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FIXED POINT THEOREMS AND STABILITY OF FIXED POINT SETS OF MULTIVALUED MAPPINGS

ROMMEL O. GREGORIO[1,2,*] AND PRISCILLA S. MACANSANTOS[1]

[1]University of the Philippines Baguio, Department of Mathematics and Computer Science,
Baguio City 2600, Philippines
[2]Philippine Science High School, Cordillera Administrative Region Campus,
Baguio City 2600, Philippines

Abstract. In this paper, we prove new fixed point theorems of multivalued mappings in partially ordered metric spaces using newly reformulated pre-order relations. As consequence, we derive fixed point theorems for single valued mappings given by Nieto and Rodriguez-Lopez [11], [12]. We also establish some results on the stability of fixed point sets of multivalued mappings in partially ordered metric spaces. General illustrative examples are also given. Essential to our results are the pre-order relations $<_1, <_2, <_3$ defined in [3], and newly reformulated pre-order relations namely $<_4, <_5, <_6$, which are obtained by imposing a distance condition to comparable elements of two non-empty, closed and bounded sets.

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1. Introduction

*Corresponding author
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One of the main areas in the study of fixed points is metric fixed point theory, where the major and classical result was given and proved by Banach [2], known as the Banach contraction principle. It essentially states that a self-contraction mapping on a complete metric space has a unique fixed point. In 1969, Nadler [10] extended the Banach contraction principle to multivalued mappings in complete metric spaces and later on, Ran and Reurings [13] weakened the contraction principle by considering single valued mappings over partially ordered complete metric spaces, and applied their results to matrix equations. This trend in the study of existence of fixed points in partially ordered sets was immediately followed by Nieto and Rodriguez-Lopez [11], and very recently by Beg and Butt [4]. The study of fixed point theorems of multivalued mappings has developed not only in theories, but in applications to control theory, convex optimization, integral and differential inclusions, computer science, and economics and game theory.

In this paper, we extend the results of Nieto and Rodriguez-Lopez to multivalued mappings by using newly reformulated pre-order relations with a weakened multivalued contraction condition, and give examples to illustrate the application of our results. We likewise establish stability theorems of fixed point sets of multivalued mappings in partially ordered complete metric spaces, which are similar to that of Lim [8].

2. Preliminaries

We recall some definitions and important results found in the literature.

Definition 2.1. A partially ordered set is a system $(X,\preceq)$ where X is a non-empty set and $\preceq$ is a binary relation on X satisfying, for all $x,y,z\in X$,

a. $x\preceq x$ (reflexivity)

b. if $x\preceq y$ and $y\preceq x$, then $x=y$ (antisymmetry)

c. if $x\preceq y$ and $y\preceq z$, then $x\preceq z$ (transitivity)

Definition 2.2. A non-empty set X together with a metric $d:X\times X\to\mathbb{R}^+\cup\{0\}$ is called a metric space if the following conditions are satisfied by any $x,y,z\in X$:

a. $d(x,y)\geq 0$ and $d(x,y)=0$ if and only if $x=y$

b. $d(x,y) = d(y,x)$

c. $d(x,z) \leq d(x,y) + d(y,z)$

Definition 2.3. Let (X,d) be a metric space. A sequence $\{x_n\} \in X$ is a Cauchy sequence if it has the property that given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ whenever $n, m \geq N$. The metric (X,d) is complete if every Cauchy sequence in X is convergent.

Definition 2.4. $(X,d,\preceq)$ is a partially ordered complete metric space if (X,d) is a complete metric space and $(X,\preceq)$ is a partially ordered set.

Definition 2.5. The distance between any point a of X and any nonempty subset B of X is defined as:

$$d(a,B) := \inf_{b \in B} d(a,b).$$

For $A, B \in CB(X)$ (set of non-empty, closed, and bounded subsets of X), let

$$D(A,B) := \max\{\sup_{b \in B} d(b,A), \sup_{a \in A} d(a,B)\}.$$

D is called the Hausdorff metric induced by d.

We use the following relations between non-empty subsets of partially ordered complete metric spaces. The first three appeared in [3], [5], [14], while the last three are newly reformulated relations.

Definition 2.6. Let $(X,d,\preceq)$ be a partially ordered complete metric space. Let $A, B \in CB(X)$, define the following pre-order relations:

(1) $A <_1 B \Leftrightarrow \forall a \in A, \exists b \in B$ such that $a \preceq b$

(2) $A <_2 B \Leftrightarrow \forall b \in B, \exists a \in A$ such that $a \preceq b$

(3) $A <_3 B \Leftrightarrow A <_1 B$ and $A <_2 B$

(4) $A <_4 B \Leftrightarrow \forall a \in A, \exists b \in B$ such that $a \preceq b$ and $d(a,b) \leq D(A,B)$

(5) $A <_5 B \Leftrightarrow \forall b \in B, \exists a \in A$ such that $a \preceq b$ and $d(a,b) \leq D(A,B)$

(6) $A <_6 B \Leftrightarrow A <_4 B$ and $A <_5 B$

Example 2.7. Consider $X = \mathbb{R}$, and let $A_1 = [0, \frac{1}{2}]$ and $B_1 = [-1, 1]$, '$\preceq$' is the usual order on X, then $A_1 <_1 B_1$ but $A_1 \not<_2 B_1$. On the other hand, if $A_2 = [0,1]$ and $B_2 = [0, \frac{1}{3}]$, then $A_2 <_2 B_2$ but $A_2 \not<_1 B_2$. Thus, $<_1$ and $<_2$ are different relations between A and B, and so are $<_4$ and $<_5$.

Remark 2.8. The relations $<_1$, $<_2$, and $<_3$ are reflexive and transitive, but are not antisymmetric. To see this, consider $X = \mathbb{R}$, $A = [0,2]$, and $B = [0,\frac{1}{2}] \cup [1,2]$, '$\preceq$' is the usual order on X, then $A <_3 B$ and $B <_3 A$, but $A \neq B$.

Definition 2.9. A point $x \in X$ is said to be a fixed point of the multivalued mapping F (single valued mapping f) if $x \in Fx$ ($x = fx$).

Lemma 2.10. [10] *Let $\{A_n\}$ be a sequence in $CB(X)$ and $\lim_{n\to\infty} D(A_n,A) = 0$ for $A \in CB(X)$. If $x_n \in A_n$ and $\lim_{n\to\infty} d(x_n,x) = 0$, then $x \in A$.*

Theorem 2.11. [11] *Let $(X,d,\preceq)$ be a partially ordered complete metric space. Let $f : X \to X$ be a monotone nondecreasing mapping such that there exists $\alpha \in [0,1)$ with $d(f(x), f(y)) \leq \alpha d(x,y)$ for all $x \preceq y$. Assume that either f is continuous or X has a property that, if there is a nondecreasing sequence $\{x_n\} \to x$ in X, then $x_n \preceq x$. Moreover, if there exists $x_0 \in X$ with $x_0 \preceq f(x_0)$, then f has a fixed point.*

Theorem 2.12. [4] *Let $(X,d,\preceq)$ be a partially ordered complete metric space. Let $F : X \to CB(X)$ be a multivalued mapping satisfying:*

i. *There exists $x_0 \in X$, and some $x_1 \in Fx_0$ with $x_0 \preceq x_1$ such that $d(x_0,x_1) < 1$.*
ii. *If $d(x,y) < \varepsilon < 1$ for some $y \in Fx$ then $x \preceq y$.*
iii. *If a nondecreasing sequence $x_n \to x$ in X, then $x_n \preceq x$, for all n.*
iv. *There exists $\alpha \in (0,1)$ with $D(Fx,Fy) \leq \alpha d(x,y)$ for all $x \preceq y$.*

Then F has a fixed point.

3. Main results

We are now ready to discuss our results on fixed point theorems of multivalued mappings in partially ordered metric spaces. In particular, we first present fixed point theorems of multivalued mappings and single valued mappings, and then apply our results with examples. We further provide stability theorems of fixed point sets of multivalued mappings which satisfy the weakened multivalued contraction condition in partially ordered complete metric spaces.

Theorem 3.1. *Let $(X,d,\preceq)$ be a partially ordered complete metric space. Let $F : X \to CB(X)$ be a multivalued mapping such that the following conditions are satisfied:*

i. *There exists $x_0 \in X$ such that $\{x_0\} <_1 Fx_0$*
ii. *For all $x,y \in X$, if $x \preceq y$ then $Fx <_4 Fy$*
iii. *If $x_n \to x$ is a nondecreasing sequence in X, then $x_n \preceq x$ for all n*
iv. *There exists $\alpha \in (0,1)$ such that $D(Fx,Fy) \leq \alpha d(x,y)$ for all $x \preceq y$*

Then F has a fixed point.

Proof. Let $x_0 \in X$ such that $\{x_0\} <_1 Fx_0$, then there exists $x_1 \in Fx_0$ such that $x_0 \preceq x_1$. If $x_0 = x_1$ then x_0 is a fixed point of F, and we are done.

Suppose that $x_0 \neq x_1$, by condition ii, $Fx_0 <_4 Fx_1$, then there exists $x_2 \in Fx_1$ such that $x_1 \preceq x_2$ and

$$d(x_1,x_2) \leq D(Fx_0,Fx_1).$$

Using condition iv, we obtain

$$\begin{aligned} d(x_1,x_2) &\leq D(Fx_0,Fx_1) \\ &\leq \alpha d(x_0,x_1). \end{aligned}$$

Again, from condition ii and iv, $Fx_1 <_4 Fx_2$, and there exists $x_3 \in Fx_2$ such that $x_2 \preceq x_3$ and

$$\begin{aligned} d(x_2,x_3) &\leq D(Fx_1,Fx_2) \\ &\leq \alpha d(x_1,x_2) \\ &\leq \alpha^2 d(x_0,x_1). \end{aligned}$$

Continuing this process, we obtain a nondecreasing sequence $\{x_n\}$ such that $x_{n+1} \in Fx_n$ and

$$\begin{aligned} d(x_n,x_{n+1}) &\leq D(Fx_{n-1},Fx_n) \\ &\leq \alpha^n d(x_0,x_1). \end{aligned}$$

Next, show that $\{x_n\}$ is a Cauchy sequence. By the way the sequence $\{x_n\}$ was generated, note that we have:

$$\begin{aligned} d(x_n,x_{n+1}) &\leq \alpha^n d(x_0,x_1) \\ d(x_{n+1},x_{n+2}) &\leq \alpha^{n+1} d(x_0,x_1) \\ d(x_{n+2},x_{n+3}) &\leq \alpha^{n+2} d(x_0,x_1). \end{aligned}$$

Now let $N \in \mathbb{N}$, and $m,n \geq N$ such that $m > n$, then,

$$\begin{aligned} d(x_n,x_m) &\leq d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + \ldots + d(x_{m-1},x_m) \\ &\leq \alpha^n d(x_0,x_1) + \alpha^{n+1} d(x_0,x_1) + \ldots + \alpha^{m-1} d(x_0,x_1) \\ &= \alpha^n d(x_0,x_1)(1 + \alpha + \ldots + \alpha^{m-1-n}) \\ &\leq \alpha^n d(x_0,x_1) \frac{1}{1-\alpha}. \end{aligned}$$

Therefore as $n \to \infty$, $d(x_n,x_m) \to 0$, and this implies that $\{x_n\}$ is a Cauchy sequence. Since X is a complete metric space, there exists $x \in X$ such that $x_n \to x$, where $\{x_n\}$ is a nondecreasing sequence. From condition iii, we have $x_n \preceq x$ for all n.

From condition iv, it follows that $D(Fx_n,Fx) \leq \alpha d(x_n,x)$, and since $x_n \to x$, this implies that $\lim_{n\to\infty} D(Fx_n,Fx) = 0$. Clearly, because $x_{n+1} \in Fx_n$ it follows from Lemma 2.10 that $x \in Fx$, that is, x is a fixed point of F. □

Remark 3.2. The definition of relation $<_4$ is very important in the proof of Theorem 3.1 because it ensures a contraction of the distance between elements of Fx and Fy whenever $x \preceq y$ for $x,y \in X$.

A dual result of Theorem 3.1 can be obtained by using relations $<_2$ and $<_5$.

Theorem 3.3. *Let $(X,d,\preceq)$ be a partially ordered complete metric space. Let $F : X \to CB(X)$ be a multivalued mapping such that the following conditions are satisfied:*

i. *There exists $x_0 \in X$ such that $Fx_0 <_2 \{x_0\}$*
ii. *For all $x,y \in X$, if $x \preceq y$ then $Fx <_5 Fy$*
iii. *If $x_n \to x$ is a nonincreasing sequence in X, then $x \preceq x_n$ for all n*

iv. *There exists* $\alpha \in (0,1)$ *such that* $D(Fx,Fy) \leq \alpha d(x,y)$ *for all* $x \preceq y$

Then F *has a fixed point.*

Proof. The proof follows that of Theorem 3.1 and by considering nonincreasing sequence. □

Remark 3.4. Completeness of the metric space X in Theorems 3.1 and 3.3 plays a crucial role, because it is possible that a multivalued mapping F does not have a fixed point if the underlying metric space is not complete even if conditions i to iv of Theorems 3.1 and 3.3 are satisfied. Indeed, contractions on incomplete metric spaces may fail to have fixed points. To see this, consider the following example.

Example 3.5. Let $X = (0,1] \subseteq \mathbb{R}$ with the usual order. Define $d(x,y) = |x-y|$ and $Fx = \left\{y : y \in \left[\frac{x}{5}, \frac{x}{4}\right]\right\}$ for all $x \in (0,1]$.

Note that if $x_0 = 1 \in X$, then $Fx_0 = [\frac{1}{5}, \frac{1}{4}]$, thus condition i of Theorem 3.3 is satisfied. Take any $x,y \in X$ such that $x \preceq y$, then $Fx = [\frac{x}{5}, \frac{x}{4}]$ and $Fy = [\frac{y}{5}, \frac{y}{4}]$. Note that for all $y' \in Fy$ there exists $x' \in Fx$ such that $x' \preceq y'$ and $d(x',y') \leq D(Fx,Fy)$, that is $Fx <_5 Fy$. Also, if $x \preceq y$, we have that $D(Fx,Fy) = \frac{y-x}{4}$ and $d(x,y) = y-x$. Clearly, F satisfies the weakened contraction condition defined in Theorem 3.3. If for instance, we did not notice that X is not a complete metric space, then by Theorem 3.3, it can be concluded that F has a fixed point. But by closer investigation on how F was defined, it can be seen that F has no fixed point. Therefore, to be able to use Theorems 3.1 and 3.3, one should check that X is indeed a complete metric space to deduce a correct and valid conclusion.

As consequence of Theorem 3.1 and Theorem 3.3, we have the following fixed point theorem for single valued mappings.

Corollary 3.6. *Let* $(X,d,\preceq)$ *be a partially ordered complete metric space. Let* $f : X \to X$ *be a single valued mapping such that the following conditions are satisfied:*

i. *There exists* $x_0 \in X$ *such that* $x_0 \preceq fx_0$ *or* $fx_0 \preceq x_0$
ii. *For all* $x,y \in X$, *if* $x \preceq y$ *then* $fx \preceq fy$
iii. *If* $x_n \to x$ *is a sequence in* X *whose consecutive elements are comparable, then* $x_n \preceq x$ *or* $x \preceq x_n$ *for all* n

iv. *There exists* $\alpha \in (0,1)$ *such that* $d(fx, fy) \leq \alpha d(x,y)$ *for all* $x \preceq y$

Then f *has a fixed point.*

Remark 3.7. In Corollary 3.6, if we replace condition (iii) by requiring f to be continuous, then the existence of fixed point of f can be proven.

In [4], Beg and Butt established fixed point theorems of multivalued mappings in partially ordered complete metric spaces. In the following example, we illustrate the applicability of Theorem 3.1 where Theorem 2.12 fails.

Example 3.8. Let $X = \{(0,0),(0,-\frac{1}{p}),(-\frac{1}{q},0),(-\frac{1}{r},\frac{1}{r}),\text{such that } p,q,r \in \mathbb{Z}^+, 1 < p < q < r \text{ and } r-q < q\} \subseteq \mathbb{R}^2$, where $\preceq$ is defined as: for $x,y \in X$ such that $x = (x_1,y_1)$, and $y = (x_2,y_2)$, $x \preceq y$ if and only if $x_1 \leq x_2$ and $y_1 \leq y_2$. Define $d : X \times X \to \mathbb{R}$ as:

$$d(x,y) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$$

where $x = (x_1,y_1), y = (x_2,y_2)$. Let $F : X \to CB(X)$ as:

$$Fx = \begin{cases} \{(0,0),(-\frac{1}{r},\frac{1}{r})\} & \text{if } x = (0,0) \\ \{(-\frac{1}{q},0)\} & \text{if } x = (0,-\frac{1}{p}) \\ \{(0,0)\} & \text{if } x = (-\frac{1}{q},0) \\ \{(0,0)\} & \text{if } x = (-\frac{1}{r},\frac{1}{r}) \end{cases}$$

Let $x = (0,-\frac{1}{p})$, then $Fx = \{(-\frac{1}{q},0)\}$. Note that for $y \in Fx$, $d(x,y) = d((0,-\frac{1}{p}),(-\frac{1}{q},0)) = \frac{1}{p} < 1$, but x and y are not comparable. Thus, we cannot use Theorem 2.12 to show that F has a fixed point.

However, it can be shown that the assumptions of Theorem 3.1 are satisfied, hence it can be invoked to conclude the existence of a fixed point of F.

Example 3.9. Let $X = [0,1]$, where $\preceq$ is the usual order in $\mathbb{R}$ and d is defined as $d(x,y) = |x-y|$ for all $x,y \in X$. Define $F : X \to CB(X)$ as:

$$Fx = \left\{ y \in X : \frac{x}{n+1} \leq y \leq \frac{x}{n} \right\}$$

for a fixed value of n, where $n \in \mathbb{Z}$ and $n \geq 2$.

X with the defined metric d is a complete metric space, and F satisfies the assumptions of Theorem 3.3, therefore we can conclude that F has a fixed point.

Lim [8] presented an interesting stability result that holds for multivalued contractions. If a sequence of multivalued mappings $\{F_n\}$, each of which satisfies Nadler's fixed point theorem, uniformly converges to a multivalued mapping F_0, then the fixed point sets of $\{F_n\}$ converges to the fixed point set of F_0. Here, we prove theorems for the stability of fixed point sets of uniformly convergent sequence of multivalued mappings in partially ordered complete metric spaces. We start with a very important lemma.

Lemma 3.10. *Let* $(X,d,\preceq)$ *is a partially ordered complete metric space. Let* $F_i : X \to CB(X), (i = 1,2)$ *satisfy the conditions given in Theorem 3.1. Denote by* $S(F_1)$ *and* $S(F_2)$ *the respective fixed point sets of* F_1 *and* F_2*, and* $S = S(F_1) \cup S(F_2)$*. If*

a. *for all* $x_0 \in S(F_1)$*,* $\{x_0\} <_4 F_2(x_0)$
b. *for all* $y_0 \in S(F_2)$*,* $\{y_0\} <_4 F_1(y_0)$

then

$$D(S(F_1),S(F_2)) \leq \sup_{x\in S} D(F_1(x),F_2(x)) \frac{1}{1-\alpha}$$

Proof. Let $x_0 \in S(F_1)$ such that $\{x_0\} <_4 F_2(x_0)$, then there exists $x_1 \in F_2(x_0)$ such that $x_0 \preceq x_1$ and $d(x_0,x_1) \leq D(x_0,F_2(x_0))$. Similar to the proof of Theorem 3.1 we can deduce a nondecreasing sequence $\{x_n\}$ such that $x_n \in F_2(x_{n-1})$ and $d(x_n,x_{n+1}) \leq \alpha^n d(x_0,x_1)$.

Let $N \in \mathbb{N}$, and $m,n \geq N$ such that $m > n$, then,

$$\begin{aligned} d(x_n,x_m) &\leq d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + \ldots + d(x_{m-1},x_m) \\ &\leq \alpha^n d(x_0,x_1) + \alpha^{n+1} d(x_0,x_1) + \ldots + \alpha^{m-1} d(x_0,x_1) \\ &= \alpha^n d(x_0,x_1)(1 + \alpha + \ldots + \alpha^{m-1-n}) \\ &\leq \alpha^n d(x_0,x_1) \frac{1}{1-\alpha} \end{aligned}$$

and this goes to zero as $n \to \infty$. Thus, $\{x_n\}$ is a Cauchy sequence. Since X is a complete metric space, there exists $z \in X$ such that $x_n \to z$, where $\{x_n\}$ is a nondecreasing sequence. Therefore, from condition iii of Theorem 3.1, $x_n \preceq z$ for all n.

From condition iv of Theorem 3.1, it follows that $D(F_2(x_n),F_2(z)) \leq \alpha d(x_n,z)$, and since $x_n \to z$, then $D(F_2(x_n),F_2(z)) = 0$ as $n \to \infty$. Clearly, because $x_{n+1} \in F_2(x_n)$ it follows from Lemma 2.10 that $z \in F_2(z)$, that is, $z \in S(F_2)$. Furthermore,

$$\begin{aligned}
d(x_0,z) &\leq d(x_0,x_1)+d(x_1,x_2)+\ldots+d(x_n,x_{n+1})+\ldots \\
&\leq d(x_0,x_1)+\alpha d(x_0,x_1)+\ldots+\alpha^n d(x_0,x_1)+\ldots \\
&= d(x_0,x_1)(1+\alpha+\ldots+\alpha^n+\ldots) \\
&\leq d(x_0,x_1)\frac{1}{1-\alpha} \\
&\leq D(x_0,F_2(x_0))\frac{1}{1-\alpha} \\
&\leq D(F_1(x_0),F_2(x_0))\frac{1}{1-\alpha}.
\end{aligned}$$

Reversing the roles of F_1 and F_2 and repeating the arguments above lead to the conclusion that for each $y_0 \in S(F_2)$ there exists $y_1 \in F_1(y_0)$ and $w \in S(F_1)$ such that

$$d(y_0,w) \leq D(F_1(y_0),F_2(y_0))\frac{1}{1-\alpha}$$

Hence,

$$D(S(F_1),S(F_2)) \leq \sup_{x\in S} D(F_1(x),F_2(x))\frac{1}{1-\alpha}.$$

□

Theorem 3.11. *Let $(X,d,\preceq)$ is a partially ordered complete metric space. Let $F_i : X \to CB(X),(i=0,1,2,\ldots)$ be a sequence of multivalued mappings, each satisfying all the conditions in Theorem 3.1. Denote by $S(F_0)$, $S(F_1)$, $S(F_2),\ldots$ the respective fixed point sets of F_0,F_1 and $F_2\ldots$, and $S=\bigcup S(F_i)$. If $\lim_{n\to\infty} D(F_n(x),F_0(x))=0$ uniformly for all $x \in S$, and*

a. *for all $x_0 \in S(F_0)$, $\{x_0\} <_4 F_i(x_0)$ for $i=1,2,\ldots$*
b. *for all $y_0 \in S(F_i)$, $\{y_0\} <_4 F_0(y_0)$ for $i=1,2,\ldots$*

then

$$\lim_{n\to\infty} D(S(F_n),S(F_0)) = 0$$

Proof. Let $\varepsilon > 0$. Since $\lim_{n\to\infty} D(F_n(x), F_0(x)) = 0$ for all $x \in S$, then we can choose $N \in \mathbb{N}$ such that for $n \geq N$, $\sup_{x\in S} D(F_n(x), F_0(x)) < (1-\alpha)\varepsilon$. By invoking Lemma 3.10, then $D(S(F_n), S(F_0)) < \varepsilon$ for all n. Therefore, $\lim_{n\to\infty} D(S(F_n), S(F_0)) = 0$. □

Remark 3.12. Dual results of Lemma 3.10 and Theorem 3.11 can be obtained by using $<_5$ relation.

Conflict of Interests

The authors declare that there is no conflict of interests.

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