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COUPLED COMMON FIXED POINT THEOREMS FOR FOUR MIXED WEAKLY MONOTONE MAPPINGS WITH TWICE POWER TYPE ϕ-CONTRACTION CONDITION

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Abstract. By using the compatibility condition and mixed weakly monotone property, in this manuscript, we prove some coupled common fixed point theorems for four mappings with twice power type ϕ-contraction condition in partially ordered complete metric spaces.Our results are generalizations of the main results of Gordji, Akbartabar and Cho and Ramezani et al.

Keywords: common fixed point, mixed weakly monotone property, compatibility, partially ordered metric space.

2000 AMS Subject Classification: 47H10, 46B20, 54H25.

1. Introduction

Existence of fixed points in partially ordered metric spaces was first investigated by Turinici [14], where he extended the Banach contraction principle in partially ordered sets. In 2004, Ran and Reurings [12] presented some applications of Turinici's theorem to matrix equations. Following these initial articles, some remarkable results were reported, e.g., [1,2,13].
Recently. the existence of coupled fixed points for some kinds of contractive - type mappings

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in partially ordered metric spces, ordered metric spaces, cone metric spaces, fuzzy metric spaces, and other spaces with applications has been investigated by some authers, for examples, Bhasker and Lakshmikantham [3], Cho *et al.* [5,6], Gordji *et al.* [7,8] and others. In [3] Bhasker and Lakshmikantham introduced the notions of a mixed monotone mapping and a coupled fixed point and proved some coupled fixed point theorems for mixed monotone mappings and discussed the existence and uniqueness of solution for periodic boundary value problems.On the other hand Jungck [10,11] introduced more generalized commuting mappings, called compatible and weakly compatible mappings, which are more general than commuting and weakly commuting maps. Especially, in [9], Gordji *et al.* introduced the notion of a mixed weakly monotone pair of mappings and proved some coupled common fixed point theorems for a contractive-type mappings.In this paper, by using the compatible condition and mixed weakly monotone property in metric spaces, we discussed the existence and uniqueness of coupled common fixed point for four mappings with twice power type ϕ-contractive condition in complete metric spaces.The results presented in this paper improves and extends some previous results.

2. Preliminaries

Definition 2.1. [3] Let (X,$\leq$) be a partially ordered set and f:X$\times$X $\to$ X be a mapping. We say that f has the mixed monotone property on X if, for any $x,y \in X$,

$$x_1,x_2 \in X, x_1 \leq x_2 \implies f(x_1,y) \leq f(x_2,y)$$

and

$$y_1,y_2 \in X, y_1 \leq y_2 \implies f(x,y_1) \geq f(x,y_2).$$

Definition 2.2. [3] An element $(x,y) \in X \times X$ is called a coupled fixed point of a mapping F:$X \times X$ $\to$ X if $x = F(x,y)$ and $y = F(y,x)$.

Definition 2.3. [9] Let (X,$\leq$) be a partially ordered set and f,g:$X\times X\rightarrow$ X be mappings.We say that a pair (f,g) has the mixed weakly monotone property on X if, for any $x,y\in X$,

$$x\leq f(x,y), y\geq f(y,x)$$

$$\Longrightarrow f(x,y)\leq g(f(x,y),f(y,x)), f(y,x)\geq g(f(y,x),f(x,y))$$

and

$$x\leq g(x,y), y\geq g(y,x)$$

$$\Longrightarrow g(x,y)\leq f(g(x,y),g(y,x)), g(y,x)\geq f(g(y,x),g(x,y)).$$

Example 2.4. [9] Consider an ordered cone metric space $(R,\leq,d)$, where $\leq$ represents the usual order relation and d is a usual metric on R and let f,g: $R\times R\rightarrow R$ be two functions defined by

$$f(x,y)=x-2y, g(x,y)=x-y.$$

Then a pair (f,g) has the mixed weakly monotone property.

Example 2.5. [9] Consider an ordered cone metric space $(R,\leq,d)$, where $\leq$ represents the usual order relation and d is a usual metric on R and let f,g :$R\times R\rightarrow$ R be two functions defined by

$$f(x,y)=x-y+1, g(x,y)=2x-3y.$$

Then both mappings f and g have the mixed monotone property, but a pair (f,g) has not the mixed weakly monotone property.

Definition 2.6. [10] Let (X, d) be a metric space and let f and g be two maps from X into itself.Then f and g are called compatible if

$$lim_{n\rightarrow\infty}d(fgx_n,gfx_n)=0$$

Definition 2.7. [4] Let ϕ be a function, we called ϕ satisfies the condition (ϕ), if the function ϕ satisfying the following condition:

$$(\phi):\phi:[0,\infty)\rightarrow[0,\infty)$$

and continuous at a point t from the right, nondecreasing and $\phi(t)<t$ for every $t>0$.

In order to prove the main results of this paper, we need the following Lemmas:

Lemma 2.8 [4] *Let the function ϕ satisfies the condition (ϕ), then we have*

(i) *For all real number $t \in [0,\infty)$, if $t \leq \phi(t)$, then $t = 0$;*

(ii) *For all nonnegative sequence $\{t_n\}$, if $t_{n+1} \leq \phi(t_n)$, n = 1,2 3....*

then $lim_{n\to\infty} t_n = 0$.

Lemma 2.9. [4] *Let (X, d) be a metric space, $\{y_n\}$ is a sequence in X which satisfies the condition $lim_{n\to\infty} d(y_n, y_{n+1}) = 0$. Suppose that $\{y_n\}$ is not a Cauchy sequence in X, then there must exist an $\varepsilon_0 > 0$, and the positive integer sequences $\{m_i\}$, $\{n_i\}$, such that*

(i) $m_i > n_i + 1$, $n_i \to \infty$ $(i \to \infty)$;

(ii) $d(y_{m_i}, y_{n_i}) \geq \varepsilon_0; d(y_{m_{i-1}}, y_{n_i}) < \varepsilon_0$ *for i = 0, 1, 2,....*

3. Main results

Theorem 3.1. *Let (X, d) be a complete metric space and let S, T, A and B be four mappings from $X \times X$ into X satisfying the conditions*

(i) $S(X \times X) \subset B(X \times X)$ *and* $T(X \times X) \subset A(X \times X)$,

(ii) *the pairs (S,A) and (B,T) has mixed weakly monotone property,*

(iii)

$$\begin{aligned}
&d^2(S(x,y), T(u,v)) \\
&\leq \phi(\max\{d(A(x,y), B(u,v))d(A(x,y), S(x,y)), \\
&d(A(x,y), T(u,v))d(B(u,v), S(x,y)), \\
&d(A(x,y), B(u,v))d(B(u,v), T(u,v))\}), \quad \forall (x,y), (u,v) \in X \times X,
\end{aligned}$$

(iv) *If A and B are continuous and the pairs (S,A) and (B,T) are compatible.*

Then S, T, A and B have a unique coupled common fixed point in X.

Proof. Suppose that $x_0 \leq S(x_0, y_0)$ and $y_0 \geq S(y_0, x_0)$ and since $S(X \times X) \subset B(X \times X)$ there exist $x_1, y_1 \in X$ such that

$$S(x_0, y_0) = B(x_1, y_1) := x_1,$$

$$S(y_0,x_0) = B(y_1,x_1) := y_1.$$

Again the pair (B, T) has mixed monotone property,

$$B(x_1,y_1) \leq T(B(x_1,y_1),B(y_1,x_1)) = T(x_1,y_1)$$

and

$$B(y_1,x_1) \geq T(B(y_1,x_1),B(x_1,y_1)) = T(y_1,x_1).$$

Since $T(X\times X) \subset A(X\times X)$, there exist $x_2,y_2 \in X$ such that

$$T(x_1,y_1) = A(x_2,y_2) := x_2,$$

$$T(y_1,x_1) = A(y_2,x_2) := y_2.$$

Again a pair (S, A) has mixed monotone property,

$$A(x_2,y_2) \leq S(A(x_2,y_2),A(y_2,x_2)) = S(x_2,y_2)$$

and

$$A(y_2,x_2) \geq S(A(y_2,x_2),A(x_2,y_2)) = S(y_2,x_2).$$

Continuing in this way, we find that

$$x_{2n} = T(x_{2n-1},y_{2n-1}) = A(x_{2n},y_{2n}), y_{2n} = T(y_{2n-1},x_{2n-1}) = A(y_{2n},x_{2n})$$

and

$$x_{2n+1} = S(x_{2n},y_{2n}) = B(x_{2n+1},y_{2n+1}), y_{2n+1} = S(y_{2n},x_{2n}) = B(y_{2n+1},x_{2n+1})$$

for all $n \geq 1$. Then we can easily verify that

$$x_0 \leq x_1 \leq x_2 \leq \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \leq x_n \leq x_{n+1} \leq \ldots\ldots\ldots$$

and

$$y_0 \geq y_1 \geq y_2 \geq \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \geq y_n \geq y_{n+1} \geq \ldots\ldots\ldots\ldots$$

Similarly, from the condition $x_0 \leq T(x_0,y_0)$ and $y_0 \geq T(y_0,x_0)$, one may easily see that the sequences $\{x_n\}$ and $\{y_n\}$ are increasing and decreasing respectively. Let $d_n = d(x_n,x_{n+1})$ and $d'_n = d(y_n,y_{n+1})$. Now we shall show that

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} d(x_n,x_{n+1}) = 0 \textit{ and } \lim_{n\to\infty} d'_n = \lim_{n\to\infty} d(y_n,y_{n+1}) = 0. \tag{1}$$

In fact, from condition (iii) and the property of ϕ, we have

$$
\begin{aligned}
& d^2(x_{2n-1}, x_{2n}) \\
& = d^2(S(x_{2n-2}, y_{2n-2}), T(x_{2n-1}, y_{2n-1})) \\
& \leq \phi(\max\{d(A(x_{2n-2}, y_{2n-2}), B(x_{2n-1}, y_{2n-1}))d(A(x_{2n-2}, y_{2n-2}), S(x_{2n-2}, y_{2n-2})), \\
& d(A(x_{2n-2}, y_{2n-2}), T(x_{2n-1}, y_{2n-1}))d(B(x_{2n-1}, y_{2n-1}), S(x_{2n-2}, y_{2n-2})), \\
& d(A(x_{2n-2}, y_{2n-2}), B(x_{2n-1}, y_{2n-1}))d(B(x_{2n-1}, y_{2n-1}), T(x_{2n-1}, y_{2n-1}))\}) \\
& = \phi(\max\{d(x_{2n-2}, x_{2n-1})d(x_{2n-2}, x_{2n-1}), d(x_{2n-2}, x_{2n})d(x_{2n-1}, x_{2n-1}), d(x_{2n-2}, x_{2n-1}) \\
& d(x_{2n-1}, x_{2n})\}) \\
& = \phi(\max\{d^2(x_{2n-2}, x_{2n-1}), d(x_{2n-2}, x_{2n-1})d(x_{2n-1}, x_{2n})\}).
\end{aligned}
$$

Now suppose that $d(x_{2n-2}, x_{2n-1}) < d(x_{2n-1}, x_{2n})$, by the property of function ϕ, we get

$$d^2(x_{2n-1}, x_{2n}) \leq \phi(d^2(x_{2n-1}, x_{2n})).$$

Therefore by virtue of this and using Lemma 2.8 (i), we have $d^2(x_{2n-1}, x_{2n}) = 0$, which implies that $d(x_{2n-1}, x_{2n}) = 0$. Thus

$$d(x_{2n-2}, x_{2n-1}) < d(x_{2n-1}, x_{2n}) = 0,$$

which is a contradiction. It follows that, in any event, we have

$$d(x_{2n-2}, x_{2n-1}) \geq d(x_{2n-1}, x_{2n}).$$

It follows that $d^2(x_{2n-1}, x_{2n}) \leq \phi(d^2(x_{2n-2}, x_{2n-1}))$. Hence by Lemma 2.8 (ii), we get

$$d^2(x_{2n-1}, x_{2n}) \to 0 (n \to \infty) \textit{ and so } \lim_{n\to\infty} d(x_{2n-1}, x_{2n}) = 0.$$

Similarly, we find that $\lim_{n\to\infty} d(y_{2n-1}, y_{2n}) = 0$. Also it can be proved that

$$\lim_{n\to\infty} d(x_{2n}, x_{2n+1}) = 0$$

and $\lim_{n\to\infty} d(y_{2n}, y_{2n+1}) = 0$. Thus, $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$ and $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$, Therefore we obtained result (1).

Next we shall prove that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X. If not, by Lemma 2.9 there exists $\varepsilon_0 > 0$ and the positive integer sequences $\{m_i\}$, $\{n_i\}$ such that

(a) $m_i > n_i + 1$, $n_i \to \infty (i \to \infty)$.

(b) $d(x_{m_i}, x_{n_i}) \geq \varepsilon_0$; $d(x_{m_i - 1}, x_{n_i}) \leq \varepsilon_0$ for i = 0, 1, 2,...

or $d(y_{m_i}, y_{n_i}) \geq \varepsilon_0$; $d(y_{m_i - 1}, y_{n_i}) \leq \varepsilon_0$ for i = 0, 1, 2,...

Letting $e_i = d(x_{m_i}, x_{n_i})$; $e'_i = d(y_{m_i}, y_{n_i})$, we get

$$\varepsilon_0 \leq e_i \leq d(x_{m_i}, x_{m_i - 1}) + d(x_{m_i - 1}, x_{n_i}) < \varepsilon_0 + d(x_{m_i - 1}, x_{m_i})$$

and

$$\varepsilon_0 \leq e'_i \leq d(y_{m_i}, y_{m_i - 1}) + d(y_{m_i - 1}, y_{n_i}) < \varepsilon_0 + d(y_{m_i - 1}, y_{m_i}).$$

Letting $i \to \infty$ in the above inequalities, we have from (1) that

$$\lim_{i\to\infty} e_i = \varepsilon_0 \text{ and } \lim_{i\to\infty} e'_i = \varepsilon_0. \tag{2}$$

On the other hand, we have

$$e_i = d(x_{m_i}, x_{n_i}) \leq d(x_{m_i}, x_{m_i+1}) + d(x_{m_i+1}, x_{n_i+1}) + d(x_{n_i+1}, x_{n_i}), \tag{3}$$

$$e'_i = d(y_{m_i}, y_{n_i}) \leq d(y_{m_i}, y_{m_i+1}) + d(y_{m_i+1}, y_{n_i+1}) + d(y_{n_i+1}, y_{n_i}). \tag{4}$$

Now we consider four possible cases for $d(x_{m_i+1}, x_{n_i+1})$ and $d(y_{m_i+1}, y_{n_i+1})$.

Case I: We assume that n_i is odd and m_i is even. By virtue of condition (iii), we have

$$\begin{aligned}
& d^2(x_{m_i+1}, x_{n_i+1}) \\
& = d^2(S(x_{m_i}, y_{m_i}), T(x_{n_i}, y_{n_i})) \\
& \leq \phi(\max\{d(A(x_{m_i}, y_{m_i}), B(x_{n_i}, y_{n_i}))d(A(x_{m_i}, y_{m_i}), S(x_{m_i}, y_{m_i})), \\
& d(A(x_{m_i}, y_{m_i}), T(x_{n_i}, y_{n_i}))d(B(x_{n_i}, y_{n_i}), S(x_{m_i}, y_{m_i})), d(A(x_{m_i}, y_{m_i}), B(x_{n_i}, y_{n_i})) \\
& d(B(x_{n_i}, y_{n_i}), T(x_{n_i}, y_{n_i}))\}) \\
& = \phi(\max\{d(x_{m_i}, x_{n_i})d(x_{m_i}, x_{m_i+1}), d(x_{m_i}, x_{n_i+1})d(x_{n_i}, x_{m_i+1}), d(x_{m_i}, x_{n_i})d(x_{n_i}, x_{n_i+1})\}) \\
& \leq \phi(\max\{e_i d_{m_i}, (e_i + d_{n_i})(e_i + d_{m_i}), e_i d_{n_i}\}),
\end{aligned}$$

and

$$d^2(y_{m_i+1}, y_{n_i+1}) \le \phi(\max\{e'_i d_{m_i}, (e'_i + d_{n_i})(e'_i + d_{m_i}), e'_i d_{n_i}\}).$$

Let $i \to \infty$ in above equations. In view of (1), (3), (4) and the assumption about $\phi(t)$ is right-continuous, we have $\lim_{i\to\infty} d^2(x_{m_i+1}, x_{n_i+1}) \le \phi(\varepsilon_0^2)$, which implies that

$$\lim_{i\to\infty} d(x_{m_i+1}, x_{n_i+1}) \le [\phi(\varepsilon_0^2)]^{\frac{1}{2}} \tag{5}$$

and

$$\lim_{i\to\infty} d(y_{m_i+1}, y_{n_i+1}) \le [\phi(\varepsilon_0^2)]^{\frac{1}{2}}. \tag{6}$$

Let $i \to \infty$ in (3) and (4). Using (1), (5) and (6), we obtain

$$\varepsilon_0 \le e_i \le 0 + [\phi(\varepsilon_0^2)]^{\frac{1}{2}} + 0 = [\phi(\varepsilon_0^2)]^{\frac{1}{2}},$$

and

$$\varepsilon_0 \le e'_i \le 0 + [\phi(\varepsilon_0^2)]^{\frac{1}{2}} + 0 = [\phi(\varepsilon_0^2)]^{\frac{1}{2}}.$$

It follows that

$$\varepsilon_0^2 \le e_i^2 \le \phi(\varepsilon_0^2) \le \varepsilon_0^2,$$

and

$$\varepsilon_0^2 \le e_i'^2 \le \phi(\varepsilon_0^2) \le \varepsilon_0^2,$$

which are contradictions.

Case II: We assume that n_i and m_i are all even. By virtue of the condition (iii), we have

$$\begin{aligned} d(x_{m_i+1}, x_{n_i+1}) &= d(S(x_{m_i}, y_{m_i}), S(x_{n_i}, y_{n_i})) \\ &\le d(S(x_{m_i}, y_{m_i}), T(x_{n_i+1}, y_{n_i+1})) + d(S(x_{n_i}, y_{n_i}), T(x_{n_i+1}, y_{n_i+1})) \end{aligned} \tag{7}$$

and

$$\begin{aligned}
&d^2(S(x_{m_i},y_{m_i}),T(x_{n_i+1},y_{n_i+1}))\\
&\le \phi(\max\{d(A(x_{m_i},y_{m_i}),B(x_{n_i+1},y_{n_i+1}))d(A(x_{m_i},y_{m_i}),S(x_{m_i},y_{m_i})),\\
&d(A(x_{m_i},y_{m_i}),T(x_{n_i+1},y_{n_i+1}))d(B(x_{n_i+1},y_{n_i+1}),S(x_{m_i},y_{m_i})),\\
&d(A(x_{m_i},y_{m_i}),B(x_{n_i+1},y_{n_i+1}))d(B(x_{n_i+1},y_{n_i+1}),T(x_{n_i+1},y_{n_i+1}))\})\\
&= \phi(\max\{d(x_{m_i},x_{n_i+1})d(x_{m_i},x_{m_i+1}),d(x_{m_i},x_{n_i+2})d(x_{n_i+1},y_{m_i+1}),\\
&d(x_{m_i},x_{n_i+1})d(x_{n_i+1},x_{n_i+2})\})\\
&\le \phi(\max\{(e_i+d_{n_i})d_{m_i},(e_i+d_{n_i}+d_{n_i+1})(e_i+d_{n_i}+d_{m_i}),\\
&d(e_i+d_{n_i})d_{n_i+1}\}).
\end{aligned} \tag{8}$$

Letting $i \to \infty$ and using (1) and the assumption about $\phi(t)$ is right-continuous, we have

$$\lim_{i\to\infty} d^2(S(x_{m_i},y_{m_i}),T(x_{n_i+1},y_{n_i+1})) \le [\phi(\varepsilon_0^2)],$$

$$\lim_{i\to\infty} d(S(x_{m_i},y_{m_i}),T(x_{n_i+1},y_{n_i+1})) \le [\phi(\varepsilon_0^2)]^{\frac{1}{2}}. \tag{9}$$

It follows from (1) that

$$\lim_{i\to\infty} d(S(x_{n_i},y_{n_i}),T(x_{n_i+1},y_{n_i+1})) = \lim_{i\to\infty} d(x_{n_i+1},x_{n_i+2}) = 0. \tag{10}$$

Let $i \to \infty$ in (8). Using (9) and (10), we obtain that

$$\lim_{i\to\infty} d(x_{m_i+1},x_{n_i+1}) \le [\phi(\varepsilon_0^2)]^{\frac{1}{2}} + 0 = [\phi(\varepsilon_0^2)]^{\frac{1}{2}}. \tag{11}$$

Let $i \to \infty$ in (3). Using (1) and (11), we obtain that

$$\varepsilon_0 \le e_i \le 0 + [\phi(\varepsilon_0^2)]^{\frac{1}{2}} = [\phi(\varepsilon_0^2)]^{\frac{1}{2}},$$

which implies that $\varepsilon_0^2 \le e_i^2 \le \phi(\varepsilon_0^2) < \varepsilon_0^2$. This is a contradiction. Similarly, we hae

$$\lim_{i\to\infty} d(y_{m_i+1},y_{n_i+1}) \le [\phi(\varepsilon_0^2)]^{\frac{1}{2}} + 0 = [\phi(\varepsilon_0^2)]^{\frac{1}{2}}.$$

Let $i \to \infty$ in (4). Using (1) and (11), we obtain that

$$\varepsilon_0 \le e'_i \le 0 + [\phi(\varepsilon_0^2)]^{\frac{1}{2}} = [\phi(\varepsilon_0^2)]^{\frac{1}{2}},$$

which implies that

$$\varepsilon_0^2 \leq e_i'^2 \leq \phi(\varepsilon_0^2) < \varepsilon_0^2.$$

This a contradiction. Similarly, we can also complete the proof when n_i and m_i are all odd, or n_i is even and m_i is odd. This is the anticipated contradiction. Hence $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X. Since X is complete, suppose that $x_n \to x* \in X$ and $y_n \to y* \in X$ then the sequences $\{x_{2n-1}\}$, $\{x_{2n}\}$ and $\{y_{2n-1}\}$, $\{y_{2n}\}$ are said to be convergent to x* and y* respectively. It follows that

$$\lim_{n\to\infty} x_{2n} = x* = \lim_{n\to\infty} T(x_{2n-1}, y_{2n-1}) = \lim_{n\to\infty} A(x_{2n}, y_{2n}).$$

$$\lim_{n\to\infty} y_{2n} = y* = \lim_{n\to\infty} T(y_{2n-1}, x_{2n-1}) = \lim_{n\to\infty} A(y_{2n}, x_{2n}).$$

$$\lim_{n\to\infty} x_{2n+1} = x* = \lim_{n\to\infty} S(x_{2n}, y_{2n}) = \lim_{n\to\infty} B(x_{2n+1}, y_{2n+1}).$$

$$\lim_{n\to\infty} y_{2n+1} = y* = \lim_{n\to\infty} S(y_{2n}, x_{2n}) = \lim_{n\to\infty} B(y_{2n+1}, x_{2n+1}).$$

Suppose B is continuous, then

$$\lim_{n\to\infty} B(x_{2n+1}, y_{2n+1}) = B(\lim_{n\to\infty} x_{2n+1}, \lim_{n\to\infty} y_{2n+1}) = B(x*, y*),$$

and

$$\lim_{n\to\infty} B(y_{2n+1}, x_{2n+1}) = B(\lim_{n\to\infty} y_{2n+1}, \lim_{n\to\infty} x_{2n+1}) = B(y*, x*).$$

and the pair (B,T) is compatible, we have

$$\lim_{n\to\infty} d(B(T(x_{2n-1}, y_{2n-1}), T(y_{2n-1}, x_{2n-1})), T(B(x_{2n-1}, y_{2n-1}), B(y_{2n-1}, x_{2n-1}))) = 0.$$

$$\Longrightarrow \lim_{n\to\infty} T(B(x_{2n-1}, y_{2n-1}), B(y_{2n-1}, x_{2n-1})) = B(x*, y*).$$

Supposing A is continuous, we have

$$\lim_{n\to\infty} A(x_{2n}, y_{2n}) = A(\lim_{n\to\infty} x_{2n}, \lim_{n\to\infty} y_{2n}) = A(x*, y*)$$

and

$$\lim_{n\to\infty} A(y_{2n}, x_{2n}) = A(\lim_{n\to\infty} y_{2n}, \lim_{n\to\infty} x_{2n}) = A(y*, x*).$$

and the pair (A,S) is compatible, we obtain

$$\lim_{n\to\infty} d(A(S(x_{2n}, y_{2n}), S(y_{2n}, x_{2n})), S(A(x_{2n-1}, y_{2n-1}), A(y_{2n-1}, x_{2n-1}))) = 0.$$

$$\implies \lim_{n\to\infty} S(A(x_{2n-1},y_{2n-1}),A(y_{2n-1},x_{2n-1})) = A(x*,y*).$$

Notice that

$$\begin{aligned}
&d^2(S(x_{2n},y_{2n}),T(x_{2n-1},y_{2n-1}))\\
&\leq \phi(\max\{d(A(x_{2n},y_{2n}),B(x_{2n-1},y_{2n-1}))d(A(x_{2n},y_{2n}),S(x_{2n},y_{2n})),\\
&d(A(x_{2n},y_{2n}),T(x_{2n-1},y_{2n-1}))d(B(x_{2n-1},y_{2n-1}),S(x_{2n},y_{2n})),\\
&d(A(x_{2n},y_{2n}),B(x_{2n-1},y_{2n-1}))d(B(x_{2n-1},y_{2n-1}),T(x_{2n-1},y_{2n-1}))\}).
\end{aligned}$$

Letting $n\to\infty$, we have

$$\begin{aligned}
d^2(A(x*,y*),B(x*,y*)) &\leq \phi(\max\{0,d^2(A(x*,y*),B(x*,y*)),0\})\\
&\leq \phi(d^2(A(x*,y*),B(x*,y*))).
\end{aligned}$$

By Lemma (1.8), we find that

$$d^2(A(x*,y*),B(x*,y*)) = 0. \implies d(A(x*,y*),B(x*,y*)) = 0.$$

Therefore, we have

$$A(x*,y*) = B(x*,y*).$$

Similarly, we can obtain that

$$A(y*,x*) = B(y*,x*).$$

Again

$$\begin{aligned}
&d^2(S(x_{2n},y_{2n}),T(x*,y*))\\
&\leq \phi(\max\{d(A(x_{2n},y_{2n}),B(x*,y*))d(A(x_{2n},y_{2n}),S(x_{2n},y_{2n})),\\
&d(A(x_{2n},y_{2n}),T(x*,y*))d(B(x*,y*),S(x_{2n},y_{2n})),\\
&d(A(x_{2n},y_{2n}),B(x*,y*))d(B(x*,y*),T(x*,y*))\}).
\end{aligned}$$

Notice that since $\phi(t)$ is non-decreasing. Letting $n\to\infty$, we find that

$$d^2(A(x*,y*),T(x*,y*)) \leq \phi(0) \leq \phi(d^2(A(x*,y*),T(x*,y*))).$$

By Lemma 2.8 (i), we have $d^2(A(x*,y*),T(x*,y*)) = 0$, which implies that $A(x*,y*) = T(x*,y*)$. Similarly, we have $A(y*,x*) = T(y*,x*)$. Also

$$\begin{aligned}
&d^2(S(x*,y*),T(x_{2n-1},y_{2n-1})\\
&\leq \phi(\max\{d(A(x*,y*),B(x_{2n-1},y_{2n-1}))d(A(x*,y*),S(x*,y*)),\\
&d(A(x*,y*),T(x_{2n-1},y_{2n-1}))d(B(x_{2n-1},y_{2n-1}),S(x*,y*)),\\
&d(A(x*,y*),B(x_{2n-1},y_{2n-1}))d(B(x_{2n-1},y_{2n-1}),T(x_{2n-1},y_{2n-1}))\}).
\end{aligned}$$

Notice that since $\phi(t)$ is non-decreasing. Letting $n \to \infty$, we have

$$d^2(S(x*,y*),B(x*,y*)) \leq \phi(0) \leq \phi(d^2(S(x*,y*),B(x*,y*))).$$

By Lemma 2.8 (i), we have $d^2(S(x*,y*),B(x*,y*)) = 0$, which implies that $S(x*,y*) = B(x*,y*)$. Similarly, we have $S(y*,x*) = B(y*,x*)$. Hence (x*,y*) is a coupled common fixed point of S,T,A and B. Next, we show the uniquess of fixed point. Suppose $(x*,y*)$ and (x',y') are the fixed points of S,T,A and B. Notice that

$$\begin{aligned}
&d^2(S(x*,y*),T(x',y'))\\
&\leq \phi(\max\{d(A(x*,y*),B(x',y'))d(A(x*,y*),S(x*,y*)),d(A(x*,y*),T(x',y'))\\
&d(B(x',y'),S(x*,y*)),d(A(x*,y*),B(x',y'))d(B(x',y'),T(x',y'))\})\\
&= \phi(\max\{d((x*,y*),(x',y'))d((x*,y*),(x*,y*)),d((x*,y*)(x',y'))\\
&d((x',y'),(x*,y*)),d((x*,y*),(x',y'))d((x',y'),(x',y'))\})\\
&\leq \phi(d^2((x*,y*),(x,y))).
\end{aligned}$$

By Lemma 2.8 (i), we have $d^2((x*,y*),(x',y')) = 0$, which implies that $d((x*,y*),(x',y')) = 0$. Hence $(x*,y*) = (x,y)$, the fixed point is unique. This completes the proof.

Corollary 3.2. *Let (X,d) be a complete metric space and let S and T be two mappings from $X \times X$ into X satisfying conditions (i) and (ii) of Theorem 3.1 and the condition:*
(iii) $\forall\ (x,y),(u,v) \in X \times X$

$$\begin{aligned}
d^2(S(x,y),T(u,v)) \leq \phi(max\{&d(x,S(x,y))d(u,T(u,v)),d(u,S(x,y))d(x,T(u,v)),\\
&d(x,u)d(S(x,y),T(u,v))\}).
\end{aligned}$$

Then S and T have a unique coupled common fixed point in X.

Next, we give some examples to support our main results.

Example 3.3. Let $X = [0,1]$ be a metric space with usual metric $d(x,y) = |x-y|$ for every $x,y \in X$. Define $\phi(t) = \frac{t}{2}$, $\forall\, t \in [0,\infty)$, let $\leq$ represents the usual order relation, the maps S,T,A and B as follows:

$$S(x,y) = \frac{x^2+y^2}{x+y}, (x+y \neq 0), T(x,y) = \frac{x+3y}{4}, A(x,y) = x, \textit{ and } B(x,y) = \frac{3x+y}{4}.$$

The compatible pairs (S,A) and (B,T) has mixed weakly monotone property.Now if we take $x = u$ and $y = v$ then

$$d(S(x,y),T(x,y)) = \frac{|x-y||3x-y|}{4|x+y|}, d(A(x,y),B(x,y)) = \frac{|x-y|}{4},$$
$$d(A(x,y),S(x,y)) = \frac{|y||x-y|}{|x+y|}, d(A(x,y),T(x,y)) = \frac{3|x-y|}{4},$$
$$d(B(x,y),S(x,y)) = |\frac{y(x-y)}{2(x+y)} - \frac{(x+y)}{4}|, d(B(x,y),T(x,y)) = \frac{|x-y|}{4}.$$

Now

$$\begin{aligned} d(S(x,y),T(x,y)) &= \frac{|x-y||3x-y|}{4|x+y|} \\ &\leq \phi(max\{\frac{|y||x-y|^2}{4|x+y|}, \frac{3|x-y|}{4}\left|\frac{y(x-y)}{2(x+y)} - \frac{x+y}{4}\right|, \frac{|x-y|^2}{16}\}). \end{aligned}$$

Let $x = u = \frac{1}{6}$, and $y = v = \frac{1}{4}$ we have

$$\begin{aligned} d(S(\frac{1}{6},\frac{1}{4}),T(\frac{1}{6},\frac{1}{4})) &= (\frac{1}{80})^2 \\ &\leq \phi(\max\{\frac{1}{960}, \frac{31}{3840}, \frac{1}{2304}\}) \\ &= \phi(\frac{31}{3840}). \end{aligned}$$

Then all the conditions of Theorem 2.1 satisfied for all $x,y \in [0,1]$. It is easy to show that the point (1,1) is unique coupled common fixed point of S,T,A and B.

Example 3.4. Let $X = [0,1]$ be a metric space with usual metric $d(x,y) = |x-y|$ for every $x,y \in X$. Define $\phi(t) = \frac{t}{2}$, $\forall\, t \in [0,\infty)$, let $\leq$ represents the usual order relation, the maps S and

T as follows:

$$S(x,y) = \frac{x^3+y^3}{x^2+y^2}, (x^2+y^2 \neq 0) \text{ and } T(x,y) = \frac{2x+3y}{5}.$$

$$d(S(x,y),T(x,y)) = \frac{|x-y||3x^2-2y^2|}{5|x^2+y^2|} \text{ and } d(x,S(x,y)) = \frac{|y|^2|x-y|}{|x^2+y^2|}.$$

$$d(x,T(x,y)) = \frac{3|x-y|}{5}, d(x,x) = 0.$$

$$\begin{aligned} d^2(S(x,y),T(x,y)) &= (\frac{|x-y||3x^2-2y^2|}{5|x^2+y^2|})^2 \\ &\leq \phi(\max\{\frac{3|x-y|^2|y|^2}{5|x^2+y^2|}, 0\}) \\ &= \phi(\frac{3|x-y|^2|y|^2}{5|x^2+y^2|}). \end{aligned}$$

A pair (S,T) has mixed weakly monotone property and satisfyig all the conditions of Corollary 2.2 for all $x,y \in [0,1]$. It is easy to show that $(1,1)$ is unique coupled common fixed point of S and T in X.

Example 3.5. Consider $(R,\leq,d)$, where $\leq$ represents the usual order relation and d is a usual metric on R and let S,T,A and $B : R\times R \to R$ be four functions defined by

$$S(x,y) = \frac{4x-2y+22}{24},$$

$$T(x,y) = \frac{6x-3y+33}{36}, A(x,y) = \frac{8x-4y+44}{48},$$

and

$$B(x,y) = \frac{10x-5y+55}{60}.$$

Then the pair (S,A) and (B,T) has mixed monotone property, and satisfying all the conditions of the theorem 2.1. Then (1,1) is a coupled common fixed point of S,T,A and B.

Conflict of Interests

The authors declare that there is no conflict of interests.

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