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# PROPERTY $P$ AND SOME FIXED POINT RESULTS ON A NEW $\varphi$-WEAKLY CONTRACTIVE MAPPING

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**Abstract.** In this paper, we prove some fixed point results for new weakly contractive maps in $G-$ metric spaces. It is proved that these maps satisfy property $P$. The results obtained in this paper generalize several well known comparable results in the literature.

**Keywords**: fixed point; coincidence point; $G-$metric spaces; contraction.

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## 1. Introduction

The study of fixed points of nonlinear mappings satisfying certain contractive conditions has been at the center of rigorous research activity, see [13, 14, 15, 16, 17, 19, 20, 22]. The notion of $D$-metric space is a generalization of usual metric spaces and it is introduced by Dhage [1, 2]. Recently, Mustafa and Sims [25, 26, 27] have shown that most of the results concerning

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Dhage's $D$-metric spaces are invalid. In [25, 28, 29, 30], they introduced a improved version of the generalized metric space structure which they called $G$-metric spaces. For more results on $G$-metric spaces and fixed point results, one can refer to the papers [3, 4, 5, 6, 7, 8, 9, 10, 11, 18, 21, 23, 31, 32, 33], some of them have given some applications to matrix equations, ordinary differential equations, and integral equations.

## 2. Preliminaries

**Definition 2.1.** [24] Let $X$ be a non-empty set, $G : X \times X \times X \to \mathbb{R}_+$ be a function satisfying the following properties:

(1) $G(x,y,z) = 0$ if $x = y = z$,
(2) $0 < G(x,x,y)$ for all $x$, $y \in X$ with $x \neq y$,
(3) $G(x,x,y) \leq G(x,y,z)$ for all $x,y,z \in X$ with $y \neq z$,
(4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = ::::$ (symmetry in all three variables),
(5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ for all $x,y,z,a \in X$ (rectangle inequality).

Then the function $G$ is called a generalized metric, or, more specially, a $G$-metric on $X$, and the pair $(X,G)$ is called a $G-$metric space.

**Definition 2.2.** [24] Let $(X,G)$ be a $G$-metric space, and let $(x_n)$ be a sequence of points of $X$. We say that $(x_n)$ is $G-$convergent to $x \in X$ if $\lim_{n,m\to\infty} G(x;x_n,x_m) = 0$, that is, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x;x_n,x_m) < \varepsilon$, for all $n;m \geq N$. We call $x$ the limit of the sequence $x_n$ and write $x_n \to x$ or $\lim_{n\to\infty} x_n = x$.

**Proposition 2.3.** [24] *Let $(X,G)$ be a G-metric space. The following are equivalent:*

(1) *$(x_n)$ is G-convergent to x;*
(2) $G(x_n,x_n,x) \to 0$ *as* $n \to \infty$*;*
(3) $G(x_n,x,x) \to 0$ *as* $n \to \infty$;
(4) $G(x_n,x_m,x) \to 0$ *as* $n,m \to \infty$.

**Definition 2.4.** [24] Let $(X,G)$ be a $G$-metric space. A squence $(x_n)$ is called a $G-$ Cauchy sequence if, for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $G(x_n,x_m,x_l) < \varepsilon$ for all $m,n,l \geq N$, that is $G(x_n,x_m,x_l) \to 0$ as $n,m,l \to \infty$.

**Proposition 2.5.** [24] *Let $(X,G)$ be a G-metric space. Then the following are equivalent:*

(1) *The sequence $(x_n)$ is G-Cauchy;*

(2) *For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n,x_m,x_m) < \varepsilon$, for all $n;m \geq N$.*

**Proposition 2.6.** [26] *Let $(X,G)$ be a G-metric space. Then for any $x,y,z,a \in X$, it follows that:*

(1) $G(x,y,z) \leq \frac{2}{3}\left[G(x,y,a)+G(x,a,z)+G(a,y,z)\right]$,

(2) $G(x,y,z) \leq G(x,a,a)+G(y,a,a)+G(z,a,a)$.

**Proposition 2.7.** [24] *Let $(X,G)$ be a G-metric space. A mapping $f : X \to X$ is $G-$continuous at $x \in X$ if and only if it is $G-$ sequentially continuous at x, that is, whenever $(x_n)$ is G-convergent to x, $f(x_n)$ is G-convergent to $f(x)$.*

**Proposition 2.8.** [24] *Let $(X,G)$ be a G-metric space. Then the function $G(x,y,z)$ is jointly continuous all three of its variables.*

**Definition 2.9.** [24] A $G$-metric space $(X,G)$ is called $G-$ complete if every $G-$Cauchy sequence is $G-$convergent in $(X,G)$.

**Definition 2.10.** [24]. Two mappings $f,g : X \to X$ are weakly compatible if they commute at their coincidence points, that is $ft = gt$ for some $t \in X$ implies that $fgt = gft$.

**Definition 2.11.** [24] Let $X$ be a non-empty set and $S,T$ self-mappings of $X$. A point $x \in X$ is called a coincidence point of $S$ and $T$ if $Sx = Tx$. A point $w \in X$ is said to be a point of coincidence of $S$ and $T$ if there exists $x \in X$ so that $w = Sx = Tx$.

**Definition 2.12.** [24]. Suppose $(X,\preceq)$ is a partially ordered set and $f,g : X \to X$ are mappings. $f$ is said to be $g-$Non decreasing if for $x,y \in X$, $gx \preceq gy$ implies $fx \preceq fy$.

Khan *et al.* [34] introduced the concept of altering distance function that is a control function employed to alter the metric distance between two points enabling one to deal wity relatively new classes of fixed point problems.

Let us denote by $\Psi$ the class of the set of altering distance functions $\psi : [0,+\infty[\to [0,+\infty[$ which satisfies the following conditions:

(1) $\psi$ is nondecreasing,

(2) $\psi$ is continuous,

(3) $\psi(t) = 0 \iff t = 0$

and by $\Phi$ the class of the set of continuous functions $\varphi : [0,+\infty[\to [0,+\infty[$ and nondecreasing.

**Definition 2.13.** Let $(X,G)$ be a $G$- a complete metric space and $T$ self-mapping of $X$. We say that $T$ satisfies the property $P$ if $F(T) = F(T^n)$ for each $n \in \mathbb{N}$, where $F(T)$ denotes the set of fixed point of $T$.

**Remark 2.14.** *In general $F(T) \neq F(T^n)$ for $n \geq 2$.*

**Example 2.15.** *We consider $X = [0,1]$ and $Tx = 1-x$. $T$ has a unique fixed point $x = \frac{1}{2}$. Every point of $X$ is a fixed point of $T^n$, $n \geq 2$.*

**Example 2.16.** *$X = [0,\pi]$ and $Tx = \cos x$, $T$ has a unique fixed point and every iterate of $T$ has the same fixed point as $T$.*

Jeong and Rhoades [32] showed that maps satisfying many contractive conditions have property $P$. An interesting fact about map satisfying property $P$ is that they have no non trivial periodic points; see [32, 34] and the references therein. In this paper, we will prove some general point theorem for a new weakly contractive maps in $G$-complete metric spaces.

## 3. Main Results

We start with the following remark.

**Remark 3.1.** *If $\psi \in \Psi$ and if $\varphi \in \Phi$ with the condition $\psi(t) > \varphi(t)$ for all $t > 0$, then $\varphi(0) = 0$.*

**Proof.** Since $\varphi(t) < \psi(t)$ for all $t > 0$, then we have

$$0 \leq \varphi(0) \leq \liminf_{t\to 0} \varphi(t) \leq \lim_{t\to 0} \psi(t) = \psi(0) = 0.$$

This completes the proof.

**Lemma 3.2.** *Let $(X,G)$ be a $G$- metric space and $(x_n)$ be a sequence in $X$ such that $G(x_{n+1}, x_{n+1}, x_n)$ is decreasing and*

$$\lim_{n\to\infty} G(x_{n+1},x_{n+1},x_n)=0. \tag{1}$$

*If $(x_{2n})$ is not a Cauchy sequence, then there exists $\varepsilon>0$ and two sequences $(m_k)$ and $(n_k)$ of positive integers such that the following four sequences tends to $\varepsilon$ as $k\to\infty$ :*

$$\begin{aligned} &G(x_{2m_k},x_{2m_k},x_{2n_k}), G(x_{2m_k},x_{2m_k},x_{2n_{k+1}}) \\ &G(x_{2m_{k-1}},x_{2m_{k-1}},x_{2n_k}), G(x_{2m_{k-1}},x_{2m_{k-1}},x_{2n_{k+1}}). \end{aligned} \tag{2}$$

**Proof.** If $(x_{2n})$ is not a Cauchy sequence, then there exists $\varepsilon>0$ and two sequences $(m_k)$ and $(n_k)$ of positive integers such that

$$n_k>m_k>k; G(x_{2m_k},x_{2m_k},x_{2n_k-2})<\varepsilon,\ \ G(x_{2m_k},x_{2m_k},x_{2n_k})\geq\varepsilon$$

for all integer $k$. Then

$$\begin{aligned} \varepsilon &\leq G(x_{2m_k},x_{2m_k},x_{2n_k})\leq G(x_{2m_k},x_{2m_k},x_{2n_k-2}) \\ &\quad +G(x_{2n_{k-2}},x_{2n_{k-2}},x_{2n_k-1})+G(x_{2n_{k-1}},x_{2m_{k-1}},x_{2n_k}) \\ &< \varepsilon+G(x_{2n_{k-2}},x_{2n_{k-2}},x_{2n_k-1})+G(x_{2n_{k-1}},x_{2n_{k-1}},x_{2n_k}). \end{aligned}$$

Using $(1)$, we conclude that

$$\lim_{k\to\infty} G(x_{2m_k},x_{2m_k},x_{2n_k})=\varepsilon. \tag{3}$$

Further, we have

$$G(x_{2m_k},x_{2m_k},x_{2n_k})\leq G(x_{2m_k},x_{2m_k},x_{2n_{k+1}})+G(x_{2n_{k+1}},x_{2n_{k+1}},x_{2n_k})$$

and

$$G(x_{2m_k},x_{2m_k},x_{2n_{k+1}})\leq G(x_{2m_k},x_{2m_k},x_{2n_k})+G(x_{2n_k},x_{2n_k},x_{2n_{k+1}}).$$

Passing to the limit when $k\to\infty$ and using $(1)$ and $(3)$, we obtain

$$\lim_{k\to\infty} G(x_{2m_k},x_{2m_k},x_{2n_{k+1}})=\varepsilon.$$

The remaining two sequences in $(2)$ tend to $\varepsilon$ can be proved in a similar way.

**Theorem 3.3** *Let $(X,G)$ be a complete G-metric space and let $f : X \to X$ be a mapping. If there exist $\psi \in \Psi$ and $\varphi \in \Phi$ with the condition $\psi(t) > \varphi(t)$ for all $t > 0$, such that*

$$\psi(G(fx,fy,fz)) \leq \varphi\left(\max\left\{\begin{array}{c} G(x,y,y), G(x,fx,fx), G(y,fy,fy), \\ G(z,fz,fz) \\ \alpha G(fx,fx,y)+(1-\alpha)G(fy,fy,z) \\ \beta G(x,fx,fx)+(1-\beta)G(y,fy,fy) \end{array}\right\}\right) \tag{4}$$

*for all $x,y,z \in X$, where $0 < \alpha, \beta < 1$. Then $f$ has a unique fixed point (say $u$), where $f$ is G-continuous at $u$.*

**Proof.** Fix $x_0 \in X$. Then construct a sequence $(x_n)$ by $x_{n+1} = fx_n = f^n x_0$. We may assume that $x_n \neq x_{n+1}$ for each $n \geq 0$. Since, if there exist $n \in \mathbb{N}$ such that $x_n = x_{n+1}$, then $x_n$ is a fixed point of $f$. From (4), substituting $x = x_{n-1}, y = z = x_n$ then, for all $n \in \mathbb{N}$, we have

$$\begin{aligned} & \psi(G(x_n,x_{n+1},x_{n+1}) \quad (5) \\ \leq\ & \varphi\left(\max\left\{\begin{array}{c} G(x_{n-1},x_n,x_n), G(x_{n-1},x_n,x_n), G(x_n,x_{n+1},x_{n+1}), \\ G(x_n,x_{n+1},x_{n+1}) \\ \alpha G(x_n,x_n,x_n)+(1-\alpha)G(x_{n+1},x_{n+1},x_n) \\ \beta G(x_{n-1},x_n,x_n)+(1-\beta)G(x_{n+1},x_{n+1},x_n) \end{array}\right\}\right) \\ \leq\ & \varphi\left(\max\left\{\begin{array}{c} G(x_{n-1},x_n,x_n), G(x_n,x_{n+1},x_{n+1}) \\ \beta G(x_{n-1},x_n,x_n)+(1-\beta)G(x_{n+1},x_{n+1},x_n) \end{array}\right\}\right). \end{aligned}$$

Let $M_n = \max\{G(x_{n-1},x_n,x_n), G(x_n,x_{n+1},x_{n+1})\}$. then , (5) gives

$$\psi(G(x_n,x_{n+1},x_{n+1}) \leq \varphi(M_n). \tag{6}$$

We have two cases, either $M_n = G(x_n,x_{n+1},x_{n+1})$ or $M_n = G(x_{n-1},x_n,x_n)$. Suppose that, for some $n \in \mathbb{N}$, $M_n = G(x_n,x_{n+1},x_{n+1})$. Then we have

$$\psi(G(x_n,x_{n+1},x_{n+1}) \leq \varphi(G(x_n,x_{n+1},x_{n+1})). \tag{7}$$

Therefore from the condition of the theorem, we have $G(x_n,x_{n+1},x_{n+1}) = 0$. Hence $x_n = x_{n+1}$. which is a contraduction with the element of $x_n$ are distinct.

Thus, $M_n = G(x_{n-1}, x_n, x_n)$, and (6) becomes

$$\psi(G(x_n, x_{n+1}, x_{n+1}) \leq \varphi(G(x_{n-1}, x_n, x_n)). \tag{8}$$

By using the condition of the theorem, we get from (8)

$$G(x_n, x_{n+1}, x_{n+1}) \leq G(x_{n-1}, x_n, x_n), \text{ for all } n \in \mathbb{N}. \tag{9}$$

Therefore, $\{G(x_n, x_{n+1}, x_{n+1})$ for all $n \in \mathbb{N}\}$ is a positive non increasing sequence. Hence there exists $r \geq 0$ such that

$$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = r. \tag{10}$$

Letting $n \to \infty$ and using (8), (10) and the continuity of $\psi$ and $\varphi$, we get

$$\psi(r) \leq \varphi(r). \tag{11}$$

Hence, using the condition of the theorem, we have $r = 0$, which implies that

$$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0. \tag{12}$$

Now we prove that $\{x_n\}$ is a Cauchy sequence. Suppose that $(x_n)$ is not a Cauchy sequence. Using Lemma, we know that there exist $\varepsilon > 0$ and two sequences $(m_k)$ and $(n_k)$ of positive integers such that the following four sequences tend to $\varepsilon$ as $k$ goes to infinity:

$$G(fx_{2m_k}, fx_{2m_k}, fx_{2n_k}), G(fx_{2m_k}, fx_{2m_k}, fx_{2n_{k+1}})$$

$$G(fx_{2m_{k-1}}, fx_{2m_{k-1}}, fx_{2n_k}), G(fx_{2m_{k-1}}, fx_{2m_{k-1}}, fx_{2n_{k+1}}).$$

Putting in the contractive condition $x = y = x_{2m_k}$ and $z = x_{2n_{k+1}}$, using (4) and we proceed as before, it follows that

$$\psi(G(x_{2m_k}, x_{2m_k}, x_{2n_{k+1}}) \leq \varphi(G(x_{2m_{k-1}}, x_{2m_{k-1}}, x_{2n_k}) \tag{13}$$

and so, by the condition of the Theorem, we have

$$\lim_{k\to\infty} G(x_{2m_{k-1}}, x_{2m_{k-1}}, x_{2n_k}) = 0.$$

Since $\psi$ is a continuous mapping, using (13) letting $k \to \infty$, we have

$$\psi(\varepsilon) \leq \varphi(\varepsilon).$$

Then, the condition of the theorem implies that $\varepsilon = 0$, which contradicts $\varepsilon > 0$. Therefore, $(x_n)$ is a Cauchy sequence in $(X,G)$. Since $(X,G)$ is a complete metric space, there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$. For $n \in \mathbb{N}$, we have

$$\begin{aligned}
\psi(G(fu,fu,x_n)) &= \psi(G(fu,fu,fx_{n-1})) \qquad (14)\\
&\leq \varphi\left(\max\left\{\begin{array}{c} G(u,u,x_{n-1}), G(u,u,fu), G(u,u,fu), \\ G(x_{n-1},x_n,x_n) \\ \alpha G(fu,fu,u)+(1-\alpha)G(fu,fu,x_{n-1}) \\ \beta G(u,fu,fu)+(1-\beta)G(u,fu,fu) \end{array}\right\}\right)\\
&\leq \varphi\left(\max\left\{\begin{array}{c} G(u,u,x_{n-1}), G(u,u,fu) \\ ,G(x_{n-1},x_n,x_n) \\ \alpha G(fu,fu,u)+(1-\alpha)G(fu,fu,x_{n-1}) \end{array}\right\}\right).
\end{aligned}$$

Letting $n \to \infty$, and the using the fact that $\psi$, $\varphi$ are continuous and $G$ is continuous on its variables, we get that $G(fu,fu,u) = 0$. Hence $fu = u$. So $u$ is a fixed point of $f$. Now to show uniqueness, let $v$ be another fixed point of $f$ with $v \neq u$. Therefore,

$$\begin{aligned}
\psi(G(u,u,v)) &= \psi(G(fu,fu,fv)) \qquad (15)\\
&\leq \varphi\left(\max\left\{\begin{array}{c} G(u,u,v), G(u,fu,fu), G(u,fu,fu), \\ G(v,fv,fv) \\ \alpha G(fu,fu,u)+(1-\alpha)G(fu,fu,v) \\ \beta G(u,fu,fu)+(1-\beta)G(v,fv,fv) \end{array}\right\}\right)\\
&= \varphi(\max\{G(u,u,v),(1-\alpha)G(fu,fu,v)\})\\
&= \varphi(G(u,u,v)).
\end{aligned}$$

Hence, we have

$$\psi(G(u,u,v)) \leq \varphi(G(u,u,v)). \qquad (16)$$

Therefore, by using the condition of the theorem, we get $G(u,u,v)$ and $u = v$.

Now we are in a position to to show that $f$ is continuous at $u$. Let $\{x_n\}$ be a sequence in $X$ such that $x_n \to u$. Using $(4)$, we have

$$\begin{aligned} \psi(G(fx_n,u,u)) &= \psi(G(fx_n,fu,fu)) \\ &\leq \varphi\left(\max\left\{\begin{array}{c} G(x_n,u,fu),G(x_n,fx_n,fx_n),G(u,u,fu), \\ G(u,u,fu) \\ \alpha G(fx_n,fx_n,u)+(1-\alpha)G(fu,fu,u) \\ \beta G(x_n,fx_n,fx_n)+(1-\beta)G(u,fu,fu) \end{array}\right\}\right) \\ &= \varphi(\max\{G(x_n,u,u),\alpha G(fx_n,fx_n,u),\beta G(x_n,fx_n,fx_n)\}) \\ &\leq \varphi\left(\max\left\{\begin{array}{c} G(x_n,u,u),\alpha G(x_{n+1},x_{n+1},u), \\ \beta G(x_n,u,u)+\beta G(u,x_{n+1},x_{n+1}) \end{array}\right\}\right). \end{aligned} \tag{17}$$

Using the condition of the theorem and (17), we get

$$G(fx_n,u,u) \leq \max\left\{\begin{array}{c} G(x_n,u,u),\alpha G(x_{n+1},x_{n+1},u), \\ \beta G(x_n,u,u)+\beta G(u,x_{n+1},x_{n+1}) \end{array}\right\}. \tag{18}$$

Therefore, we obtain $\lim_{n\to\infty} G(fx_n,u,u) = 0$. Using the continuity of $G$, we obtain $\lim_{n\to\infty} fx_n = f(u)$. This completes the proof.

**Corollary 3.4.** *Let $(X,G)$ be a complete G-metric space and Let $f$ be a map satisfying*

$$G(fx,fy,fz) \leq \lambda\left(\max\left\{\begin{array}{c} G(x,y,y),G(x,fx,fx),G(y,fy,fy), \\ G(z,fz,fz) \\ \alpha G(fx,fx,y)+(1-\alpha)G(fy,fy,z) \\ \beta G(x,fx,fx)+(1-\beta)G(y,fy,fy) \end{array}\right\}\right) \tag{19}$$

*for all $x,y,z \in X$, where $0 < \alpha,\beta,\lambda < 1$, Then $f$ has a unique fixed point (say u), where $f$ is G-continuous at u.*

**Proof.** We obtain the result by taking $\psi(t) = t$ and $\varphi(t) = \lambda t$, in Theorem 3.3.

**Corollary 3.5.** *Let $(X,G)$ be a complete G-metric space, Let $f$ be a map satisfying*

$$G(fx,fy,fz) \leq \lambda\left(\max\left\{\begin{array}{c} G(x,y,y),G(x,fx,fx),G(y,fy,fy),G(z,fz,fz) \\ \alpha G(fx,fx,y)+(1-\alpha)G(fy,fy,z) \\ \beta G(x,fx,fx)+(1-\beta)G(y,fy,fy) \end{array}\right\}\right) \tag{20}$$

*for all $x,y,z \in X$, where $0<\alpha,\beta,\lambda<1$. Then $f$ has a unique fixed point (say u), where $f$ is G-continuous at u.*

**Proof.** We obtain the result by taking $\psi(t)=t$ and $\varphi(t)=\lambda t, \alpha=\beta=\frac{1}{2}$ in Theorem 3.3

**Corollary 3.6.** *Let $(X,G)$ be a complete G-metric space. Let $f$ be a map satisfying*

$$\psi(G(fx,fy,fz)) \leq \psi\left(\max\left\{\begin{array}{c} G(x,y,y),G(x,fx,fx),G(y,fy,fy), \\ G(z,fz,fz) \\ \alpha G(fx,fx,y)+(1-\alpha)G(fy,fy,z) \\ \beta G(x,fx,fx)+(1-\beta)G(y,fy,fy) \end{array}\right\}\right) - \phi\left(\max\left\{\begin{array}{c} G(x,y,y),G(x,fx,fx),G(y,fy,fy), \\ G(z,fz,fz) \\ \alpha G(fx,fx,y)+(1-\alpha)G(fy,fy,z) \\ \beta G(x,fx,fx)+(1-\beta)G(y,fy,fy) \end{array}\right\}\right) \tag{21}$$

*for all $x,y,z \in X$, where $0<\alpha,\beta<1$, $\psi\in\Psi$ and $\phi\in\Phi$ with $\varphi(t)=0 \Longleftrightarrow t=0$. Then $f$ has a unique fixed point (say u), where $f$ is G-continuous at u.*

**Proof.** We obtain the result by taking $\varphi(t)=\psi(t)-\phi(t)$, in Theorem 3.3.

**Theorem 3.7.** *Under the condition of theorem 3.3, $f$ has the property P.*

**Proof.** Note that $f$ has a fixed point. Therefore $F(f^n)\neq\phi$ for each $n>1$, assume that $u\in F(f^n)$. We claim that $u\in F(f)$. To prove the claim, suppose that $u\neq fu$. Using (4), we have

$$\begin{aligned} \psi(G(u,fu,fu)) &= \psi\left(G(f^nu,f^{n+1}u,f^{n+1}u)\right) \\ &= \psi\left(G(ff^{n-1}u,ff^nu,ff^nu)\right) \\ &\leq \varphi\left(\max\left\{\begin{array}{c} G(f^{n-1}u,u,u),G(u,fu,fu) \\ \alpha G(u,u,u)+(1-\alpha)G(fu,fu,u) \\ \beta G(f^{n-1},u,u)+(1-\beta)G(u,fu,fu) \end{array}\right\}\right) \\ &= \varphi\left(\max\left\{G(f^{n-1}u,u,u),G(u,fu,fu))\right\}\right). \end{aligned} \tag{22}$$

Letting $M=\max\left\{G(f^{n-1}u,u,u),G(u,fu,fu))\right\}$, we deduce from (22)

$$\psi(G(u,fu,fu)) \leq \varphi(\max\{M)\}). \tag{23}$$

If $M = G(u, fu, fu)$, then

$$\psi(G(u, fu, fu)) \leq \varphi(G(u, fu, fu)).$$

Then by using the condition of the theorem, we get $G(u, fu, fu) = 0$, therefore $u = fu$, which is a contradiction. On the other hand, if $M = G(f^{n-1}u, u, u)$, then (4) gives that

$$\begin{aligned} \psi(G(u, fu, fu)) &= \psi\left(G(f^n u, f^{n+1} u, f^{n+1} u)\right) \\ &\leq \varphi\left(G(f^{n-1} u, f^n u, f^n u)\right). \end{aligned} \tag{24}$$

By using the condition of the theorem, we have

$$G(f^n u, f^{n+1} u, f^{n+1} u) \leq G(f^{n-1} u, f^n u, f^n u).$$

Therefore, $\left\{G(f^n u, f^{n+1} u, f^{n+1} u) \text{ for all } n \in \mathbb{N}\right\}$ is a positive non increasing sequence. Hence there exists $\gamma \geq 0$ such that

$$\lim_{n\to\infty} G(f^n u, f^{n+1} u, f^{n+1} u) = \gamma. \tag{25}$$

Letting $n \to \infty$ in (24), using (25) and the continuity of $\psi$ and $\varphi$, we get

$$\psi(\gamma) \leq \varphi(\gamma). \tag{26}$$

Hence, using the condition of the theorem, we have $\gamma = 0$, which implies that

$$\lim_{n\to\infty} G(f^n u, f^{n+1} u, f^{n+1} u) = 0. \tag{27}$$

Thus $G(u, fu, fu) = 0$, and we have $u = fu$, which is a contradiction. Therefore, $u \in F(f)$ and $f$ has the property $P$.

Let

$$M_{\alpha,\beta}(x, y, z) = \max \left\{ \begin{array}{c} G(x, y, y), G(x, fx, fx), G(y, fy, fy), \\ G(z, fz, fz) \\ \alpha G(fx, fx, y) + (1-\alpha) G(fy, fy, z) \\ \beta G(x, fx, fx) + (1-\beta) G(y, fy, fy) \end{array} \right\}, \tag{28}$$

where $\alpha, \beta \in (0, 1]$.

**Example 3.8.** *Let* $X=[0,1]$ *and* $G(x,y,z)=\max\{|x-y|,|y-z|,|x-z|\}$ *be a* $G-$*metric space on* $X$. *Define* $f:X\to X$ *by* $f(x)=\frac{1}{8}t$. *We take* $\psi(t)=t$ *and* $\varphi(t)=\frac{1}{8}t$, *for* $t\in[0,\infty)$ *and* $\alpha,\beta\in(0,1]$. *So that*

$$\psi\left(M_{\alpha,\beta}(x,y,z)\right)=M_{\alpha,\beta}(x,y,z). \tag{29}$$

*We have*

$$\begin{aligned}\psi(G(fx,fy,fz)) &= \max\left\{\frac{|x-y|}{8},\frac{|y-z|}{8},\frac{|x-z|}{8}\right\}=\frac{1}{8}G(x,y,z) \\ &= \frac{1}{8}M_{\alpha,\beta}(x,y,z) \\ &\leq \varphi\left(M_{\alpha,\beta}(x,y,z)\right).\end{aligned} \tag{30}$$

*From theorem 3.3, we deduce that* $f$ *has a unique fixed point* $u=0$ *and* $f$*satisfies the property P.*

## 4. Applications

Denote by $\Lambda$ the set of functions $\chi:[0,\infty)\to[0,\infty)$ satisfying the following hypotheses.

(1) $\chi$ is a Lebesgue integrable on each compact of $[0,\infty)$,

(2) For every $\varepsilon>0$,we have $\int_0^t\chi(s)ds>0$.

It is an easy matter to see that the mapping $\psi:[0,\infty)\to[0,\infty)$, defined by $\psi(t)=\int_0^t\chi(s)ds$ is an altering distance function.

**Theorem 4.1.** *Let* $(X,G)$ *be a complete G-metric space and* $f:X\to X$ *be a mapping. If there exist* $\psi\in\Psi$ *and* $\varphi\in\Phi$ *with the condition* $\psi(t)>\varphi(t)$ *for all* $t>0$, *such that*

$$\int_0^{\psi(G(fx,fy,fz))}\chi(t)\,dt\leq\int_0^{\varphi\left(M_{\alpha,\beta}(x,y,z)\right)}\chi(t)\,dt$$

*for all* $x,y,z\in X$, *where* $0<\alpha,\beta<1$. *Then* $f$ *has a unique fixed point (say u), where* $f$ *is G-continuous at u.*

### Conflict of Interests

The authors declare that there is no conflict of interests.

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