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A RELATED FIXED POINT THEOREM IN THREE MENGER SPACES

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Abstract. The aim of the present paper is to establish a fixed point theorem for six set-valued mappings in three complete Menger spaces. The results presented in this article mainly generalize the corresponding results in [1].

Keywords: Menger spaces; multi-valued maps; fixed point.

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1. Introduction

The literature in related fixed point theorems have been developed by many authors; [1], [2], [4]-[9] and the references therein. The result of Fisher on two metric spaces [4] was generalized to three metric spaces by Jain, Sahu and Fisher [8]. The result in [8] was generalized to set-valued mapings by Jain and Fisher [7]. Recently Beg and Chauhan extended the result in [7] in Menger spaces and obtained related fixed point theorems for three mappings; for more details, see [1]. In this paper, a related fixed point theorem for six set-valued mapings in three Menger spaces is obtained based on the result in [1].

2. Preliminaries

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In this paper, we always use R to denote the set of real numbers and R^+ to denote the set of non-negative real numbers. Next, we give some definitions and lemmas which play an important role in this paper.

Definition 2.1. A mapping $F : R \to R^+$ is called a distribution function if it is non-decreasing and left continuous with $\inf_{t\in R} F(t) = 0$ and $\sup_{t\in R} F(t) = 1$.Let D denotes the set of all distribution functions whereas H stands for specific distribution function(also known as Heaviside function) defined as

$$H(t) = \begin{cases} 0, & t \leq 0; \\ 1, & t > 0. \end{cases}$$

Definition 2.2. A PM-space is an ordered pair (X,F)consisting of non- empty set $X$and a mapping F from $X \times X$ into D.The value of F at $(x,y) \in X$ is represented by $F_{x,y}$.The functions $F_{x,y}$are assumed to satisfy the following conditions:

(i) $F_{x,y}(t) = 1$ for all $t > 0$ if and only if $x = y$;

(ii) $F_{x,y}(0) = 0$;

(iii) $F_{x,y}(t) = F_{y,x}(t)$;

(iv) if $F_{x,y}(t) = 1$ and $F_{y,z}(s) = 1$,then $F_{x,z}(t+s) = 1$for all $x,y \in X$ and $t,s \geq 0$.

Every metric (X,d) space can always be realized as a PM-space by considering F from $X \times X$ into D as $F_{u,v}(s) = H(s - d(u,v))$ for all $u,v \in X$.

Definition 2.3. A mapping $\Delta : [0,1] \times [0,1] \to [0,1]$ is called a triangular norm (briefly t-norm) if the following conditions are satisfied:

(i) $\Delta(a,1) = a \ \ for \ \ all \ \ a \in [0,1]$;

(ii) $\Delta(a,b) = \Delta(b,a)$;

(iii) $\Delta(c,d) \geq \Delta(a,b)$ for $c \geq a, d \geq b$;

(iv) $\Delta(\Delta(a,b),c) = \Delta(a,\Delta(b,c))$ for all $a,b,c,d \in [0,1]$.

Examples of t-norm are $\Delta(a,b) = \min(a,b)$, $\Delta(a,b) = ab$ and $\Delta(a,b) = \min(a+b-1,0)$ etc.

Definition 2.4. A Menger space is a triplet(X,F,Δ),where$(X,F)$is a PM-space,$\Delta$is t-norm and the following condition hold:

$$F_{x,z}(t+s) \geq \Delta(F_{x,y}(t), F_{y,z}(s)), \forall x,y,z \in X, t,s \geq 0.$$

Definition 2.5. A sequence $\{p_n\}$in a Menger space (X,F,Δ) is said to converge to a point p in X if for every $\varepsilon > 0$ and $\lambda > 0$,there is an integer $N(\varepsilon,\lambda)$ such that $F_{p_n,p}(\varepsilon) > 1-\lambda$,for all $n \geq N(\varepsilon,\lambda)$.The sequence is said to be Cauchy sequence if for every $\varepsilon > 0$ and $\lambda > 0$,there is an integer $N(\varepsilon,\lambda)$ such that $F_{p_n,p_m}(\varepsilon) > 1-\lambda$, for all $n,m \geq N(\varepsilon,\lambda)$.

Throughout this paper, $B(X)$ is denoted by the set of all non-empty bounded subsets of Menger space X.

For all $A,B \in B(X)$ and for all $t > 0$, we define

$\delta F_{A,B}(t) = \inf\{F_{a,b}(t) : a \in A, b \in B\}$.

If $A = \{a\}$, then $\delta F_{A,B}(t) = \delta F_{a,B}(t)$.

If we have also $B = \{b\}$, then $\delta F_{A,B}(t) = F_{a,b}(t)$.

It follows from the definition that $\delta F_{A,B}(t) = 1 \Leftrightarrow A = B = \{a\}$.

Let $\{A_n\}$ be a sequence in $B(X)$. we say that $\{A_n\}$ δ-converges to a set A in X if for every $\varepsilon > 0$ we have

$$\lim_{n\to\infty} \delta F_{A_n,A}(\varepsilon) = 1.$$

Lemma 2.1 [3] *Let* $(X,F,\min)$ *be a Menger space. Let* $A,G,H \in B(X)$*. Then for* $t_1,t_2 > 0$ *we have*

$$\delta F_{A,H}(t_1+t_2) \geq \min\{\delta F_{A,G}(t_1), \delta F_{G,H}(t_2)\}.$$

Lemma 2.2 [10] *Let* $(X,F,\min)$ *be a Menger space. If the sequence* $\{a_n\}$ *converges to a and the sequence* $\{b_n\}$ *converges to b, then for* $t > 0$ *we have*

$$\liminf_{n\to\infty} F_{a_n,b_n}(t) = F_{a,b}(t).$$

Lemma 2.3 [3] *Let $(X,F,\min)$ be a Menger space. If the sequence $\{A_n\}$ δ-converges to a and the sequence $\{B_n\}$ δ-converges to b, then for $t>0$ we have*

$$\liminf_{n\to\infty} \delta F_{A_n,B_n}(t) = F_{a,b}(t).$$

3. Main result

Now, we are in a position to state the main results of the paper.

Theorem 3.1 *Let $(X,F_1,\min)$, $(Y,F_2,\min)$ and $(Z,F_3,\min)$ be three complete Menger spaces. If F and P are continuous mappings of X into $B(Y)$, G and Q are continuous mappings of Y into $B(Z)$ and H and R are mappings of Z into $B(X)$ satisfying the inequalities*

$$\begin{aligned}\delta_1 F_{1HGFx,RQPx'}(ct) \geq \min\{&F_{1x,x'}(t), \delta_1 F_{1x,HGFx}(t), \delta_1 F_{1x',RQPx'}(t),\\ &\delta_3 F_{3GFx,QPx'}(t), \delta_2 F_{2Fx,Px'}(t)\} \qquad (1)\end{aligned}$$

$$\begin{aligned}\delta_2 F_{2FRQy,PHGy'}(ct) \geq \min\{&F_{2y,y'}(t), \delta_2 F_{2y,FRQy}(t), \delta_2 F_{2y',PHGy'}(t),\\ &\delta_1 F_{1RQy,HGy'}(t), \delta_3 F_{3Qy,Gy'}(t)\} \qquad (2)\end{aligned}$$

$$\begin{aligned}\delta_3 F_{3GFRz,QPHz'}(ct) \geq \min\{&F_{3z,z'}(t), \delta_3 F_{3z,GFRz}(t), \delta_3 F_{3z',QPHz'}(t),\\ &\delta_2 F_{2FRz,PHz'}(t), \delta_1 F_{1Rz,Hz'}(t)\} \qquad (3)\end{aligned}$$

for all x,x'in X,y,y' in Y and z,z' in Z and $c\in(0,1)$, Then HGF and RQP has a unique fixed point u in X, FRQ and PHG has a unique fixed point v in Y and GFR and QPH has a unique fixed point w in Z. Further, $Fu = Pu = \{v\}, Gv = Qv = \{w\}$ and $Hw = Rw = \{u\}$.

Proof. Let x_1 be an arbitrary point in X. Define sequences $\{x_n\}$ in X, $\{y_n\}$ in Y, $\{z_n\}$ in Z by

$$y_{2n+1} \in Fx_{2n+1}, \quad y_{2n+2} \in Px_{2n+2},$$

$$z_{2n+1} \in Gy_{2n+1}, \quad z_{2n+2} \in Qy_{2n+2},$$

$$x_{2n+2} \in Hz_{2n+1}, \quad x_{2n+3} \in Rz_{2n+2},$$

for $n = 0, 1, 2....$ Using inequality (1), we get that

$$\begin{aligned} F_{1_{x_{2n+2},x_{2n+3}}}(ct) &\geq \delta_1 F_{1_{RQPx_{2n+2},HGFx_{2n+1}}}(ct) \\ &\geq \min\{F_{1_{x_{2n+2},x_{2n+1}}}(t), \delta_1 F_{1_{x_{2n+2},RQPx_{2n+2}}}(t), \delta_1 F_{1_{x_{2n+1},HGFx_{2n+1}}}(t), \\ &\quad \delta_3 F_{3_{QPx_{2n+2},GFx_{2n+1}}}(t), \delta_2 F_{2_{Px_{2n+2},Fx_{2n+1}}}(t)\} \\ &\geq \min\{\delta_1 F_{1_{HGFx_{2n+1},RQPx_{2n}}}(t), \delta_1 F_{1_{HGFx_{2n+1},RQPx_{2n+2}}}(t), \\ &\quad \delta_1 F_{1_{RQPx_{2n},HGFx_{2n+1}}}(t), \\ &\quad \delta_3 F_{3_{QPHz_{2n+1},GFRz_{2n}}}(t), \delta_2 F_{2_{PHGy_{2n+1},FRQy_{2n}}}(t)\} \\ &\geq \min\{\delta_1 F_{1_{HGFx_{2n+1},RQPx_{2n}}}(t), \delta_3 F_{3_{QPHz_{2n+1},GFRz_{2n}}}(t), \\ &\quad \delta_2 F_{2_{PHGy_{2n+1},FRQy_{2n}}}(t)\}. \end{aligned} \tag{4}$$

In view of (2), we have

$$\begin{aligned} F_{2_{y_{2n+2},y_{2n+3}}}(ct) &\geq \delta_2 F_{2_{FRQy_{2n+2},PHGy_{2n+1}}}(ct) \\ &\geq \min\{F_{2_{y_{2n+2},y_{2n+1}}}(t), \delta_2 F_{2_{y_{2n+2},FRQy_{2n+2}}}(t), \\ &\quad \delta_2 F_{2_{y_{2n+1},PHGy_{2n+1}}}(t), \delta_1 F_{1_{RQy_{2n+2},HGy_{2n+1}}}(t), \\ &\quad \delta_3 F_{3_{Qy_{2n+2},Gy_{2n+1}}}(t)\} \\ &\geq \min\{\delta_2 F_{2_{PHGy_{2n+1},FRQy_{2n}}}(t), \delta_2 F_{2_{PHGy_{2n+1},FRQy_{2n+2}}}(t), \\ &\quad \delta_2 F_{2_{FRQy_{2n},PHGy_{2n+1}}}(t), \delta_1 F_{1_{RQPx_{2n+2},HGFx_{2n+1}}}(t), \\ &\quad \delta_3 F_{3_{QPHz_{2n+1},GFRz_{2n}}}(t)\}. \end{aligned}$$

It follows from (4) that

$$\begin{aligned} F_{2_{y_{2n+2},y_{2n+3}}}(ct) \geq \min\{&\delta_2 F_{2_{PHGy_{2n+1},FRQy_{2n}}}(t), \delta_1 F_{1_{HGFx_{2n+1},RQPx_{2n}}}(t), \\ &\delta_3 F_{3_{QPHz_{2n+1},GFRz_{2n}}}(t)\}. \end{aligned} \tag{5}$$

Using inequality (3), we have

$$\begin{aligned} &F_{3_{z_{2n+2},z_{2n+3}}}(ct) \geq \delta_3 F_{3_{GFRz_{2n+2},QPHz_{2n+1}}}(ct) \geq \min\{F_{3_{z_{2n+1},z_{2n+2}}}(t), \delta_3 F_{3_{z_{2n+2},GFRz_{2n+2}}}(t), \\ &\delta_3 F_{3_{z_{2n+1},QPHz_{2n+1}}}(t), \delta_2 F_{2_{FRz_{2n+2},PHz_{2n+1}}}(t), \delta_1 F_{1_{Rz_{2n+2},Hz_{2n+1}}}(t)\} \\ &\geq \min\{\delta_3 F_{3_{QPHz_{2n+1},GFRz_{2n}}}(t), \delta_3 F_{3_{QPHz_{2n+1},GFRz_{2n+2}}}(t), \\ &\delta_3 F_{3_{GFRz_{2n},QPHz_{2n+1}}}(t), \delta_2 F_{2_{FRQy_{2n+2},PHGy_{2n+1}}}(t), \delta_1 F_{1_{RQPx_{2n+2},HGFx_{2n+1}}}(t)\}. \end{aligned}$$

In view of (4) and (5), we find that

$$F_{3_{z_{2n+2},z_{2n+3}}}(ct) \geq \min\{\delta_3 F_{3_{QPHz_{2n+1},GFRz_{2n}}}(t), \delta_2 F_{2_{FRQy_{2n},PHGy_{2n+1}}}(t), \delta_1 F_{1_{HGFx_{2n+1},RQPx_{2n}}}(t)\} \tag{6}$$

Combining (4), (5) and (6), we have

$$F_{1_{x_{2n+2},x_{2n+3}}}(t) \geq \delta_1 F_{1_{RQPx_{2n+2},HGFx_{2n+1}}}(ct) \geq \min\{\delta_1 F_{1_{HGFx_1,RQPx_2}}(\tfrac{t}{c^{2n+1}}), \delta_2 F_{2_{PHGy_1,FRQy_2}}(\tfrac{t}{c^{2n+1}}), \delta_3 F_{3_{QPHz_1,GFRz_2}}(\tfrac{t}{c^{2n+1}})\} \tag{7}$$

$$F_{2_{y_{2n+2},y_{2n+3}}}(t) \geq \delta_2 F_{2_{FRQy_{2n+2},PHGy_{2n+1}}}(ct) \geq \min\{\delta_1 F_{1_{HGFx_1,RQPx_2}}(\tfrac{t}{c^{2n+1}}), \delta_2 F_{2_{PHGy_1,FRQy_2}}(\tfrac{t}{c^{2n+1}}), \delta_3 F_{3_{QPHz_1,GFRz_2}}(\tfrac{t}{c^{2n+1}})\} \tag{8}$$

$$F_{3_{z_{2n+2},z_{2n+3}}}(t) \geq \delta_3 F_{3_{GFRz_{2n+2},QPHz_{2n+1}}}(ct) \geq \min\{\delta_1 F_{1_{HGFx_1,RQPx_2}}(\tfrac{t}{c^{2n+1}}), \delta_2 F_{2_{FRQy_2,PHGy_1}}(\tfrac{t}{c^{2n+1}}), \delta_3 F_{3_{QPHz_1,GFRz_2}}(\tfrac{t}{c^{2n+1}})\} \tag{9}$$

Now for $r = 2, 4, 6..$ and $m \geq n$, we from Lemma 2.1 find that

$$F_{1_{x_{2n+r},x_{2m+r+1}}}(\varepsilon) \geq \delta_1 F_{1_{RQPx_{2m+r},HGFx_{2n+r-1}}}(\varepsilon) \geq min\{\delta_1 F_{1_{HGFx_{2n+r-1},RQPx_{2n+r}}}(\varepsilon - c\varepsilon), \delta_1 F_{1_{RQPx_{2n+r},RQPx_{2m+r}}}(c\varepsilon)\}$$

It follows from (7) that

$$\geq \min\{\delta_1 F_{1_{HGFx_1,RQPx_2}}(\tfrac{\varepsilon-c\varepsilon}{c^{2n+r-2}}), \delta_2 F_{2_{PHGy_1,FRQy_2}}(\tfrac{\varepsilon-c\varepsilon}{c^{2n+r-2}}), \delta_3 F_{3_{QPHz_1,GFRz_2}}(\tfrac{\varepsilon-c\varepsilon}{c^{2n+r-2}})\}, \min\{F_{1_{RQPx_{2n+r},HGFx_{2n+r+1}}}(c\varepsilon - c^2\varepsilon), F_{1_{HGFx_{2n+r+1},RQPx_{2m+r}}}(c^2\varepsilon)\}\}$$

$$\geq \min\{\delta_1 F_{1_{HGFx_1,RQPx_2}}(\tfrac{\varepsilon-c\varepsilon}{c^{2n+r-2}}), \delta_2 F_{2_{PHGy_1,FRQy_2}}(\tfrac{\varepsilon-c\varepsilon}{c^{2n+r-2}}), \delta_3 F_{3_{QPHz_1,GFRz_2}}(\tfrac{\varepsilon-c\varepsilon}{c^{2n+r-2}})\}, \min\{\delta_1 F_{1_{HGFx_1,RQPx_2}}(\tfrac{c\varepsilon-c^2\varepsilon}{c^{2n+r-1}}), \delta_2 F_{2_{PHGy_1,FRQy_2}}(\tfrac{c\varepsilon-c^2\varepsilon}{c^{2n+r-1}}), \delta_3 F_{3_{QPHz_1,GFRz_2}}(\tfrac{c\varepsilon-c^2\varepsilon}{c^{2n+r-1}})\}, F_{1_{HGFx_{2n+r+1},RQPx_{2m+r}}}(c^2\varepsilon)\}\}$$

$$\geq \min\{\delta_1 F_{1_{HGFx_1,RQPx_2}}(\tfrac{\varepsilon-c\varepsilon}{c^{2n+r-2}}), \delta_2 F_{2_{PHGy_1,FRQy_2}}(\tfrac{\varepsilon-c\varepsilon}{c^{2n+r-2}}), \delta_3 F_{3_{QPHz_1,GFRz_2}}(\tfrac{\varepsilon-c\varepsilon}{c^{2n+r-2}}), F_{1_{HGFx_{2n+r+1},RQPx_{2m+r}}}(c^2\varepsilon)\}\}$$

Continuing in this process, we have

$$\geq \min\{\delta_1 F_{1_{HGFx_1,RQPx_2}}(\tfrac{\varepsilon-c\varepsilon}{c^{2n+r-2}}), \delta_2 F_{2_{PHGy_1,FRQy_2}}(\tfrac{\varepsilon-c\varepsilon}{c^{2n+r-2}}),$$

$\delta_3 F_{3QPHz_1,GFRz_2}(\frac{\varepsilon-c\varepsilon}{c^{2n+r-2}}), F_{1HGFx_{2m+r-1},RQPx_{2m+r}}(c^{2m-2n}\varepsilon)\}\}$

$\geq \min\{\delta_1 F_{1HGFx_1,RQPx_2}(\frac{\varepsilon-c\varepsilon}{c^{2n+r-2}}), \delta_2 F_{2PHGy_1,FRQy_2}(\frac{\varepsilon-c\varepsilon}{c^{2n+r-2}}),$
$\delta_3 F_{3QPHz_1,GFRz_2}(\frac{\varepsilon-c\varepsilon}{c^{2n+r-2}}), \delta_1 F_{1HGFx_1,RQPx_2}(\frac{c^{2m-2n}\varepsilon}{c^{2m+r-2}}),$
$\delta_2 F_{2PHGy_1,FRQy_2}(\frac{c^{2m-2n}\varepsilon}{c^{2m+r-2}}), \delta_3 F_{3QPHz_1,GFRz_2}(\frac{c^{2m-2n}\varepsilon}{c^{2m+r-2}})\}$

$\geq \min\{\delta_1 F_{1HGFx_1,RQPx_2}(\frac{\varepsilon-c\varepsilon}{c^{2n+r-2}}), \delta_2 F_{2PHGy_1,FRQy_2}(\frac{\varepsilon-c\varepsilon}{c^{2n+r-2}}),$
$\delta_3 F_{3QPHz_1,GFRz_2}(\frac{\varepsilon-c\varepsilon}{c^{2n+r-2}})\}$

Now for n greater than some N we can have some $\lambda > 0$ such that

$$F_{1x_{2n+r},x_{2m+r+1}}(\varepsilon) \geq \delta_1 F_{1RQPx_{2m+r},HGFx_{2n+r-1}}(\varepsilon) \geq 1-\lambda, n \geq N. \tag{10}$$

This show $\{x_n\}$ is a Cauchy sequence in complete Menger space X.Let it converges to some point u in X.Similarly,we can show sequences $\{y_n\}$ and $\{z_n\}$ are Cauchy sequences with limits v and w in complete Menger spaces Y and Z respectively. It follows from (10) that

$$\delta_1 F_{1x_{2n+3},x_{2n+2}}(\varepsilon) \geq \delta_1 F_{1RQPx_{2n+2},HGFx_{2n+1}}(\varepsilon) \geq 1-\lambda, n \geq N.$$

This gives that

$$\begin{aligned}\lim_{n\to\infty} x_{2n+2} = \lim_{n\to\infty} x_{2n+3} = \lim_{n\to\infty} HGFx_{2n+1} = \lim_{n\to\infty} RQPx_{2n+2} = \{u\} \\ = \lim_{n\to\infty} HGy_{2n+1} = \lim_{n\to\infty} RQy_{2n+2}.\end{aligned} \tag{11}$$

Similarly we have

$$\begin{aligned}\lim_{n\to\infty} y_{2n+2} = \lim_{n\to\infty} y_{2n+3} = \lim_{n\to\infty} FRQy_{2n+2} = \lim_{n\to\infty} PHGy_{2n+1} = \{v\} \\ = \lim_{n\to\infty} FRz_{2n+2} = \lim_{n\to\infty} PHz_{2n+1} \\ = \lim_{n\to\infty} Fx_{2n+3} = \lim_{n\to\infty} Px_{2n+2}\end{aligned} \tag{12}$$

and

$$\begin{aligned}\lim_{n\to\infty} z_{2n+2} = \lim_{n\to\infty} z_{2n+3} = \lim_{n\to\infty} GFRz_{2n+2} = \lim_{n\to\infty} QPHz_{2n+1} = \{w\} \\ = \lim_{n\to\infty} GFx_{2n+3} = \lim_{n\to\infty} QPx_{2n+2} = \lim_{n\to\infty} Gy_{2n+3} = \lim_{n\to\infty} Qy_{2n+2}.\end{aligned} \tag{13}$$

Notice that F, P, G and Q are continuous. From (12) and (23), we have

$$\lim_{n\to\infty} y_{2n+3} = Fu = Pu = \{v\}, \tag{14}$$

$$\lim_{n\to\infty} z_{2n+3} = Gv = Qv = \{w\}. \tag{15}$$

Combining (14) with (15), we see that

$$GFu = GPu = QFu = QPu = Gv = Qv = \{w\}. \tag{16}$$

In view of Lemma 2.3, we find from (1) that

$$\begin{aligned}
\delta_1 F_{1u,HGFu}(ct) &= \liminf_{n\to\infty} \delta_1 F_{1x_{2n+3},HGFu}(ct) \\
&\geq \liminf_{n\to\infty} \delta_1 F_{1RQPx_{2n+2},HGFu}(ct) \\
&\geq \liminf_{n\to\infty} \min\{F_{1x_{2n+2},u}(t), \delta_1 F_{1x_{2n+2},RQPx_{2n+2}}(t), \delta_1 F_{1u,HGFu}(t), \\
&\quad \delta_3 F_{3QPx_{2n+2},GFu}(t), \delta_2 F_{2Px_{2n+2},Fu}(t)\}.
\end{aligned}$$

Using (11), (12), (13), (14), (16), Lemma 2.2 and Lemma 2.3, we have

$$\delta_1 F_{1u,HGFu}(ct) \geq \delta_1 F_{1u,HGFu}(t).$$

It gives that

$$HGFu = \{u\}. \tag{17}$$

Again using Lemma 2.3 and from (1), we have

$$\begin{aligned}
\delta_1 F_{1u,RQPu}(ct) &= \liminf_{n\to\infty} \delta_1 F_{1x_{2n+2},RQPu}(ct) \\
&\geq \liminf_{n\to\infty} \delta_1 F_{1HGFx_{2n+1},RQPu}(ct) \\
&\geq \liminf_{n\to\infty} min\{F_{1x_{2n+1},u}(t), \delta_1 F_{1x_{2n+1},HGFx_{2n+1}}(t), \\
&\quad \delta_1 F_{1u,RQPu}(t), \delta_3 F_{3GFx_{2n+1},QPu}(t), \delta_2 F_{2Fx_{2n+1},Pu}(t)\}.
\end{aligned}$$

Using (11), (12), (13), (14), (16), Lemma 2.2 and Lemma 2.3, we have

$\delta_1 F_{1u,RQPu}(ct) \geq \delta_1 F_{1u,RQPu}(t)$

It gives $RQPu = \{u\}$ (18)

By (17), (14) we have $PHGv = PHGFu = Pu = \{v\}$. (19)

By (18), (14) we have $FRQv = FRQPu = Fu = \{v\}$. (20)

By (15), (20) we have $GFRw = GFRQv = Gv = \{w\}$.

By (15), (19) we have $QPHw = QPHGv = Qv = \{w\}$.

By (16), (17), (18) we have $Hw = \{u\}$ and $Rw = \{u\}$.

Uniqueness of u:

Let u'be another fixed point different from u such that

$$HGFu' = \{u'\}, RQPu' = \{u'\}. \tag{21}$$

From inequality (3) and using (21),we have

$$\begin{aligned}
\delta_3 F_{3GFu',QPu'}(ct) &= \delta_3 F_{3GFRQPu',QPHGFu'}(ct) \\
&\geq \min\{F_{3QPu',GFu'}(t), \delta_3 F_{3GFu',QPu'}(t), \\
&\quad \delta_3 F_{3GFu',QPu'}(t), \delta_2 F_{2Fu',Pu'}(t), \delta_1 F_{1u',u'}(t)\} \\
&\geq \delta_2 F_{2Fu',Pu'}(t).
\end{aligned} \tag{22}$$

From inequality (2) and using (21), we have

$$\begin{aligned}
\delta_2 F_{2Fu',Pu'}(ct) = \delta_2 F_{2FRQPu',PHGFu'}(ct) &\geq \min\{F_{2Pu',Fu'}(t), \delta_2 F_{2Pu',Fu'}(t), \\
&\quad \delta_2 F_{2Fu',Pu'}(t), \delta_1 F_{1u',u'}(t), \delta_3 F_{3QPu',GFu'}(t)\} \\
&\geq \delta_3 F_{3QPu',GFu'}(t).
\end{aligned} \tag{23}$$

From (22) and (23), we have

$$\delta_3 F_{3GFu',QPu'}(t) \geq \delta_2 F_{2Fu',Pu'}(\tfrac{t}{c}) \geq \delta_3 F_{3GFu',QPu'}(\tfrac{t}{c^2}) \geq .... \geq \delta_3 F_{3GFu',QPu'}(\tfrac{t}{c^{2k}}).$$

Taking $k \to \infty$ where $k = 1,2,3..$ we have

$$\delta_3 F_{3GFu',QPu'}(t) \geq 1$$

$GFu' = QPu'$ and GFu'and QPu' are singleton. (24)

Again from (22) and (23), we have

$$\delta_2 F_{2Fu',Pu'}(t) \geq \delta_3 F_{3QPu',GFu'}(\tfrac{t}{c}) \geq \delta_2 F_{2Fu',Pu'}(\tfrac{t}{c^2}) \geq .... \geq \delta_2 F_{2Fu',Pu'}(\tfrac{t}{c^{2k}})$$

Taking $k \to \infty$, we find

$\delta_2 F_{2Fu',Pu'}(t) \geq 1$

$Fu' = Pu'$ and $Fu'$and Pu' are singleton (25)

Using (17) and (21) and from (1), we have

$$\delta_1 F_{1u,u'}(ct) = \delta_1 F_{1HGFu,RQPu'}(ct) \geq mini\{F_{1u,u'}(t), \delta_1 F_{1u,HGFu}(t), \delta_1 F_{1u',RQPu'}(t), \delta_3 F_{3GFu,QPu'}(t), \delta_2 F_{2Fu,Pu'}(t)\}$$

Using (17) and (21), we have

$$\geq mini\{\delta_3 F_{3GFu,QPu'}(t), \delta_2 F_{2Fu,Pu'}(t)\} \quad (26)$$

From (3) and Using (18) and (21),we have

$$\delta_3 F_{3GFu,QPu'}(ct) = \delta_3 F_{3GFRQPu,QPHGFu'}(ct) \geq mini\{F_{3QPu,GFu'}(t), \delta_3 F_{3QPu,GFRQPu}(t), \delta_2 F_{2FRQPu,PHGFu'}(t), \delta_1 F_{1RQPu,HGFu'}(t), \delta_3 F_{3GFu',QPHGFu'}(t)\}.$$

Using (16), (18), (21) and (24),we have

$$\geq mini\{\delta_2 F_{2Fu,Pu'}(t), \delta_1 F_{1u,u'}(t)\}. \quad (27)$$

From (26) and (27),we have

$$\delta_1 F_{1u,u'}(ct) \geq \delta_2 F_{2Fu,Pu'}(t). \quad (28)$$

From (2) and using (18) and (21), we have

$$\delta_2 F_{2Fu,Pu'}(ct) \geq mini\{F_{2Pu,Fu'}(t), \delta_2 F_{2Pu,FRQPu}(t), \delta_2 F_{2Fu',PHGFu'}(t), \delta_1 F_{1RQPu,HGFu'}(t), \delta_3 F_{3QPu,GFu'}(t)\}.$$

Using (14), (18), (25) and (21), we have

$$\geq mini\{\delta_1 F_{1u,u'}(t), \delta_3 F_{3QPu,GFu'}(t)\}.$$

Using (16) and (24), we have

$$\delta_2 F_{2Fu,Pu'}(ct) \geq mini\{\delta_1 F_{1u,u'}(t), \delta_3 F_{3GFu,QPu'}(t)\} \quad (29)$$

Using (29) in (27), we have

$$\delta_3 F_{3GFu,QPu'}(ct) \geq \delta_1 F_{1u,u'}(t). \quad (30)$$

From (29) and (30), we have

$$\delta_2 F_{2Fu,Pu'}(ct) \geq \delta_1 F_{1u,u'}(t). \quad (31)$$

From (28), (31), we have

$$\delta_1 F_{1u,u'}(ct) \geq \delta_1 F_{1u,u'}(t).$$

This gives $u = u'$. Hence u is unique. Similarly uniqueness of v and w can be proved.

Remark 3.2. If we put $F = P, G = Q, H = R$ in Theorem 3.1, then we get result of Beg and Chauhan [1] immediately.

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] I. beg and S. Chauhan, Related fixed point of set-valued mappings in three Menger spaces, Int. J. Anal. 2013 (2013), Article ID 736451.

[2] V.K. Chourasia and B. Fisher, Related fixed points for two pairs of set-valued mappings on two metric spaces, Hacettepe J. Math. Stat. 32 (2013), 27-32.

[3] C.M. Chen and T.H. Chang, Common fixed point theorems in Menger spaces, Int. J. Math. Math. Sci. 2006 (2006), Article ID 75931.

[4] B. Fisher, Related fixed points on two metric spaces, Math. Sem. Notes, Kobe Uni. 10 (1982),17-26.

[5] B. Fisher, K.P.R. Rao, A related fixed point theorem on three metric spaces,Hacettepe J. Math. stat. 36 (2007), 143-146.

[6] B. Fisher and D.Türkoḡlu, Related fixed points for set-valued mappings on two metric spaces, Int.J. Math. Math. Sci. 23 (2000), 205-210.

[7] S. Jain and B. Fisher, A related fixed point theorem for three metric spaces, Hacettepe J. Math. Stat. 31 (2002), 19-24.

[8] R.K. Jain,H.K. Sahu and B. Fisher, Related fixed point theorems for three metric spaces, Novi Sad J. Math. 26 (1996) 11-17.

[9] K. Kikina and L. Kikina, Generalized fixed point theorem in three metric spaces, Int. J. Math. Anal. 4 (2010), 1995-2004.

[10] B. Schweizer and A. Sklar, Statistical metric spaces, Pacific J. Math. 10 (1960), 313-334.