\|(z_n-y_n)+(p-z)\| \\
\le\, & \alpha_n\|f(K_nx_n)-p\|^2-\alpha_n\|x_n-p\|^2+\|x_n-x_{n+1}\| \\
& \times(\|x_n-p\|+\|x_{n+1}-p\|)+2\mu_1\gamma_n\|F_1z_n-F_1z\|\|(z_n-y_n)+(p-z)\|. \qquad (3.21)
\end{aligned}
$$

Since $\alpha_n\to 0,\ \ \|x_n-x_{n+1}\|\to 0$ and $\|F_1z_n-F_1z\|\to 0$ as $n\to\infty$, we have

$$
\lim_{n\to\infty}\|(z_n-y_n)+(p-z)\|=0. \qquad (3.22)
$$

Also, from (3.4) and (3.19) , we obtain that

$$
\begin{aligned}
\|x_{n+1}-p\|^2 \le\, & \alpha_n\|f(K_nx_n)-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n\|y_n-p\|^2 \\
\le\, & \alpha_n\|f(K_nx_n)-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n(\|z_n-z\|^2-\|(z_n-y_n)+(p-z)\|^2 \\
& +2\mu_1\|F_1z_n-F_1z\|\|(z_n-y_n)+(p-z)\|) \\
=\, & \alpha_n\|f(K_nx_n)-p\|^2+\beta_n\|x_n-p\|^2-\gamma_n\|(z_n-y_n)+(p-z)\|^2 \\
& +2\gamma_n\mu_1\|F_1z_n-F_1z\|\|(z_n-y_n)+(p-z)\|+\gamma_n(\|x_n-p\|^2 \\
& -\|(u_n-z_n)-(p-z)\|^2+2\mu_2\|(u_n-z_n)-(p-z)\|\|F_2u_n-F_2p\|)
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\gamma_n\|(u_n-z_n)-(p-z)\|^2 \le\, & \alpha_n\|f(K_nx_n)-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n\|x_n-p\|^2 \\
& -\|x_{n+1}-p\|^2-\gamma_n\|(z_n-y_n)+(p-z)\|^2 \\
& +2\gamma_n\mu_1\|F_1z_n-F_1z\|\|(z_n-y_n)+(p-z)\| \\
& +2\gamma_n\mu_2\|(u_n-z_n)-(p-z)\|\|F_2u_n-F_2p\| \qquad (3.23) \\
\le\, & \alpha_n\|f(K_nx_n)-p\|^2-\alpha_n\|x_n-p\|^2+\|x_n-x_{n+1}\|(\|x_n-p\| \\
& +\|x_{n+1}-p\|)-\gamma_n\|(z_n-y_n)+(p-z)\|^2+2\gamma_n\mu_1\|F_1z_n-F_1z\| \\
& \times\|(z_n-y_n)+(p-z)\|+2\gamma_n\mu_2\|(u_n-z_n)-(p-z)\|\|F_2u_n-F_2p\|. \qquad (3.24)
\end{aligned}
$$

Since $\alpha_n \to 0$, $\|x_n - x_{n+1}\| \to 0$, $\|F_2u_n - F_2p\| \to 0$ and $\|(z_n - y_n) + (p - z)\| \to 0$ as $n \to \infty$, we have

$$\lim_{n\to\infty} \|(u_n - z_n) - (p - z)\| = 0. \tag{3.25}$$

In addition, from the firm nonexpansivity of $T_{r_n}^{(\Phi,\varphi)}$, we have

$$\begin{aligned}
\|u_n - p\|^2 &= \|T_{r_n}^{(\Phi,\varphi)}(x_n - r_nAx_n) - T_{r_n}^{(\Phi,\varphi)}(p - r_nAp)\|^2 \\
&\leq \langle u_n - p, (x_n - r_nAx_n) - (p - r_nAp)\rangle \\
&= \frac{1}{2}(\|(x_n - r_nAx_n) - (p - r_nAp)\|^2 + \|u_n - p\|^2 - \|(x_n - r_nAx_n) - (p - r_nAp) - (u_n - p)\|^2) \\
&\leq \frac{1}{2}(\|x_n - p\|^2 + \|u_n - p\|^2 - \|(x_n - u_n) - r_n(Ax_n - Ap)\|^2) \\
&= \frac{1}{2}(\|x_n - p\|^2 + \|u_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\langle Ax_n - Ap, x_n - u_n\rangle - r_n^2\|Ax_n - Ap\|^2,
\end{aligned}$$

which implies that

$$\|u_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\|Ax_n - Ap\|\|x_n - u_n\|. \tag{3.26}$$

From (3.1), (3.4) and (3.26), we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \alpha_n\|f(K_nx_n) - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|y_n - p\|^2 \\
&\leq \alpha_n\|f(K_nx_n) - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|u_n - p\|^2 \\
&\leq \alpha_n\|f(K_nx_n) - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n(\|x_n - p\|^2 \\
&\quad - \|x_n - u_n\|^2 + 2r_n\|Ax_n - Ap\|\|x_n - u_n\|).
\end{aligned}$$

It follows that

$$\begin{aligned}\gamma_n\|x_n-u_n\|^2 &\le \alpha_n\|f(K_nx_n)-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n\|x_n-p\|^2-\|x_{n+1}-p\|^2\\ &\quad+2\gamma_nr_n\|Ax_n-Ap\|\|x_n-u_n\|\\ &=\alpha_n\|f(K_nx_n)-p\|^2-\alpha_n\|x_n-p\|^2+\|x_n-p\|^2-\|x_{n+1}-p\|^2\\ &\quad+2\gamma_nr_n\|Ax_n-Ap\|\|x_n-u_n\|\\ &\le \alpha_n\|f(K_nx_n)-p\|^2-\alpha_n\|x_n-p\|^2+(\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|\\ &\quad+2\gamma_nr_n\|Ax_n-Ap\|\|x_n-u_n\|.\end{aligned}$$

Since $\alpha_n\to 0, \|Ax_n-Ap\|\to 0$ and $\|x_n-x_{n+1}\|\to 0$ as $n\to\infty$, we obtain that

$$\lim_{n\to\infty}\|x_n-u_n\|=0. \tag{3.27}$$

From (3.22), (3.25) and (3.27), we obtain that

$$\begin{aligned}\lim_{n\to\infty}\|u_n-y_n\| &= \lim_{n\to\infty}\|(u_n-z_n)-(p-z)+(z_n-y_n)+(p-z)\|\\ &\le \lim_{n\to\infty}\|(u_n-z_n)-(p-z)\|+\lim_{n\to\infty}\|(z_n-y_n)+(p-z)\|\\ &=0 \end{aligned} \tag{3.28}$$

and

$$\lim_{n\to\infty}\|x_n-y_n\|\le\lim_{n\to\infty}\|x_n-u_n\|+\lim_{n\to\infty}\|u_n-y_n\|=0. \tag{3.29}$$

Since $\|K_ny_n-y_n\|\le\|K_ny_n-x_n\|+\|x_n-y_n\|$, by (3.16) and (3.29), we have

$$\lim_{n\to\infty}\|K_ny_n-y_n\|=0. \tag{3.30}$$

Step 4. We shall show that

$$\limsup_{n\to\infty}\langle f(x^*)-x^*,x_n-x^*\rangle\le 0,$$

where $x^*=P_\Delta f(x^*)$. To show this inequality, we can choose a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that

$$\lim_{i\to\infty}\langle f(x^*)-x^*,y_{n_i}-x^*\rangle=\limsup_{n\to\infty}\langle f(x^*)-x^*,y_n-x^*\rangle. \tag{3.31}$$

Since $\{y_{n_i}\}$ is bounded, there exists a subsequence $\{y_{n_{ij}}\}$ of $\{y_{n_i}\}$ which converges weakly to ω. Without loss of generality, we can assume that $y_{n_i} \rightharpoonup \omega$. Let us show $\omega \in \Delta$.

First, we show that $\omega \in O$. Utilizing Lemma 2.6 , we have for all $x, y \in C$

$$
\begin{aligned}
\|\Gamma(x)-\Gamma(y)\|^2 &= \|T_{\mu_1}^{G_1}[T_{\mu_2}^{G_2}(x-\mu_2F_2x)-\mu_1F_1T_{\mu_2}^{G_2}(x-\mu_2F_2x)] \\
&\quad - T_{\mu_1}^{G_1}[T_{\mu_2}^{G_2}(y-\mu_2F_2y)-\mu_1F_1T_{\mu_2}^{G_2}(y-\mu_2F_2y)]\|^2 \\
&\leq \|T_{\mu_2}^{G_2}(x-\mu_2F_2x)-T_{\mu_2}^{G_2}(y-\mu_2F_2y)-\mu_1(F_1T_{\mu_2}^{G_2}(x-\mu_2F_2x) \\
&\quad -F_1T_{\mu_2}^{G_2}(y-\mu_2F_2y))\|^2 \\
&= \|T_{\mu_2}^{G_2}(x-\mu_2F_2x)-T_{\mu_2}^{G_2}(y-\mu_2F_2y)\|^2-2\mu_1\langle T_{\mu_2}^{G_2}(x-\mu_2F_2x) \\
&\quad -T_{\mu_2}^{G_2}(y-\mu_2F_2y), F_1T_{\mu_2}^{G_2}(x-\mu_2F_2x)-F_1T_{\mu_2}^{G_2}(y-\mu_2F_2y)\rangle \\
&\quad +\mu_1^2\|F_1T_{\mu_2}^{G_2}(x-\mu_2F_2x)-F_1T_{\mu_2}^{G_2}(y-\mu_2F_2y)\|^2 \\
&\leq \|T_{\mu_2}^{G_2}(x-\mu_2F_2x)-T_{\mu_2}^{G_2}(y-\mu_2F_2y)\|^2-2\mu_1\zeta_1\|F_1T_{\mu_2}^{G_2}(x-\mu_2F_2x) \\
&\quad -F_1T_{\mu_2}^{G_2}(y-\mu_2F_2y)\|^2+\mu_1^2\|F_1T_{\mu_2}^{G_2}(x-\mu_2F_2x)-F_1T_{\mu_2}^{G_2}(y-\mu_2F_2y)\|^2 \\
&= \|T_{\mu_2}^{G_2}(x-\mu_2F_2x)-T_{\mu_2}^{G_2}(y-\mu_2F_2y)\|^2+\mu_1(\mu_1-2\zeta_1)\|F_1T_{\mu_2}^{G_2}(x-\mu_2F_2x) \\
&\quad -F_1T_{\mu_2}^{G_2}(y-\mu_2F_2y)\|^2 \\
&\leq \|T_{\mu_2}^{G_2}(x-\mu_2F_2x)-T_{\mu_2}^{G_2}(y-\mu_2F_2y)\|^2 \\
&\leq \|(x-\mu_2F_2x)-(y-\mu_2F_2y)\|^2 \\
&= \|(x-y)-\mu_2(F_2x-F_2y)\|^2 \\
&\leq \|x-y\|^2+\mu_2(\mu_2-2\zeta_2)\|F_2x-F_2y\|^2 \\
&\leq \|x-y\|^2.
\end{aligned}
$$

This implies that $\Gamma : C \to C$ is nonexpansive. Note that

$$\|y_n-\Gamma(y_n)\| = \|\Gamma(u_n)-\Gamma(y_n)\| \leq \|u_n-y_n\|$$

from (3.28), we have $\lim_{n\to\infty}\|y_n-\Gamma(y_n)\| \leq \lim_{n\to\infty}\|u_n-y_n\| = 0$. According to Lemma 2.8 and Lemma 2.10, we obtain $\omega \in O$.

Next, we show that $\omega \in \Omega$. Since $u_n = T_{r_n}^{(\Phi,\varphi)}(x_n - r_n A x_n)$, for any $y \in C$ we have

$$\Phi(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Ax_n, y - u_n \rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0.$$

From (A2) we have

$$\varphi(y) - \varphi(u_n) + \langle Ax_n, y - u_n \rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq -\Phi(u_n, y) \geq \Phi(y, u_n),$$

and hence

$$\varphi(y) - \varphi(u_{n_i}) + \langle Ax_{n_i}, y - u_{n_i} \rangle + \langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \rangle \geq \Phi(y, u_{n_i}). \tag{3.32}$$

Put $u_t = ty + (1-t)\omega$ for all $t \in (0,1]$ and $y \in C$. Then we have $u_t \in C$. From (3.32) we have

$$\begin{aligned}
&\varphi(u_t) - \varphi(u_{n_i}) + \langle u_t - u_{n_i}, Au_t \rangle \\
&\qquad \geq \langle u_t - u_{n_i}, Au_t \rangle - \langle u_t - u_{n_i}, Ax_{n_i} \rangle - \langle u_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \rangle + \Phi(u_t, u_{n_i}) \\
&\qquad = \langle u_t - u_{n_i}, Au_t - Au_{n_i} \rangle + \langle u_t - u_{n_i}, Au_{n_i} - Ax_{n_i} \rangle - \langle u_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \rangle \\
&\qquad\quad + \Phi(u_t, u_{n_i}).
\end{aligned}$$

Since $\|u_{n_i} - x_{n_i}\| \to 0$, we have $\|Au_{n_i} - Ax_{n_i}\| \to 0$. Further, from monotonicity of A, we have $\langle u_t - u_{n_i}, Au_t - Au_{n_i} \rangle \geq 0$.

From (A4), the weakly semicontinuity of φ , $u_{n_i} - x_{n_i} \to 0$ and $u_{n_i} \rightharpoonup \omega$, we have

$$\varphi(u_t) - \varphi(\omega) + \langle u_t - \omega, Au_t \rangle \geq \Phi(u_t, \omega) \text{ as } i \to \infty. \tag{3.33}$$

From (A1), (A4), (3.33) and the convexity of φ , we obtain

$$\begin{aligned}
0 &= \Phi(u_t, u_t) + \varphi(u_t) - \varphi(u_t) \\
&= \Phi(u_t, (ty + (1-t)\omega)) + \varphi(ty + (1-t)\omega) - \varphi(u_t) \\
&\leq t\Phi(u_t, y) + (1-t)\Phi(u_t, \omega) + t\varphi(y) + (1-t)\varphi(\omega) - \varphi(u_t) \\
&\leq t\Phi(u_t, y) + (1-t)(\varphi(u_t) - \varphi(\omega) + \langle u_t - \omega, Au_t \rangle) + t\varphi(y) + (1-t)\varphi(\omega) - \varphi(u_t) \\
&= t\Phi(u_t, y) - t\varphi(u_t) + (1-t)\langle u_t - \omega, Au_t \rangle + t\varphi(y) \\
&= t[\Phi(u_t, y) - \varphi(u_t) + \varphi(y)] + (1-t)t\langle y - \omega, Au_t \rangle, \qquad (3.34)
\end{aligned}$$

and hence

$$\Phi(u_t, y) - \varphi(u_t) + \varphi(y) + (1-t)\langle y - \omega, Au_t\rangle \geq 0, \quad \forall y \in C.$$

Letting $t \to 0$, it follows from (A3) and the weakly semicontinuity of φ that

$$\Phi(\omega, y) - \varphi(\omega) + \varphi(y) + \langle y - \omega, A\omega\rangle \geq 0, \quad \forall y \in C. \tag{3.35}$$

This implies that $\omega \in \Omega$. Next, we show that $\omega \in \cap_{i=1}^{N} F(T_i)$. Assume that there exists $j \in \{1, 2, \ldots, N\}$ such that $\omega \neq T_j\omega$. By Lemma 2.12, we have $\omega \neq K\omega$.
Since $y_{n_i} \rightharpoonup \omega$ and $\omega \neq K\omega$, by Opial's condition[10] and (3.30) and Lemma 2.13, we have

$$\begin{aligned}
\liminf_{i\to\infty} \|y_{n_i} - \omega\| &< \liminf_{i\to\infty} \|y_{n_i} - K\omega\| \\
&\leq \liminf_{i\to\infty} (\|y_{n_i} - K_{n_i}y_{n_i}\| + \|K_{n_i}y_{n_i} - K_{n_i}\omega\| + \|K_{n_i}\omega - K\omega\|) \\
&\leq \liminf_{i\to\infty} \|y_{n_i} - \omega\|,
\end{aligned}$$

which derives a contradiction. This implies that $\omega = K\omega$. It follows from $\omega \in F(K) = \cap_{i=1}^{N} F(T_i)$, that $\omega \in \cap_{i=1}^{N} F(T_i)$. Hence $\omega \in \Delta$.
Since $x^* = P_\Delta f(x^*)$, we have

$$\begin{aligned}
\limsup_{n\to\infty} \langle f(x^*) - x^*, x_n - x^*\rangle &= \lim_{i\to\infty} \langle f(x^*) - x^*, x_{n_i} - x^*\rangle \\
&= \lim_{i\to\infty} \langle f(x^*) - x^*, y_{n_i} - x^*\rangle \\
&= \langle f(x^*) - x^*, \omega - x^*\rangle \leq 0. \qquad (3.36)
\end{aligned}$$

Step 5. Finally, we prove that $\{x_n\}$ converge strongly to x^*.

From (3.1), we obtain

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \langle \alpha_n f(K_nx_n)+\beta_n x_n+\gamma_n K_ny_n-x^*, x_{n+1}-x^*\rangle \\ &= \alpha_n\langle f(K_nx_n)-x^*, x_{n+1}-x^*\rangle+\beta_n\langle x_n-x^*, x_{n+1}-x^*\rangle+\gamma_n\langle K_ny_n-x^*, x_{n+1}-x^*\rangle \\ &\leq \alpha_n\langle f(K_nx_n)-f(x^*), x_{n+1}-x^*\rangle+\alpha_n\langle f(x^*)-x^*, x_{n+1}-x^*\rangle \\ &\quad +\frac{1}{2}\beta_n(\|x_n-x^*\|^2+\|x_{n+1}-x^*\|^2)+\frac{1}{2}\gamma_n(\|K_ny_n-x^*\|^2+\|x_{n+1}-x^*\|^2) \\ &\leq \frac{1}{2}(1-\alpha_n)(\|x_n-x^*\|^2+\|x_{n+1}-x^*\|^2)+\alpha_n\langle f(x^*)-x^*, x_{n+1}-x^*\rangle \\ &\quad +\frac{1}{2}\alpha_n(\|f(K_nx_n)-f(x^*)\|^2+\|x_{n+1}-x^*\|^2) \\ &\leq \frac{1}{2}(1-\alpha_n)(\|x_n-x^*\|^2+\|x_{n+1}-x^*\|^2)+\alpha_n\langle f(x^*)-x^*, x_{n+1}-x^*\rangle \\ &\quad +\frac{1}{2}\alpha_n\rho^2\|x_n-x^*\|^2+\frac{1}{2}\alpha_n\|x_{n+1}-x^*\|^2 \\ &= \frac{1}{2}(1-\alpha_n(1-\rho^2))\|x_n-x^*\|^2+\frac{1}{2}\|x_{n+1}-x^*\|^2+\alpha_n\langle f(x^*)-x^*, x_{n+1}-x^*\rangle, \end{aligned} \tag{3.37}$$

which implies that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\leq (1-\alpha_n(1-\rho^2))\|x_n-x^*\|^2+2\alpha_n\langle f(x^*)-x^*, x_{n+1}-x^*\rangle \\ &= (1-\alpha_n(1-\rho^2))\|x_n-x^*\|^2+\alpha_n(1-\rho^2)\cdot\frac{2}{(1-\rho^2)}\langle f(x^*)-x^*, x_{n+1}-x^*\rangle \\ &= (1-\delta_n)\|x_n-x^*\|^2+\delta_n\sigma_n, \end{aligned} \tag{3.38}$$

where $\delta_n=\alpha_n(1-\rho^2)$ and $\sigma_n=\frac{2}{(1-\rho^2)}\langle f(x^*)-x^*, x_{n+1}-x^*\rangle$. It is easy to see that $\sum_{n=1}^{\infty}\delta_n=\infty$ and $\limsup_{n\to\infty}\sigma_n\leq 0$. Applying Lemma 2.4 to (3.38), we conclude that $x_n\to x^*$ as $n\to\infty$. This completes the proof. □

Corollary 3.2. *Let H be a real Hilbert space, C a closed convex nonempty subset of H. Let $\Phi, G_1, G_2 : C\times C\to\mathbb{R}$ be three bifunctions which satisfying (A1)-(A4) and $\varphi : C\to\mathbb{R}$ a lower semicontinuous and convex function. Let $F_1, F_2 : C\to H$ be ζ_1-inverse strongly monotone and ζ_2-inverse strongly monotone, respectively. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself such that $\cap_{i=1}^N F(T_i)\cap MEP(\Phi,\varphi)\cap O\neq\emptyset$ and f a $\rho-$contraction of C*

into itself. Assume that either (B1) or (B2) holds. Let $x_1 \in C$ *and let* $\{x_n\}$ *be a sequence defined by*

$$\begin{cases} \Phi(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = T_{\mu_1}^{G_1}[T_{\mu_2}^{G_2}(u_n - \mu_2 F_2 u_n) - \mu_1 F_1 T_{\mu_2}^{G_2}(u_n - \mu_2 F_2 u_n)] \\ x_{n+1} = \alpha_n f(K_n x_n) + \beta_n x_n + \gamma_n K_n y_n, \forall n \geq 1, \end{cases}$$

where K_n *be a* $K-$*mapping generated by* $T_1, T_2, \ldots, T_N$ *and* $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$ *and* $\{\alpha_n\}, \{\beta_n\}$ *and* $\{\gamma_n\}$ *are three sequences in* $(0,1)$ *with* $\alpha_n + \beta_n + \gamma_n = 1$, $\{\lambda_{n,i}\}_{i=1}^N$ *a sequence in* $[a,b]$ *with* $0 < a \leq b < 1$, $\{r_n\} \subset (0,\infty)$, *for all* $n \in \mathbb{N}$, $\mu_1 \in (0, 2\zeta_1), \mu_2 \in (0, 2\zeta_2)$ *satisfy the following conditions:*

(i) *the sequence* $\{r_n\}$ *satisfies*

(C1) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < \infty$*; and*

(C2) $\sum_{n=1}^{N} |r_{n+1} - r_n| < \infty$*;*

(ii) *the sequence* $\{\alpha_n\}$ *satisfies*

(D1) $\lim_{n\to\infty} \alpha_n = 0$*; and*

(D2) $\sum_{n=0}^{\infty} \alpha_n = \infty$*;*

(iii) *the sequence* $\{\beta_n\}$ *satisfies*

(E1) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$*;*

(iv) *the finite family of sequences* $\{\lambda_{n,i}\}_{i=1}^N$ *satisfies*

(F1) $\lim_{n\to\infty} |\lambda_{n+1,i} - \lambda_{n,i}| = 0$ *for every* $i \in \{1, 2, \ldots, N\}$.

Then $\{x_n\}$ *converge strongly to* $x^* = P_{\cap_{i=1}^N F(T_i)\cap MEP(\Phi,\varphi)\cap O} f(x^*)$ *and* (x^*, y^*) *is a solution of problem* (1.7) *where* $y^* = T_{\mu_2}^{G_2}(x^* - \mu_2 F_2 x^*)$.

Proof. In Theorem 3.1, for all $n \geq 0$, $u_n = T_{r_n}^{(\Phi,\varphi)}(x_n - r_n A x_n)$ is equivalent to

$$\Phi(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C. \tag{3.39}$$

Putting $A \equiv 0$, we obtain

$$\Phi(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C.$$

□

Corollary 3.3. *Let H be a real Hilbert space, C a closed convex nonempty subset of H. Let $\Phi, G_1, G_2 : C \times C \to \mathbb{R}$ be three bifunctions which satisfying (A1)-(A4) and Let $A, F_1, F_2 : C \to H$ be $\eta-$inverse strongly monotone, ζ_1-inverse strongly monotone and ζ_2-inverse strongly monotone, respectively. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself such that $\cap_{i=1}^N F(T_i) \cap EP \cap O \neq \emptyset$ and f a $\rho-$contraction of C into itself. Assume that either (B1) or (B2) holds. Let $x_1 \in C$ and let $\{x_n\}$ be a sequence defined by*

$$\begin{cases} \Phi(u_n, y) + \langle Ax_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C. \\ y_n = T_{\mu_1}^{G_1} [T_{\mu_2}^{G_2}(u_n - \mu_2 F_2 u_n) - \mu_1 F_1 T_{\mu_2}^{G_2}(u_n - \mu_2 F_2 u_n)], \\ x_{n+1} = \alpha_n f(K_n x_n) + \beta_n x_n + \gamma_n K_n y_n, \forall n \geq 1, \end{cases}$$

where K_n be a $K-$mapping generated by $T_1, T_2, \ldots, T_N$ and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$ and $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in $(0,1)$ with $\alpha_n + \beta_n + \gamma_n = 1$, $\{\lambda_{n,i}\}_{i=1}^N$ a sequence in $[a,b]$ with $0 < a \leq b < 1$, $\{r_n\}$ a sequence in $[0, 2\eta]$ for all $n \in \mathbb{N}$, $\mu_1 \in (0, 2\zeta_1), \mu_2 \in (0, 2\zeta_2)$ satisfy the following conditions:

(i) *the sequence $\{r_n\}$ satisfies*

(C1) $0 < c \leq r_n \leq d < 2\eta$*; and*

(C2) $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$*;*

(ii) *the sequence $\{\alpha_n\}$ satisfies*

(D1) $\lim_{n\to\infty} \alpha_n = 0$*; and*

(D2) $\sum_{n=0}^{\infty} \alpha_n = \infty$*;*

(iii) *the sequence $\{\beta_n\}$ satisfies*

(E1) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$*;*

(iv) *the finite family of sequences $\{\lambda_{n,i}\}_{i=1}^N$ satisfies*

(F1) $\lim_{n\to\infty} |\lambda_{n+1,i} - \lambda_{n,i}| = 0$ *for every $i \in \{1, 2, \ldots, N\}$.*

Then $\{x_n\}$ converge strongly to $x^* = P_{\cap_{i=1}^N F(T_i) \cap EP \cap O} f(x^*)$ and (x^*, y^*) is a solution of problem (1.7) *where $y^* = T_{\mu_2}^{G_2}(x^* - \mu_2 F_2 x^*)$.*

Proof. Put $\varphi \equiv 0$ in Theorem 3.1. Then we have from (3.39) that

$$\Phi(u_n, y) + \langle Ax_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C.$$

□

Corollary 3.4. *Let H be a real Hilbert space, C a closed convex nonempty subset of H. Let $\Phi, G_1, G_2 : C \times C \to \mathbb{R}$ be three bifunctions which satisfying (A1)-(A4). Let $F_1, F_2 : C \to H$ be ζ_1-inverse strongly monotone and ζ_2-inverse strongly monotone, respectively. Let $\{T_i\}_{i=1}^{N}$ be a finite family of nonexpansive mappings of C into itself such that $\cap_{i=1}^{N} F(T_i) \cap EP(\Phi) \cap O \neq \emptyset$ and f a $\rho-$contraction of C into itself. Assume that either (B1) or (B2) holds. Let $x_1 \in C$ and let $\{x_n\}$ be a sequence defined by*

$$\begin{cases} \Phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C. \\ y_n = T_{\mu_1}^{G_1}[T_{\mu_2}^{G_2}(u_n - \mu_2 F_2 u_n) - \mu_1 F_1 T_{\mu_2}^{G_2}(u_n - \mu_2 F_2 u_n)], \\ x_{n+1} = \alpha_n f(K_n x_n) + \beta_n x_n + \gamma_n K_n y_n, \forall n \geq 1, \end{cases}$$

where K_n be a $K-$mapping generated by $T_1, T_2, \ldots, T_N$ and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$ and $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in $(0,1)$ with $\alpha_n + \beta_n + \gamma_n = 1$, $\{\lambda_{n,i}\}_{i=1}^{N}$ a sequence in $[a,b]$ with $0 < a \leq b < 1$, $\{r_n\} \subset (0,\infty)$, for all $n \in \mathbb{N}$, $\mu_1 \in (0, 2\zeta_1), \mu_2 \in (0, 2\zeta_2)$ satisfy the following conditions:

(i) *the sequence $\{r_n\}$ satisfies*

(C1) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < \infty$*; and*

(C2) $\sum_{n=1}^{N} |r_{n+1} - r_n| < \infty$*;*

(ii) *the sequence $\{\alpha_n\}$ satisfies*

(D1) $\lim_{n\to\infty} \alpha_n = 0$*; and*

(D2) $\sum_{n=0}^{\infty} \alpha_n = \infty$*;*

(iii) *the sequence $\{\beta_n\}$ satisfies*

(E1) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$*;*

(iv) *the finite family of sequences $\{\lambda_{n,i}\}_{i=1}^{N}$ satisfies*

(F1) $\lim_{n\to\infty} |\lambda_{n+1,i} - \lambda_{n,i}| = 0$ *for every $i \in \{1, 2, \ldots, N\}$.*

Then $\{x_n\}$ converge strongly to $x^* = P_{\cap_{i=1}^{N} F(T_i) \cap EP(\Phi) \cap O} f(x^*)$ and (x^*, y^*) is a solution of problem (1.7) where $y^* = T_{\mu_2}^{G_2}(x^* - \mu_2 F_2 x^*)$.

Proof. Put $\varphi \equiv 0$ and $A \equiv 0$ in Theorem 3.1. Then we have from (3.39) that

$$\Phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C.$$

□

Corollary 3.5. *Let H be a real Hilbert space, C a closed convex nonempty subset of H. Let $G_1, G_2 : C \times C \to \mathbb{R}$ be two bifunctions which satisfying (A1)-(A4) and let $A, F_1, F_2 : C \to H$ be $\eta-$inverse strongly monotone, ζ_1-inverse strongly monotone and ζ_2-inverse strongly monotone, respectively. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself such that $\cap_{i=1}^N F(T_i) \cap VI(A,C) \cap O \neq \emptyset$ and f a $\rho-$contraction of C into itself. Assume that either (B1) or (B2) holds. Let $x_1 \in C$ and let $\{x_n\}$ be a sequence defined by*

$$\begin{cases} u_n = P_C(x_n - r_n A x_n), \\ y_n = T_{\mu_1}^{G_1}[T_{\mu_2}^{G_2}(u_n - \mu_2 F_2 u_n) - \mu_1 F_1 T_{\mu_2}^{G_2}(u_n - \mu_2 F_2 u_n)], \\ x_{n+1} = \alpha_n f(K_n x_n) + \beta_n x_n + \gamma_n K_n y_n, \forall n \geq 1, \end{cases}$$

where K_n be a $K-$mapping generated by $T_1, T_2, \ldots, T_N$ and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$ and $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in $(0,1)$ with $\alpha_n + \beta_n + \gamma_n = 1$, $\{\lambda_{n,i}\}_{i=1}^N$ a sequence in $[a,b]$ with $0 < a \leq b < 1$, $\{r_n\}$ a sequence in $[0, 2\eta]$ for all $n \in \mathbb{N}$, $\mu_1 \in (0, 2\zeta_1), \mu_2 \in (0, 2\zeta_2)$ satisfy the following conditions:

(i) *the sequence $\{r_n\}$ satisfies*

(C1) $0 < c \leq r_n \leq d < 2\eta$*; and*

(C2) $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$*;*

(ii) *the sequence $\{\alpha_n\}$ satisfies*

(D1) $\lim_{n\to\infty} \alpha_n = 0$*; and*

(D2) $\sum_{n=0}^{\infty} \alpha_n = \infty$*;*

(iii) *the sequence $\{\beta_n\}$ satisfies*

(E1) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$*;*

(iv) *the finite family of sequences $\{\lambda_{n,i}\}_{i=1}^N$ satisfies*

(F1) $\lim_{n\to\infty} |\lambda_{n+1,i} - \lambda_{n,i}| = 0$ *for every $i \in \{1, 2, \ldots, N\}$.*

Then $\{x_n\}$ *converge strongly to* $x^* = P_{\cap_{i=1}^N F(T_i)\cap VI(A,C)\cap O} f(x^*)$ *and* (x^*, y^*) *is a solution of problem* (1.7) *where* $y^* = T_{\mu_2}^{G_2}(x^* - \mu_2 F_2 x^*)$.

Proof. Put $\Phi \equiv 0$ and $\varphi \equiv 0$ in Theorem 3.1. Then we have from (3.39) that

$$\langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C.$$

That is,

$$\langle y - u_n, x_n - r_n A x_n - u_n\rangle \leq 0, \quad \forall y \in C.$$

It follows that $u_n = P_C(x_n - r_n A x_n)$ for all $n \geq 1$. Hence the corollary is obtained by Theorem 3.1. □

Conflict of Interests

The authors declare that there is no conflict of interests.

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