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FIXED POINT THEOREM FOR SET VALUED MAPS IN G-METRIC SPACE

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Abstract. In this paper, some results on fixed points for a set valued maps in complete G-metric space are established.

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1. Introduction

Unless mentioned or defined otherwise, for all terminology and notation in this paper, the reader is referred to [5,7,9,10,14]. There are several reasons for the acceleration of interest in fixed point theory. One way to study a fixed point is through set valued maps. For such fixed point study, Nadler [10] introduced a important notion of set valued contraction and proved a set valued version of the Banach contraction principle. In a related vein, several authors studied many fixed point results for set valued contraction mappings; see [1,2,8,13] and the references therein. In [11] and [12], Popa initiated the study of fixed point for mappings satisfying implicit

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relations satisfying ϕ-map. Afterwards, Berinde [4] proved some constructive fixed point theorems for almost contractions satisfying an implicit relation, which generalize related results; see [2,5,6] and the references therein.

Throughout this paper, let (X,G) be G-metric space, $CB(X)$ denotes the collection of all nonempty closed bounded subsets of X.

Let $H(.,.,.)$ be the Hausdorff G-distance on $CB(X)$, i.e, for $A,B,C \in CB(X)$ and $x \in X$

$$D_G(A,B,C) := \inf\{G(a,b,c) : a \in A, b \in B, c \in C\}$$

$$\delta_G(A,B,C) := \sup\{G(a,b,c) : a \in A, b \in B, c \in C\}$$

and in [8] Kaewcharoen and Kaewkhao defined Hausdorff G-metric as,

$$H_G(A,B,C) := \max\{\sup_{x\in A} G(x,B,C), \sup_{x\in B} G(x,C,A), \sup_{x\in C} G(x,A,B)\},$$

where,

$$G(x,B,C) = d_G(x,B) + d_G(B,C) + d_G(x,C),$$

$$d_G(x,B) = \inf\{d_G(x,y) : y \in B\},$$

$$d_G(A,B) = \inf\{d_G(a,b) : a \in A, b \in B\}.$$

In this paper, we establish some results on fixed points for a set valued maps in complete G-metric space.

2. Preliminaries

Before going to the main theorem it is necessary to present a formidable number of definitions, basic concepts and terminology, which will be use in sequel.

In [9], Mustafa and Sims introduced the more appropriate notion of generalized metric space called G-metric spaces as follows.

Definition 2.1. [9] Let X be a nonempty set, and let $G : X \times X \times X \to R^+ \cup \{0\}$ be a function satisfying the following axioms:

- (G1) $G(x,y,z) = 0$ if $x = y = z$;
- (G2) $G(x,x,y) > 0$, for all $x,y \in X$ with $x \neq y$;

- (G3) $G(x,x,y) \leq G(x,y,z)$, for all $x,y,z \in X$ with $z \neq y$;
- (G4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \cdots$ (symmetry in all three variables);
- (G5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ for all $x,y,z,a \in X$, (rectangle inequality).

Then the function G is called a generalized metric, or more specifically a G-metric on X and the pair (X,G) is called a G-metric space.

Definition 2.2. [9] Let (X,G) be a G-metric space and $\{x_n\}$ be a sequence of points in X, a point $x \in X$ is said to be the limit of the sequence $\{x_n\}$ if, $\lim_{n\to\infty} G(x,x_n,x_m) = 0$, and the sequence $\{x_n\}$ is G-convergent to x.

Proposition 2.3. [9] Let (X,G) be a G-metric space. Then the following are equivalent:

- $\{x_n\}$ is G-convergent to x;
- $G(x_n,x_n,x) \to 0$ as $n \to \infty$;
- $G(x_n,x,x) \to 0$ as $n \to \infty$;
- $G(x_m,x_n,x) \to 0$ as $m,n \to \infty$.

Definition 2.4. [9] Let (X,G) be a G-metric space. A sequence $\{x_n\}$ is called G-Cauchy if, for each $\varepsilon > 0$ there exists a positive integer N such that $G(x_n,x_m,x_l) < \varepsilon$ for all $n,m,l \geq N$; i.e., $G(x_n,x_m,x_l) \to 0$ as $n,m,l \to \infty$.

Definition 2.5. [9] A G-metric space (X,G) is said to be G-complete if every G-Cauchy sequence in (X,G) is G-convergent in X.

Proposition 2.6. [9] *Let (X,G) be a G-metric space. For any $x,y,z,a \in X$, it follows that:*

- *If* $G(x,y,z) = 0,$ *then* $x = y = z$;
- $G(x,y,z) \leq G(x,x,y) + G(x,x,z)$;
- $G(x,y,y) \leq 2G(y,x,x)$,
- $G(x,y,z) \leq G(x,a,z) + G(a,y,z)$;
- $G(x,y,z) \leq \frac{2}{3}(G(x,y,a) + G(x,a,z) + G(a,y,z))$;
- $G(x,y,z) \leq G(x,a,a) + G(y,a,a) + G(z,a,a)$.

Definition 2.7. An element $x \in X$ is said to be fixed point of set valued mapping $T : X \to CB(X)$, if $x \in Tx$.

Example 2.8 Consider $X=[0,+\infty)$ and define $T:X\to CB(X)$ as

$$Tx=\left\{\begin{array}{c}\{x\},\ \text{if } x\in[0,1)\\ ([0,1]),\ \text{if } x\in[1,+\infty)\end{array}\right\}.$$

Clearly, T is set valued mapping.

Theorem 2.9. [10] *Let (X,d) be a complete metric space and $T:X\to CB(X)$ be a set valued map satisfying*

$$H(Tx,Ty)\le qd(x,y)\ \forall\, x,y\in X,$$

where $q\in[0,1]$ then T has a fixed point.

Proposition 2.10 *Let X be a nonempty set. Assume that $g:X\to X$ and $T:X\to 2^X$ are weakly compatible mappings. If g and T have a unique point of coincidence $w=gx\in Tx$, then w is the unique common fixed point of g and T.*

Proof. Assume that g and T have a unique point of coincidence $w=gx\in Tx$. Therefore $gw=g(gx)\in gT(x)\subseteq Tg(x)=Tw$. This implies that gw is a point of coincidence of g and T. Thus, $w=gw\in Tw$, since g and T have unique point of coincident, w is a common fixed point of g and T. Now we shall show that w is the unique common fixed point. To do so, suppose z be another fixed point distinct from w, which gives $z=gz\in Tz$, implies that z is point of coincidence of g and T. But w is unique point of coincidence, hence $z=w$, which gives that w is the unique common fixed point of g and T.□

In order to establish the main result we need to state the following Lemma 2.11, which is more general form of lemma 2.1 used to prove the theorem 2.1 in [15]. Its proof is a simple consequence of the definition of the Hausdorff G-distance

Lemma 2.11. *Let (X,G) be a complete G-metric space and $A,B\in CB(X)$, then for each $a\in A$ and $\varepsilon>0$, there exist $b\in B$ such that*

$$G(a,b,b)\le hH_G(A,B,B),\ h>1 \text{ and } b=b(a).$$

3. Main Result

Theorem 3.1. *Let (X,G) be a complete G-metric space and let $T : X \to CB(X)$ be a set valued map such that the contraction condition*
$H_G(Tx,Ty,Tz) \leq \alpha(G(x,y,z)) + \beta[G(x,Tx,Tx) + G(y,Ty,Ty) + G(z,Tz,Tz)]$
$+\gamma[G(x,Ty,Ty)+G(x,Tz,Tz)+G(y,Tx,Tx)+G(y,Tz,Tz)+G(z,Tx,Tx)+G(z,Ty,Ty)]$ *holds $\forall\ x,y\ \in X$, where $\alpha,\beta,\gamma > 0$ and $\alpha+3\beta+4\gamma < 1$. Then T has a fixed point.*

Proof. In view of lemma 2.11 and the assumption $0 < \alpha+3\beta+4\gamma < 1$, we see that there exists $r > 0$ such that

$$0 < \alpha+3\beta+4\gamma < \sqrt{r} < 1.$$

Let us choose $\lambda = \frac{\alpha+\beta+2\gamma}{\sqrt{r}-(2\beta+2\gamma)}$, clearly $0 < \lambda < 1$.

Let $x_0 \in X$ be arbitrary. Then there exist $x_1 \in X$ such that $x_1 \in Tx_0$. Now using Lemma 2.11, $h = \frac{1}{\sqrt{r}}$, it follows that

$$\left\{\begin{array}{c} \exists\ x_2 \in Tx_1;\ G(x_1,x_2,x_2) \leq \frac{1}{\sqrt{r}}H_G(Tx_0,Tx_1,Tx_1) \\ \exists\ x_3 \in Tx_2;\ G(x_2,x_3,x_3) \leq \frac{1}{\sqrt{r}}H_G(Tx_1,Tx_2,Tx_2) \\ \exists\ x_4 \in Tx_3;\ G(x_3,x_4,x_4) \leq \frac{1}{\sqrt{r}}H_G(Tx_2,Tx_3,Tx_3) \\ .\quad .\quad . \\ .\quad .\quad . \\ .\quad .\quad . \\ \exists\ x_{n+1} \in Tx_n;\quad G(x_n,x_{n+1},x_{n+1}) \leq \frac{1}{\sqrt{r}}H_G(Tx_{n-1},Tx_n,Tx_n) \end{array}\right\}.$$

Hence, we have

$$G(x_n,x_{n+1},x_{n+1}) \leq \frac{1}{\sqrt{r}}H_G(Tx_{n-1},Tx_n,Tx_n).$$

Using contraction condition, one can obtain

$$\leq \frac{1}{\sqrt{r}}\left\{\begin{array}{c} \alpha(G(x_{n-1},x_n,x_n)) + \beta[(G(x_{n-1},Tx_{n-1},Tx_{n-1})) + G(x_n,Tx_n,Tx_n) \\ +G(x_n,Tx_n,Tx_n)] + \gamma[(G(x_{n-1},Tx_n,Tx_n)) + G(x_{n-1},Tx_n,Tx_n) \\ +G(x_n,Tx_n,Tx_n) + G(x_n,Tx_{n-1},Tx_{n-1}) + (G(x_n,Tx_n,Tx_n)) \\ +G(x_n,Tx_{n-1},Tx_{n-1})] \end{array}\right\}.$$

Notie that $x_n \in Tx_{n-1}$ implies

$$G(x_{n-1}, Tx_{n-1}, Tx_{n-1}) \leq G(x_{n-1}, x_n, x_n)$$

and $x_{n+1} \in Tx_n$ implies

$$G(x_n, Tx_n, Tx_n) \leq G(x_n, x_{n+1}, x_{n+1}).$$

Thus we get

$$\leq \frac{1}{\sqrt{r}} \left\{ \begin{array}{c} \alpha(G(x_{n-1},x_n,x_n)) + \beta[(G(x_{n-1},x_n,x_n)) + G(x_n,x_{n+1},x_{n+1}) \\ +G(x_n,x_{n+1},x_{n+1})] + \gamma[(G(x_n,x_{n+1},x_{n+1})) + G(x_n,x_{n+1},x_{n+1}) \\ +G(x_n,x_n,x_n) + G(x_n,x_n,x_n) + (G(x_{n-1},x_n,x_n)) \\ +G(x_{n-1},x_n,x_n)] \end{array} \right\}.$$

$$\leq \frac{1}{\sqrt{r}} \left\{ \begin{array}{c} \alpha(G(x_{n-1},x_n,x_n)) + \beta[(G(x_{n-1},x_n,x_n)) + G(x_n,x_{n+1},x_{n+1}) \\ +G(x_n,x_{n+1},x_{n+1})] + 2\gamma[(G(x_n,x_{n+1},x_{n+1})) + G(x_n,x_n,x_n) \\ +(G(x_{n-1},x_n,x_n))] \end{array} \right\}.$$

$$\leq \frac{1}{\sqrt{r}} \left\{ (\alpha+\beta+2\gamma)G(x_{n-1},x_n,x_n) + (2\beta+2\gamma)G(x_n,x_{n+1},x_{n+1}) \right\},$$

implies

$$G(x_n,x_{n+1},x_{n+1}) \leq \frac{\alpha+\beta+2\gamma}{\sqrt{r}-(2\beta+2\gamma)} G(x_{n-1},x_n,x_n) \ \forall\, n \geq 1.$$

On substituting the value of λ we get,

$$G(x_n,x_{n+1},x_{n+1}) \leq \lambda G(x_{n-1},x_n,x_n),\ 0 < \lambda < 1.$$

Repeating the above process, we have

$$G(x_n,x_{n+1},x_{n+1}) \leq \lambda^n G(x_0,x_1,x_1),\ \forall\, n \geq 1. \tag{3.1}$$

Now we claim that $\{x_n\}$ is cauchy sequence. Towards this, we need to show that there is a $(+ive)$ integer $n_0 = n_0(\varepsilon)$, $\varepsilon > 0$ such that

$$G(x_n,x_{n+p},x_{n+p}) \leq \varepsilon \text{ for every } n \geq n_0 \text{ uniformly on } p \in N.$$

By the rectangular inequality

$$G(x_n,x_{n+p},x_{n+p}) \leq G(x_n,x_{n+1},x_{n+1}) + G(x_{n+1},x_{n+2},x_{n+2}) + \cdots + G(x_{n+p-1},x_{n+p},x_{n+p}).$$

Making use of (3.1), we find that the expression reduces to

$$G(x_n, x_{n+p}, x_{n+p}) \leq \lambda^n G(x_0, x_1, x_1) + \lambda^{n+1} G(x_0, x_1, x_1) + \cdots + \lambda^{n+p-1} G(x_0, x_1, x_1)$$
$$= \lambda^n [1 + \lambda + \lambda^2 + \cdots + \lambda^{p-1}] G(x_0, x_1, x_1).$$

With the help of geometric progression, the above inequality becomes,

$$G(x_n, x_{n+p}, x_{n+p}) \leq \frac{\lambda^n}{1-\lambda} G(x_0, x_1, x_1), \ \forall\, n \in N \text{ uniformly on p } \in N. \tag{3.2}$$

Since $0 < \lambda < 1$ and $n \to \infty$, there exist a (+ive) integer n_0 such that

$$\frac{\lambda^n}{1-\lambda} G(x_0, x_1, x_1) < \varepsilon, \ \forall\, n \geq n_0 \tag{3.3}$$

In view of equation (3.2) and (3.3), it is easy to see that the sequence $\{x_n\}$ is Cauchy. By the completeness of (X, G) there exists $z \in X$ such that $\lim_{n\to\infty} x_n = z$

Now we shall show that z is fixed point of T. Note that

$$G(z, Tz, Tz) \leq G(z, z_{n+1}, z_{n+1}) + G(z_{n+1}, Tz, Tz)$$
$$\leq G(z, z_{n+1}, z_{n+1}) + H_G(Tz_n, Tz, Tz).$$

Using contraction condition, the expression turns out to be

$$G(z, z_{n+1}, z_{n+1}) + \alpha(G(z_n, z, z)) + \beta[(G(z_n, Tz_n, Tz_n)) + G(z, Tz, Tz) + G(z, Tz, Tz)]$$
$$+ \gamma[(G(z_n, Tz, Tz)) + (G(z_n, Tz, Tz)) + (G(z, Tz, Tz)) + (G(z, Tz_n, Tz_n))$$
$$+ (G(z, Tz, Tz)) + (G(z, Tz_n, Tz_n))],$$

which implies

$$G(z, z_{n+1}, z_{n+1}) + \alpha(G(z_n, z, z)) + \beta[(G(z_n, z_{n+1}, z_{n+1})) + 2G(z, Tz, Tz)]$$
$$+ \gamma[(G(z_n, Tz, Tz)) + (G(z_n, Tz, Tz)) + 2(G(z, Tz, Tz)) + 2(G(z, Tz_n, Tz_n))]$$
$$\leq G(z, z_{n+1}, z_{n+1}) + \alpha(G(z_n, z, z)) + \beta(G(z_n, z_{n+1}, z_{n+1})) + [2\beta + 2\gamma] G(z, Tz, Tz)$$
$$+ 2\gamma(G(z_n, Tz, Tz)) + 2\gamma G(z, z_{n+1}, z_{n+1}).$$

This holds for all n, now proceeding the limit $n \to \infty$ in above expression, we get

$$G(z, Tz, Tz) \leq [2\beta + 2\gamma] G(z, Tz, Tz).$$

Since $[2\beta+2\gamma]<1$, we get

$$[\beta+\gamma]<\frac{1}{2},$$

which gives

$$G(z,Tz,Tz)=0.$$

It follows that $z\in Tz$. Hence z is a fixed point of T. □

Conflict of Interests

The authors declare that there is no conflict of interests.

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