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COMPOSITION OF WEIGHTED PSEUDO-ALMOST AUTOMORPHIC FUNCTIONS AND HYPERBOLIC DIFFERENTIAL EQUATIONS

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Abstract. This paper is concerned with weighted pseudo-almost automorphic functions, which are more general and complicated than pseudo-almost automorphic functions. New results, concerning the composition of weighted pseudo-almost automorphic functions and the existence of weighted pseudo-almost automorphic solutions to the class of perturbed hyperbolic differential equations, are established. Our results improve and generalize some recent results.

Keywords: pseudo-almost automorphic; weighted pseudo-almost automorphic; uniform continuity; C_0-semigroup; fixed point.

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1. Introduction-Preliminaries

Almost automorphic functions were first introduced by S.Bochner [2] as a natural generalization of the classical concept of almost periodic function. Almost automorphic function is an attractive topic in the qualitative theory

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of differential equations due to their significance and applications in physics, mathematical biology, control theory and others.

Recently, the theory of almost automorphic functions and some composition theorems have been developed extensively (see e.g.[1, 2, 3, 4, 5, 6], [8, 9, 11, 12, 13, 14]). However, to the best of the authors' knowledge, results for weighted almost automorphic functions, which are more general and complicated than pseudo-almost automorphic functions, are rare. Actually, there are even no results available in the literature on the composition of weighted pseudo-almost automorphic functions. From the work of [3], one can see the basic and key role of the composition of almost automorphic functions in discussing the existence of almost automorphic solutions to differential equations and semilinear equations. In this paper, we study the the composition of weighted pseudo-almost automorphic functions and obtain the existence of weighted pseudo-almost automorphic solutions to the class of perturbed hyperbolic differential equations.

Throughout this paper, we always assume that $\mathbb{H}$ is Hilbert space and $(\mathbb{X}, \|\cdot\|)$ is Banach space. Let $BC(\mathbb{R},\mathbb{X})$ (respectively, $BC(\mathbb{R}\times\mathbb{X},\mathbb{X})$) be the space of bounded continuous functions $f:\mathbb{R}\to\mathbb{X}$ (respectively, $f:\mathbb{R}\times\mathbb{X}\to\mathbb{X}$), and $BC(\mathbb{R},\mathbb{X})$ equipped with the sup norm defined by

$$\|f\| = \sup_{t\in\mathbb{R}} \|f(t)\|,$$

is a Banach space.

Let U denote the collection of functions (weights) $\rho:\mathbb{R}\to(0,+\infty)$, which are locally integrable over $\mathbb{R}$. If $\rho\in U$ and for $T>0$, we then set

$$\mu(T,\rho) := \int_{-T}^{T} \rho(t)dt.$$

Denote

$$U_\infty := \left\{\rho\in U : \lim_{T\to\infty} \mu(T,\rho) = \infty\right\},$$

and

$$U_B := \left\{\rho\in U_\infty : \rho \text{ is bounded with } \inf_{x\in\mathbb{R}} \rho(x) > 0\right\}.$$

Obviously, $U_B\subset U_\infty\subset U$, with strict inclusions.

Definition 1.1. [11]

(i) *A continuous function $f:\mathbb{R}\to\mathbb{X}$ is said to be almost automorphic if for each sequence of real numbers $\{s_n\}_{n=1}^{\infty}$, we can extract a subsequence $\{\tau_n\}_{n=1}^{\infty}$ such that*

$$g(t) = \lim_{n\to\infty} f(t+\tau_n)$$

is well-defined in $t\in\mathbb{R}$, and

$$\lim_{n\to\infty} g(t-\tau_n) = f(t)$$

for each $t\in\mathbb{R}$. Denote by $AA(\mathbb{R},\mathbb{X})$ the set of all such functions.

(ii) *A continuous function $f : \mathbb{R}\times\mathbb{X}\to\mathbb{X}$ is said to be almost automorphic if $f(t,x)$ is almost automorphic in $t\in\mathbb{R}$ uniformly for all $x\in\mathbb{K}$, where $\mathbb{K}$ is any bounded subset of $\mathbb{X}$. That is to say, for each sequence of real numbers $\{s_n\}_{n=1}^{\infty}$, we can extract a subsequence $\{\tau_n\}_{n=1}^{\infty}$ such that*

$$g(t,x)=\lim_{n\to\infty} f(t+\tau_n,x)$$

is well-defined in $t\in\mathbb{R}$ for all $x\in\mathbb{K}$, and

$$\lim_{n\to\infty} g(t-\tau_n,x)=f(t,x)$$

for all $t\in\mathbb{R}$ and $x\in\mathbb{K}$. Denote by $AA(\mathbb{R}\times\mathbb{X},\mathbb{X})$ the set of all such functions.

For $\rho\in U_\infty$, the *weighted ergodic* space $WAA_0(\mathbb{R},\mathbb{X},\rho)$ and $WAA_0(\mathbb{R}\times\mathbb{X},\mathbb{X},\rho)$ are defined by

$$WAA_0(\mathbb{R},\mathbb{X},\rho):=\left\{f\in BC(\mathbb{R},\mathbb{X}) : \lim_{T\to\infty}\frac{1}{\mu(T,\rho)}\int_{-T}^{T}\|f(t)\|\rho(t)dt=0\right\},$$

$$WAA_0(\mathbb{R}\times\mathbb{X},\mathbb{X},\rho):=\left\{\begin{array}{c} F\in BC(\mathbb{R}\times\mathbb{X},\mathbb{X}) : F(\cdot,x) \text{ is bounded for each } x\in\mathbb{X} \text{ and} \\ \lim\limits_{T\to\infty}\frac{1}{\mu(T,\rho)}\int_{-T}^{T}\|F(t,x)\|\rho(t)dt=0 \text{ uniformly in } x\in\mathbb{X} \end{array}\right\}.$$

Now we are ready to introduce the set *PAA* of weighted pseudo-almost automorphic functions.

Definition 1.2.

(i) *Let $\rho\in U_\infty$. A function $f\in BC(\mathbb{R},\mathbb{X})$ is called weighted pseudo-almost automorphic if it can be expressed as $f=g+\phi$, where $g\in AA(\mathbb{R},\mathbb{X})$ and $\phi\in WAA_0(\mathbb{R},\mathbb{X},\rho)$. Denote by $PAA(\mathbb{R},\mathbb{X},\rho)$ the set of all such functions.*

(ii) *Let $\rho\in U_\infty$. A function $F\in BC(\mathbb{R}\times\mathbb{X},\mathbb{X})$ is called weighted pseudo-almost automorphic in $t\in\mathbb{R}$ and uniformly in $x\in\mathbb{X}$ if it can be expressed as $F=G+\Phi$, where $G\in AA(\mathbb{R}\times\mathbb{X},\mathbb{X})$ and $\Phi\in WAA_0(\mathbb{R}\times\mathbb{X},\mathbb{X},\rho)$. Denote by $PAA(\mathbb{R}\times\mathbb{X},\mathbb{X},\rho)$ the set of all such functions.*

The functions g and ϕ (or G and Φ) in Definition 1.2 are called the *almost automorphic* and the *weighted ergodic perturbation* components of f (or F), respectively. Moreover, the decomposition $g+\phi$ of f (or $G+\Phi$ of F) is unique, and $WAA_0(\mathbb{R},\mathbb{X},\rho)$ and $PAA(\mathbb{R},\mathbb{X},\rho)$ both are Banach spaces with the norm inherited from $BC(\mathbb{R},\mathbb{X})$ (see [5]).

Definition 1.3. [15] *The Bocher transform $f^b(t,s), t\in\mathbb{R}, s\in[0,1]$ of a function $f:\mathbb{R}\mapsto\mathbb{X}$ is defined by*

$$f^b(t,s):=f(t+s).$$

Definition 1.4. [7] *Let $p\in[1,\infty)$. The space $BS^p(\mathbb{X})$ of all Stepanov bounded functions, with the exponent p, consists of all measurable functions $f:\mathbb{R}\to\mathbb{X}$ such that $f^b\in L^\infty(\mathbb{R},L^p((0,1),\mathbb{X}))$. This is a Banach space with* *the norm*

$$\|f\|_{S^p} := \|f^b\|_{L^\infty(\mathbb{R},L^p)} = \sup_{t\in\mathbb{R}}\Big(\int_t^{t+1}\|f(\tau)\|^p d\tau\Big)^{\frac{1}{p}}.$$

Definition 1.5. [14] *The space $AS^p(\mathbb{X})$ of Stepanov-like almost automorphic functions consists of all $f \in BS^p(\mathbb{X})$ such that $f^b \in AA(\mathbb{R}, L^p((0,1),\mathbb{X}))$. That is, a function $f \in L^p_{loc}(\mathbb{R},\mathbb{X})$ is said to be S^p-almost automorphic if its Bochner transform $f^b : \mathbb{R} \to L^p((0,1),\mathbb{X})$ is almost automorphic in the sense that for every sequence of real numbers $(s'_n)_{n\in N}$, there exist a subsequence $(s_n)_{n\in N}$ and a function $g \in L^p_{loc}(\mathbb{R},\mathbb{X})$ such that*

$$\Big[\int_t^{t+1}\|f(s_n+s)-g(s)\|^p ds\Big]^{\frac{1}{p}} \to 0, \quad \text{and}$$

$$\Big[\int_t^{t+1}\|g(s-s_n)-f(s)\|^p ds\Big]^{\frac{1}{p}} \to 0, \quad \text{as} \quad n\to\infty \quad \text{pointwise on} \quad \mathbb{R}.$$

Definition 1.6. *Let $\rho \in U_\infty$. A function $F : \mathbb{R}\times\mathbb{X} \to \mathbb{X}, (t,u) \mapsto F(t,u)$ with $F(\cdot,u) \in L^p_{loc}(\mathbb{R},\mathbb{X})$ for each $u \in \mathbb{X}$, is said to be S^p-weighted pseudo-almost automorphic if there exists two functions $G,\Phi : \mathbb{R}\times\mathbb{X} \to \mathbb{X}$ such that $F = G+\Phi$, where $G^b \in AA(\mathbb{R}\times L^p((0,1),\mathbb{X}),\mathbb{X})$ and $\Phi^b \in WAA_0(\mathbb{R}\times L^p((0,1),\mathbb{X}),\mathbb{X},\rho)$.*

The collection of those S^p-weighted pseudo-almost automorphic functions $F : \mathbb{R}\times\mathbb{X} \to \mathbb{X}$ will be denoted $PAAS^p(\mathbb{R}\times\mathbb{X},\mathbb{X},\rho,p)$. Note that if $\rho \in U_\infty$ and if the limits

$$\lim_{t\to\infty}\frac{\rho(t+\tau)}{\rho(t)} \quad \text{and} \quad \lim_{T\to\infty}\frac{\mu(T+\tau,\rho)}{\mu(T,\rho)}$$

exist for all $\tau \in \mathbb{R}$, then $PAA(\mathbb{R}\times\mathbb{X},\mathbb{X},\rho)$ is translation-invariant.

Let U^{inv}_∞ denote the collection of all weights $\rho \in U_\infty$ such $PAA(\mathbb{R}\times\mathbb{X},\mathbb{X},\rho)$ is translation-invariant.

2. Composition theorems of weighted pseudo-almost automorphic functions

Let us give the following assumptions on F:

- (H_1) $F(t,x)$ is uniformly continuous in each bounded subset $\mathbb{K} \subset \mathbb{X}$ uniformly for $t \in \mathbb{R}$. More explicitly, given $\varepsilon > 0$ and $K \subset \mathbb{X}$ bounded, there exists $\delta > 0$ such that, $x,y \in K$ and $\|x-y\| < \delta$ imply that $\|F(t,x)-F(t,y)\| < \varepsilon$ for all $t \in \mathbb{R}$.
- (H_2) $F(\mathbb{R},\mathbb{K}) = \{F(t,x) : t \in \mathbb{R}, x \in \mathbb{K}\}$ is bounded for every bounded subset $K \subset \mathbb{X}$.

The following lemmas will be used in the proof of the composition theorems.

Lemma 2.1. *Let $\rho \in U_\infty$ and $f \in WAA_0(\mathbb{R},\mathbb{X},\rho)$. Then, given $\varepsilon > 0$,*

$$\lim_{T\to\infty} \frac{1}{\mu(T,\rho)} \int_{M(T,\varepsilon,f)} \rho(t)dt = 0,$$

where $M(T,\varepsilon,f) = \{t \in [-T,T] : \|f(t)\| \geq \varepsilon\}$.

PROOF. Suppose on the contrary, that there exists $\varepsilon_0 > 0$ such that

$$\frac{1}{\mu(T,\rho)} \int_{M(T,\varepsilon_0,f)} \rho(t)dt$$

does not converge to 0 as $T \to \infty$. Since ρ is positive, there exists $\delta > 0$ such that for each n,

$$\frac{1}{\mu(T_n,\rho)} \int_{M(T_n,\varepsilon_0,f)} \rho(t)dt \geq \delta \quad \text{for some} \quad T_n \geq n.$$

Then

$$\begin{aligned} & \frac{1}{\mu(T_n,\rho)} \int_{-T_n}^{T_n} \|f(t)\|\rho(t)dt \\ \geq \quad & \frac{1}{\mu(T_n,\rho)} \int_{M(T_n,\varepsilon_0,f)} \|f(t)\|\rho(t)dt \\ \geq \quad & \frac{\varepsilon_0}{\mu(T_n,\rho)} \int_{M(T_n,\varepsilon_0,f)} \rho(t)dt \\ \geq \quad & \varepsilon_0\delta, \end{aligned}$$

which contradicts the fact that $f \in WAA_0(\mathbb{R},\mathbb{X},\rho)$, and the proof is complete.

Lemma 2.2. [11] *If $f : \mathbb{R}\times\mathbb{X} \to \mathbb{X}$ is almost automorphic, and assume that $f(t,x)$ is uniformly continuous on each bounded subset $\mathbb{K} \subset \mathbb{X}$ uniformly for $t \in \mathbb{R}$. Let $\phi : \mathbb{R} \to \mathbb{X}$ be almost automorphic. Then the function $F : \mathbb{R} \to \mathbb{X}$ defined by $F(t) = f(t,\phi(t))$ is almost automorphic.*

We are now ready to study the composition theorems of weighted pseudo-almost automorphic functions.

Theorem 2.3. *Let $F \in PAA(\mathbb{R}\times\mathbb{X},\mathbb{X},\rho)$ and $h \in PAA(\mathbb{R},\mathbb{X},\rho)$ with $\rho \in U_\infty$. Assume that conditions* (H$_1$)-(H$_2$) *and the following condition hold:*

(H$_3$) *$F_{aa}(t,x)$ is uniformly continuous in any bounded subset $\mathbb{K} \subset \mathbb{X}$ uniformly for $t \in \mathbb{R}$.*

Then $F(\cdot,h(\cdot)) \in PAA(\mathbb{R}\times\mathbb{X},\mathbb{X},\rho)$ with almost automorphic component $F_{aa}(\cdot,h_{aa}(\cdot))$, where F_{aa} and h_{aa} are the almost automorphic components of F and h, respectively.

PROOF. Since $F \in PAA(\mathbb{R}\times\mathbb{X},\mathbb{X},\rho)$ and $h \in PAA(\mathbb{R},\mathbb{X},\rho)$, we know that $F(\cdot,h(\cdot)) \in BC(\mathbb{R},\mathbb{X})$. Let $F = F_{aa} + F_e$ and $h = h_{aa} + h_e$ with $F_{aa} \in AA(\mathbb{R}\times\mathbb{X},\mathbb{X}), F_e \in WAA_0(\mathbb{R}\times\mathbb{X},\mathbb{X},\rho), h_{aa} \in AA(\mathbb{R},\mathbb{X})$ and $h_e \in WAA_0(\mathbb{R},\mathbb{X},\rho)$. Then

$$F(t,h(t)) = F_{aa}(t,h_{aa}(t)) + F(t,h(t)) - F(t,h_{aa}(t)) + F_e(t,h_{aa}(t)).$$

Let

$$G(t) = F_{aa}(t, h_{aa}(t)),$$

$$\Phi(t) = F(t, h(t)) - F(t, h_{aa}(t)) + F_e(t, h_{aa}(t)).$$

Using assumption (H_3), we see $G(t) \in AA(\mathbb{R}, \mathbb{X})$ by Lemma 2.2. So we only need to show that $\Phi(t) \in WAA_0(\mathbb{R}, \mathbb{X}, \rho)$.

This will be approached by the following two steps.

Step 1. We prove that $F(\cdot, h(\cdot)) - F(\cdot, h_{aa}(\cdot)) \in WAA_0(\mathbb{R} \times \mathbb{X}, \mathbb{X}, \rho)$.

Let $K \subset \mathbb{X}$ be bounded such that $h(\mathbb{R}), h_{aa}(\mathbb{R}) \subset K$. Then, by assumption ($H_2$), there exists $S > 0$ such that

$$\|F(t, h(t)) - F(t, h_{aa}(t))\| \leq S \quad \text{for all } t \in \mathbb{R}. \tag{1}$$

Meanwhile, by condition (H_1), given $\varepsilon > 0$, there exists $\delta > 0$, such that for $x, y \in K$ with $\|x - y\| < \delta$, we have

$$\|F(t, x) - F(t, y)\| < \frac{\varepsilon}{2} \quad \text{for all } t \in \mathbb{R}. \tag{2}$$

It follows from Lemma 2.1 that

$$\lim_{T \to \infty} \frac{1}{\mu(T, \rho)} \int_{M(T, \delta, h_e)} \rho(t) dt = 0,$$

where $M(T, \delta, h_e) = \{t \in [-T, T] : \|h_e(t)\| \geq \delta\}$. Thus, there exists $T_0 > 0$ such that

$$\frac{1}{\mu(T, \rho)} \int_{M(T, \delta, h_e)} \rho(t) dt < \frac{\varepsilon}{2S} \quad \text{for all } T > T_0. \tag{3}$$

Noticing that $\|h(t) - h_{aa}(t)\| = \|h_e(t)\| < \delta$ for all $t \in [-T, T] \setminus M(T, \delta, h_e)$, by (1) – (3) we have, for $T > T_0$,

$$\begin{aligned}
& \frac{1}{\mu(T, \rho)} \int_{-T}^{T} \|F(t, h(t)) - F(t, h_{aa}(t))\| \rho(t) dt \\
= \; & \frac{1}{\mu(T, \rho)} \int_{M(T, \delta, h_e)} \|F(t, h(t)) - F(t, h_{aa}(t))\| \rho(t) dt \\
& + \frac{1}{\mu(T, \rho)} \int_{[-T, T] \setminus M(T, \delta, h_e)} \|F(t, h(t)) - F(t, h_{aa}(t))\| \rho(t) dt \\
\leq \; & \frac{S}{\mu(T, \rho)} \int_{M(T, \delta, h_e)} \rho(t) dt + \frac{\varepsilon}{2\mu(T, \rho)} \int_{[-T, T] \setminus M(T, \delta, h_e)} \rho(t) dt \\
< \; & S \frac{\varepsilon}{2S} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

This shows that $F(\cdot, h(\cdot)) - F(\cdot, h_{aa}(\cdot)) \in WAA_0(\mathbb{R} \times \mathbb{X}, \mathbb{X}, \rho)$.

Step 2. We prove that $F_e(\cdot, h_{aa}(\cdot)) \in WAA_0(\mathbb{R} \times \mathbb{X}, \mathbb{X}, \rho)$.

It follows from assumptions (H_1) and (H_3) that $F_e(t, x) = F(t, x) - F_{aa}(t, x)$ is uniformly continuous in $x \in h_{aa}([-T, T])$ uniformly in t. That is, given $\varepsilon > 0$, there exists $\delta > 0$ such that, for $x, y \in h_{aa}([-T, T])$ with $\|x - y\| < \delta$,

$$\|F_e(t, x) - F_e(t, y)\| < \frac{\varepsilon}{2} \quad \text{for all } t \in \mathbb{R}. \tag{4}$$

Meanwhile, since $F_e(t, h_{aa}(t))$ is continuous in $[-T, T]$, it is uniformly continuous in $[-T, T]$. Set $I = h_{aa}([-T, T])$. Then I is compact in $\mathbb{R}$ since the image of a compact set under a continuous mapping is compact, and so one can find a finite δ-net of I. Namely, there exist finite number of points $x_1, x_2, \cdots, x_m \in I$ such that, for any $y \in I$, we have $\|y - x_k\| < \delta$ for some $1 \le k \le m$. Let

$$\mathscr{O}_k = \{t \in [-T,T] : \|h_{aa}(t) - x_k\| < \delta\}, k = 1,2,\cdots,m.$$

Then $I \subset \bigcup_{k=1}^{m} \mathscr{O}_k$. Let

$$\mathscr{B}_1 = \mathscr{O}_1, \mathscr{B}_k = \mathscr{O}_k \setminus \left(\bigcup_{i=1}^{k-1} \mathscr{O}_i\right), k = 2,3,\cdots,m.$$

The set $\mathscr{B}_k = \{t \in [-T,T] : h_{aa}(t) \in \mathscr{O}_k\}$ is open in $[-T,T]$ and $[-T,T] = \bigcup_{k=1}^{m} \mathscr{B}_k$. Then $\mathscr{B}_j \cap \mathscr{B}_k = \emptyset$ when $j \ne k, 1 \le k \le m$. Moreover, by (4) we have

$$\|F_e(t, h_{aa}(t)) - F_e(t, x_k)\| < \frac{\varepsilon}{2} \quad \text{for all } t \in \mathscr{B}_k, 1 \le k \le m. \tag{5}$$

Since $F_e \in WAA_0(\mathbb{R} \times \mathbb{X}, \mathbb{X}, \rho)$, there exists $T_0 > 0$ such that

$$\frac{1}{\mu(T,\rho)} \int_{-T}^{T} \|F_e(t, x_k)\| \rho(t) dt < \frac{\varepsilon}{2m} \quad \text{for } T > T_0, 1 \le k \le m. \tag{6}$$

Now by (4)–(6), for $T > T_0$, we have

$$\begin{aligned}
& \frac{1}{\mu(T,\rho)} \int_{-T}^{T} \|F_e(t, h_{aa}(t))\| \rho(t) dt \\
\le\ & \frac{1}{\mu(T,\rho)} \sum_{k=1}^{m} \int_{\mathscr{B}_k \bigcap [-T,T]} (\|F_e(t, h_{aa}(t)) - F_e(t, x_k)\| + \|F_e(t, x_k)\|) \rho(t) dt \\
\le\ & \frac{1}{\mu(T,\rho)} \sum_{k=1}^{m} \int_{\mathscr{B}_k \bigcap [-T,T]} \frac{\varepsilon \rho(t)}{2} dt + \frac{1}{\mu(T,\rho)} \sum_{k=1}^{m} \int_{\mathscr{B}_k \bigcap [-T,T]} \|F_e(t, x_k)\| \rho(t) dt \\
=\ & \frac{\varepsilon}{2\mu(T,\rho)} \int_{-T}^{T} \rho(t) dt + \sum_{k=1}^{m} \frac{1}{\mu(T,\rho)} \int_{\mathscr{B}_k \bigcap [-T,T]} \|F_e(t, x_k)\| \rho(t) dt \\
<\ & \frac{\varepsilon}{2} + m \frac{\varepsilon}{2m} = \varepsilon,
\end{aligned}$$

which yields that $F_e(\cdot, h_{aa}(\cdot)) \in WAA_0(\mathbb{R} \times \mathbb{X}, \mathbb{X}, \rho)$. The proof is completed.

Theorem 2.4. *Let $F \in PAAS^p(\mathbb{R} \times \mathbb{X}, \mathbb{X}, \rho)$ and $h \in PAA(\mathbb{R}, \mathbb{X}, \rho)$ with $\rho \in U_\infty$. Assume that conditions* (H_1)-(H_3). *Then $F(\cdot, h(\cdot)) \in PAAS^p(\mathbb{R} \times \mathbb{X}, \mathbb{X}, \rho)$.*

PROOF. Since $F \in PAAS^p(\mathbb{R} \times \mathbb{X}, \mathbb{X}, \rho)$ and $h \in PAA(\mathbb{R}, \mathbb{X}, \rho)$, we have $F = F_{aa} + F_e$ and $h = h_{aa} + h_e$, where $F_{aa}^b \in AA(\mathbb{R} \times \mathbb{X}, L^p((0,1), \mathbb{X})), F_e^b \in WAA_0(\mathbb{R} \times \mathbb{X}, L^p((0,1), \mathbb{X}), \rho)$, $h_{aa} \in AA(\mathbb{R}, \mathbb{X})$ and $h_e \in WAA_0(\mathbb{R}, \mathbb{X}, \rho)$. Now by an argument the same as the proofs of Theorem 3.2 in [7], we can prove that $h_{aa}^b \in AA(\mathbb{R}, L^p((0,1), \mathbb{X}))$.

It is obvious to see that $F^b(\cdot, h^b(\cdot)) : \mathbb{R} \mapsto L^p((0,1), \mathbb{X})$. Now decompose F^b as follows

$$\begin{aligned}
F^b(\cdot, h^b(\cdot)) &= F_{aa}^b(\cdot, h_{aa}^b(\cdot)) + F^b(\cdot, h^b(\cdot)) - F_{aa}^b(\cdot, h_{aa}^b(\cdot)) \\
&= F_{aa}^b(\cdot, h_{aa}^b(\cdot)) + F^b(\cdot, h^b(\cdot)) - F^b(\cdot, h_{aa}^b(\cdot)) + F_e^b(\cdot, h_{aa}^b(\cdot)).
\end{aligned}$$

Using the Theorem 2.3, it is easy to see that $F^b_{aa}(\cdot,h^b_{aa}(\cdot)) \in AA(\mathbb{R}\times\mathbb{X},L^p((0,1),\mathbb{X}))$. So by an argument the same as the proofs of Theorem 2.3, we can show that $F^b(\cdot,h^b(\cdot)) - F^b(\cdot,h^b_{aa}(\cdot)) + F^b_e(\cdot,h^b_{aa}(\cdot)) \in WAA_0(\mathbb{R}\times\mathbb{X},,L^p((0,1),\mathbb{X}),\rho)$. Hence, $F(\cdot,h(\cdot)) \in PAAS^p(\mathbb{R}\times\mathbb{X},\mathbb{X},\rho)$.

Remark 2.5. *In particular, if $\rho = 1$-that is, if the composition of pseudo-almost automorphic functions are considered-Theorem 2.3 is the same as Theorem 2.4 in* [11]

For example, let us consider the function

$$\phi(t) = max_{k\in\mathbb{Z}}\{e^{-(t\pm k^2)^3}\}, \quad t\in\mathbb{R}.$$

For any $T>0$, set $\rho(t) = e^t, l = [\sqrt{T}]+1$. Then we have

$$\begin{aligned}
&\lim_{T\to\infty}\frac{1}{\mu(T,\rho)}\int_{-T}^{T}\|max_{k\in\mathbb{Z}}\{e^{-(t\pm k^2)^3}\}\|\rho(t)dt\\
=\ &\lim_{T\to\infty}\frac{1}{\mu(T,\rho)}\int_{-T}^{T}\|max_{-l\le k\le l}\{e^{-(t\pm k^2)^3}\}\|e^t dt\\
\le\ &\lim_{T\to\infty}\frac{2le^{\frac{1}{4}}}{\mu(T,\rho)}\int_{-\infty}^{\infty}e^{-(t-\frac{1}{2})^2}dt\\
=\ &\lim_{T\to\infty}\frac{2le^{\frac{1}{4}}\sqrt{\pi}}{\mu(T,\rho)} = 0
\end{aligned}$$

Therefore,

$$\phi(t) = max_{k\in\mathbb{Z}}\{e^{-(t\pm k^2)^3}\} \in WAA_0(\mathbb{R},\mathbb{X},\rho)$$

Set

$$f(t,x) = xsin\frac{1}{cos^2t+cos^2\pi t} + \phi(t)sinx, \quad t,x\in\mathbb{R}.$$

Clearly,

$$xsin\frac{1}{cos^2t+cos^2\pi t} \in AA(\mathbb{R}\times\mathbb{X},\mathbb{X}),$$

$$\phi(t)sinx \in WAA_0(\mathbb{R}\times\mathbb{X},\mathbb{X},\rho).$$

If we put $x(t) = cost + \frac{1}{1+t^2}$ as an element of $PAA(\mathbb{R},\mathbb{X},\rho)$, then the composition

$$f(t,x) = x(t)sin\frac{1}{cos^2t+cos^2\pi t} + \phi(t)sinx(t), \quad t\in\mathbb{R}.$$

is a weighted pseudo-almost automorphic function by Theorem 2.3.

3. Existence of weighted pseudo-almost automorphic solutions

This section is devoted to the search of the existence of weighted pseudo-almost automorphic solution to the class of perturbed hyperbolic differential equations

$$u'(t) = Au(t) + F(t,Bu(t)), \quad t\in\mathbb{R}. \tag{7}$$

where A is the infinitesimal generator of C_0-semigroup $(T(t))_{t\geq 0}$, and F satisfy assumptions(H_1)-(H_3) and we will use the following assumptions:

(H_4) $P, Q = I - P$ are projections, $T(t)$ is compact for $t > 0$ and there exist constants $M, c, d > 0$ such that

$$\|T(t)P\| \leq Me^{-ct}, \quad \text{for} \quad t \geq 0.$$

$$\|T(t)Q\| \leq Me^{-dt}, \quad \text{for} \quad t \geq 0.$$

(H_5) the operator $B : \mathbb{H} \mapsto \mathbb{H}$ is bounded.

(H_6) $F \in PAAS^p(\mathbb{R}\times\mathbb{H}, \mathbb{H}, \rho) \bigcap C(\mathbb{R}\times\mathbb{H}, \mathbb{H})$ for $p > 1$ and $\rho \in U_\infty$.

(H_7) there exists $L > 0$ such that

$$S_L := \sup_{t\in\mathbb{R}, \|Bu\|\leq L} \left(\int_t^{t+1} \|F(s, Bu(s))\|^p ds\right)^{\frac{1}{p}} \leq \frac{L}{K_c + K_d},$$

where

$$K_c = M\left(\frac{e^{qc}-1}{qc}\right)^{\frac{1}{q}} \sum_{n=1}^{\infty} e^{-cn}, \quad K_d = M\left(\frac{e^{qd}-1}{qd}\right)^{\frac{1}{q}} \sum_{n=1}^{\infty} e^{-dn},$$

where $M, c, d > 0$ are given in (H_4)and $q \geq 1$ such that $\frac{1}{q} + \frac{1}{p} = 1$.

(H_8) Let $\{u_n\} \subset PAA(\mathbb{R}, \mathbb{H}, \rho)$ be uniformly bounded in $\mathbb{R}$ and uniformly convergent in each compact subset of $\mathbb{R}$. Then $F(\cdot, Bu_n(\cdot))$ is relatively compact in $PAAS^p(\mathbb{R}\times\mathbb{H}, \mathbb{H}, \rho)$.

Definition 3.1. *A bounded continuous function $u : \mathbb{R} \to \mathbb{H}$ is said to be a mild solution to Eq.(7) provided that the function $s \to T(t-s)PF(s, Bu(s))$ is integrable on $(-\infty, t)$, $s \to T(t-s)QF(s, Bu(s))$ is integrable on $(-\infty, t)$ for each $t \in \mathbb{R}$, and*

$$u(t) = \int_{-\infty}^{t} T(t-s)PF(s, Bu(s))ds + \int_{-\infty}^{t} T(t-s)QF(s, Bu(s))ds \quad \textit{for each} \quad t \in \mathbb{R}.$$

Throughout the rest of the paper we denote by V and W, the two nonlinear integral operators defined by

$$(Vu)(t) := \int_{-\infty}^{t} T(t-s)PF(s, Bu(s))ds, \quad (Wu)(t) := \int_{-\infty}^{t} T(t-s)QF(s, Bu(s))ds,$$

for $t \in \mathbb{R}$, respectively. It is easy to see that the two integrals above are uniformly convergent in $t \in \mathbb{R}$. A function $u \in BC(\mathbb{R}, \mathbb{H})$ is called a mild solution of (7) if u can be expressed as: $u = Vu + Wu$. In addition, if $u \in PAA(\mathbb{R}, \mathbb{H}, \rho)$, u is called a mild weighted pseudo-almost automorphic solution.

The following lemma will be used in the proof of the weighted pseudo-almost automorphic solution to the differential equation.

Lemma 3.2. *Let $\rho \in U_\infty$, $\{f_n\}_{n\in N} \subset WAA_0(\mathbb{R}, \mathbb{X}, \rho)$ be a sequence of functions. If f_n converges uniformly to some f, then $f \in WAA_0(\mathbb{R}, \mathbb{X}, \rho)$.*

PROOF. Note that f is necessarily a bounded continuous function from $\mathbb{R}$ into $\mathbb{X}$, and $\|f_n - f\|_\infty \to 0$ as $n \to \infty$. Then we have

$$\begin{aligned}
\frac{1}{\mu(T,\rho)}\int_{-T}^{T}\|f(t)\|\rho(t)dt &\leq \frac{1}{\mu(T,\rho)}\int_{-T}^{T}\|f(t)-f_n(t)\|\rho(t)dt + \frac{1}{\mu(T,\rho)}\int_{-T}^{T}\|f_n(t)\|\rho(t)dt \\
&\leq \|f_n(t)-f(t)\|_\infty + \frac{1}{\mu(T,\rho)}\int_{-T}^{T}\|f_n(t)\|\rho(t)dt,
\end{aligned}$$

and hence $f \in WAA_0(\mathbb{R},\mathbb{X},\rho)$.

Theorem 3.3. *Let $\rho \in U_\infty^{inv}$. Assume* (H_1)-(H_8). *Then (7) has a mild weighted pseudo-almost automorphic solution u such that $\|Bu(t)\| \leq L$ for $t \in \mathbb{R}$.*

PROOF. Let $\mathscr{B} = \{u \in PAA(\mathbb{R},\mathbb{H},\rho) : \|Bu\| \leq L\}$. Then $\mathscr{B}$ is a closed convex subset. We claim that $(V+W)\mathscr{B} \subset \mathscr{B}$. In fact, for $u \in \mathscr{B}$ and $t \in \mathbb{R}$, by (H_4) and (H_7), we have

$$\begin{aligned}
& \|(Vu)(t)+(Wu)(t)\| \\
= \ & \|\int_{-\infty}^{t} T(t-s)PF(s,Bu(s))ds + \int_{-\infty}^{t} T(t-s)QF(s,Bu(s))ds\| \\
\leq \ & \Sigma_{n=1}^{\infty}\|\int_{t-n}^{t-n+1} T(t-s)PF(s,Bu(s))ds + \int_{t-n}^{t-n+1} T(t-s)QF(s,Bu(s))ds\| \\
\leq \ & \Sigma_{n=1}^{\infty}\int_{t-n}^{t-n+1} Me^{-c(t-s)}\|F(s,Bu(s))\|ds + \Sigma_{n=1}^{\infty}\int_{t-n}^{t-n+1} Me^{-d(t-s)}\|F(s,Bu(s))\|ds \\
\leq \ & \Sigma_{n=1}^{\infty}M(\int_{t-n}^{t-n+1} e^{-qc(t-s)}ds)^{\frac{1}{q}}(\int_{t-n}^{t-n+1}\|F(s,Bu(s))\|^p ds)^{\frac{1}{p}} \\
+ \ & \Sigma_{n=1}^{\infty}M(\int_{t-n}^{t-n+1} e^{-qd(t-s)}ds)^{\frac{1}{q}}(\int_{t-n}^{t-n+1}\|F(s,Bu(s))\|^p ds)^{\frac{1}{p}} \\
\leq \ & (\Sigma_{n=1}^{\infty}M(\frac{e^{qc}-1}{qc})^{\frac{1}{q}}e^{-cn} + \Sigma_{n=1}^{\infty}M(\frac{e^{qd}-1}{qd})^{\frac{1}{q}}e^{-dn})S_L \\
= \ & (K_c+K_d)S_L \leq L. \qquad (8)
\end{aligned}$$

Then we show that V and W are continuous mappings from $PAA(\mathbb{R},\mathbb{H},\rho)$ to $PAA(\mathbb{R},\mathbb{H},\rho)$. For $u \in \mathscr{B}$, we have $u \in PAA(\mathbb{R},\mathbb{H},\rho)$ with $u = u_1+u_2$, where $u_1 \in AA(\mathbb{R},\mathbb{H}), u_2 \in WAA_0(\mathbb{R},\mathbb{H},\rho)$. So it follows that $Bu \in PAA(\mathbb{R},\mathbb{H},\rho)$ and $Bu_1 \in AA(\mathbb{R},\mathbb{H})$ from (H_5). Setting $h(t) = F(t,Bu(t))$ and using Theorem 2.4, we have $h \in PAAS^p(\mathbb{R}\times\mathbb{H},\mathbb{H},\rho)$. Hence, we have $h = G+\Phi$, where $G^b \in AA(\mathbb{H},L^p((0,1),\mathbb{H}))$, $\Phi^b \in WAA_0(\mathbb{H},L^p((0,1),\mathbb{H}),\rho)$. Let $\Gamma_1(t) = G(t,Bu_1(t)), \Gamma_2(t) = \bigwedge_1(t)+\bigwedge_2(t)$, $\bigwedge_1(t) = F(t,Bu(t))-F(t,Bu_1(t))$, $\bigwedge_2(t) = \Phi(t,Bu_1(t))$. Then $h(t) = F(t,Bu(t)) = \Gamma_1(t)+\Gamma_2(t)$. From Theorem 2.4 and its proof, we know that $\Gamma_1^b \in AA(\mathbb{R}\times\mathbb{H},L^p((0,1),\mathbb{H}))$, $\Gamma_2^b \in WAA_0(\mathbb{R}\times\mathbb{H},L^p((0,1),\mathbb{H}),\rho)$.

Then

$$\begin{aligned}(Vu)(t) &= \int_{-\infty}^{t} T(t-s)PF(s,Bu(s))ds \\ &= \int_{-\infty}^{t} T(t-s)P\Gamma_1(s)ds + \int_{-\infty}^{t} T(t-s)P\Gamma_2(s)ds \\ &\triangleq V_1(t)+V_2(t).\end{aligned}$$

Since $h = G+\Phi$, where $G^b \in AA(\mathbb{H}, L^p((0,1),\mathbb{H}))$, $\Phi^b \in WAA_0(\mathbb{H}, L^p((0,1),\mathbb{H}),\rho)$. Now let us consider for each $n = 1,2,\cdots$, the integral

$$\begin{aligned}v_n(t) &= \int_{n-1}^{n} T(t-(t-\xi))h(t-\xi)d\xi \\ &= \int_{n-1}^{n} T(\xi)G(t-\xi)d\xi + \int_{n-1}^{n} T(\xi)\Phi(t-\xi)d\xi\end{aligned}$$

and set $Y_n(t) = \int_{n-1}^{n} T(\xi)PG(t-\xi)d\xi$ and $X_n(t) = \int_{n-1}^{n} T(\xi)P\Phi(t-\xi)d\xi$.

Let us show that $Y_n \in AA(\mathbb{R},\mathbb{H})$. For that, letting $r = t-\xi$ one obtain

$$Y_n(t) = -\int_{t-n}^{t-n+1} T(t-r)PG(r)dr \quad \text{for each} \quad t \in \mathbb{R}.$$

It follows that the function $r \mapsto T(t-r)PG(r)dr$ is integrable over $(-\infty,t)$ for each $t \in \mathbb{R}$ from (H_4).

Furthermore, using the H$\ddot{o}$lder inequality, it follows that

$$\begin{aligned}\|Y_n(t)\| &\leq M\int_{t-n}^{t-n+1} e^{-c(t-r)}\|G(r)\|dr \\ &\leq M[\int_{t-n}^{t-n+1} e^{-qc(t-r)}dr]^{\frac{1}{q}}[\int_{t-n}^{t-n+1} \|G(r)\|^p dr]^{\frac{1}{p}} \\ &\leq M[\int_{t-n}^{t-n+1} e^{-qc(t-r)}dr]^{\frac{1}{q}}\|G\|_{S^p} \\ &\leq [e^{-cn}M\sqrt[q]{\frac{1+e^{qc}}{qc}}]\|G\|_{S^p}.\end{aligned}$$

Now since $M\sqrt[q]{\frac{1+e^{qc}}{qc}}\Sigma_{n=1}^{\infty}e^{-cn} < \infty$, we deduce from the well-known Weirstrass theorem that the series $\Sigma_{n=1}^{\infty}Y_n(t)$ is uniformly convergent on $\mathbb{R}$. Furthermore,

$$V_1(t) := \int_{-\infty}^{t} T(t-s)PG(s)ds = \Sigma_{n=1}^{\infty}Y_n(t),$$

$V_1 \in C(\mathbb{R},\mathbb{X})$, and

$$\|V_1(t)\| \leq \Sigma_{n=1}^{\infty}\|Y_n(t)\| \leq K_q^{c,M}\|G\|_{S^p},$$

where $K_q^{c,M} > 0$ is a constant, which depends on the parameters q,c and M only.

Now let $(s_m)_{m\in N}$ be a sequence of real numbers. Since $G \in AS^p(\mathbb{H})$, there exist a subsequence $(s_{m_k})_{k\in N}$ of $(s_m)_{m\in N}$ and a function $v \in AS^p(\mathbb{H})$ such that

$$[\int_{t}^{t+1} \|G(s_{m_k}+\sigma)-v(\sigma)\|^p d\sigma]^{\frac{1}{p}} \to 0 \quad \text{as} \quad k \to \infty.$$

Define $Z_n(t) = \int_{n-1}^{n} T(\xi)v(t-\xi)d\xi$.

Then using the Hölder inequality we get

$$\begin{aligned}\|Y_n(t+s_{m_k})-Z_n(t)\| &= \|\int_{n-1}^{n} T(\xi)P[G(t+s_{m_k}-\xi)-\nu(t-\xi)]d\xi\| \\ &\leq M\int_{n-1}^{n} e^{-c\xi}\|G(t+s_{m_k}-\xi)-\nu(t-\xi)\|d\xi \\ &\leq M_q^{c,M}[\int_{n-1}^{n}\|G(t+s_{m_k}-\xi)-\nu(t-\xi)\|^p d\xi]^{\frac{1}{p}}.\end{aligned}$$

where $M_q^{c,M} = M\sqrt[q]{\frac{1+e^{qc}}{qc}}$.

Obviously,

$$\|Y_n(t+s_{m_k})-Z_n(t)\| \to 0 \quad \text{as} \quad k\to\infty.$$

Similarly, we can prove that

$$\|Z_n(t+s_{m_k})-Y_n(t)\| \to 0 \quad \text{as} \quad k\to\infty.$$

Therefore each $Y_n \in AA(\mathbb{R},\mathbb{H})$ for each n and hence their uniform limit $V_1 \in AA(\mathbb{R},\mathbb{H})$, by using Theorem 2.1.10 of [13].

Let us show that each $X_n \in WAA_0(\mathbb{R},\mathbb{H},\rho)$. For this, note that

$$\begin{aligned}\|X_n(t)\| &\leq M\int_{t-n}^{t-n+1} e^{-c(t-r)}\|\Phi(r)\|dr \\ &\leq M[\int_{t-n}^{t-n+1} e^{-qc(t-r)}dr]^{\frac{1}{q}}[\int_{t-n}^{t-n+1}\|\Phi(r)\|^p dr]^{\frac{1}{p}} \\ &\leq [e^{-cn}M\sqrt[q]{\frac{1+e^{qc}}{qc}}][\int_{t-n}^{t-n+1}\|\Phi(r)\|^p dr]^{\frac{1}{p}} \\ &\leq [M\sqrt[q]{\frac{1+e^{qc}}{qc}}][\int_{t-n}^{t-n+1}\|\Phi(r)\|^p dr]^{\frac{1}{p}},\end{aligned}$$

and hence $X_n \in WAA_0(\mathbb{R},\mathbb{H},\rho)$, as $\Phi^b \in WAA_0(L^p(0,1),\mathbb{H},\rho)$. Furthermore,

$$V_2(t) := \int_{-\infty}^{t} T(t-s)P\Phi(s)ds = \Sigma_{n=1}^{\infty}X_n(t),$$

$V_2 \in C(\mathbb{R},\mathbb{H})$, and

$$\|V_2(t)\| \leq \Sigma_{n=1}^{\infty}\|X_n(t)\| \leq C_q^{c,M}\|\Phi\|_{S^p},$$

where $C_q^{c,M} > 0$ is a constant, which depends on the parameters q, c and M only. By Lemma 3.2 the uniform limit $V_2 \in WAA_0(\mathbb{R},\mathbb{H},\rho)$. Hence, we have $Vu \in PAA(\mathbb{R},\mathbb{H},\rho)$. Similarly, we can get that $Wu \in PAA(\mathbb{R},\mathbb{H},\rho)$ and we omit the details. Therefore, $V+W \in PAA(\mathbb{R},\mathbb{H},\rho)$. This together with (8) yields that $(V+W)\mathscr{B} \subset \mathscr{B}$.

Now by an argument the same as the proofs of Lemma 3.4 in [10], we can prove that V and W are continuous mappings and the following statements are true.

(*i*) $\{(Vu)(t) : u \in \mathscr{B}\}$ and $\{(Wu)(t) : u \in \mathscr{B}\}$ both are relatively compact subsets of $\mathbb{H}$ for each $t \in \mathbb{R}$.

(*ii*) $\{Vu : u \in \mathscr{B}\} \subset PAA(\mathbb{R},\mathbb{H},\rho)$ and $\{(Wu) : u \in \mathscr{B}\} \subset PAA(\mathbb{R},\mathbb{H},\rho)$ both are equicontinuous.

Now using (H_8), by an argument the same as the proofs of Theorem 3.1 in [9], we can prove that $V+W$ has a fixed point in $\overline{co}(V+W)\mathscr{B}$(here we omit the details). That is Eq (7) has a mild weighted pseudo-almost automorphic solution u such that $\|Bu(t)\| \leq L$ for $t \in \mathbb{R}$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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