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# COUPLED FIXED POINT RESULTS UNDER TVS-CONE METRIC AND W-CONE-DISTANCE

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**Abstract.** In this paper, we first give an answer to a question concerning quasicontractions under w-cone-distance, posed in the recent paper [Lj. Ćirić, H. Lakzian, V. Rakočević, Fixed point theorems for w-cone distance contraction mappings in tvs-cone metric spaces, Fixed Point Theory Appl. 2012:3 (2012), doi:10.1186/1687-1812-2012-3]. Then, we prove some coupled and b-coupled fixed point results in tvs-cone metric spaces, which improve several known corresponding results. Finally, we derive several (common) coupled fixed point results under $w$-cone distance in tvs-cone metric spaces, thus also improving some known theorems. Examples are given to show the usage of these results, and that our improvements are proper.

**Keywords**: Tvs-cone metric space; coupled fixed point; w-cone-distance.

**2010 AMS Subject Classification:** 47H10, 54H25

## 1. Introduction

Cone metric spaces were considered by Huang and Zhang in [1], who re-introduced the concept which has been known since the middle of 20th century (see, e.g., [2, 3]).

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Topological vector space-valued version of these spaces was treated in [4]–[10]; see also [11] for a survey of fixed point results in these spaces.

Fixed point theorems in metric spaces with the so-called $w$-distance were obtained for the first time by Kada et al. in [12] where nonconvex minimization problems were treated. Further results were given, e.g., in [13, 14]. Cone metric version of this notion (sometimes called a $c$-distance) was used, e.g., in [15]–[19].

Coupled fixed points in ordered metric spaces were treated, e.g., in [20]–[26] and many other papers. Such problems in cone metric spaces were investigated, e.g., in [27]–[34].

In this paper, after some preliminaries, we first give an answer (Theorem 2.2) to the Question concerning quasicontractions under w-cone-distance, posed in the recent paper [19]. In Section 3, we prove some coupled and b-coupled fixed point results in tvs-cone metric spaces, which improve corresponding results from [34]. An example shows that this improvement is proper. Several other results can be improved in a similar way, some of which are also mentioned.

In Section 4, we derive several (common) coupled fixed point results under $w$-cone distance in tvs-cone metric spaces, thus also improving some known theorems. Again, examples show the usage of these results, and that the respective theorems do not hold when the underlying cone metric is used instead of a $w$-cone distance in the contractive condition.

For the sake of simplicity, we work in spaces without order, but most of the results can be easily re-formulated for the case of ordered tvs-cone metric spaces.

## 2. Preliminaries and auxiliary results

### 2.1. Cones and tvs-cone metric spaces

Let $E$ be a real Hausdorff topological vector space (*tvs* for short) with the zero vector $\theta$. A proper nonempty and closed subset $K$ of $E$ is called a *cone* if $K + K \subset K, \lambda K \subset K$ for $\lambda \geq 0$ and $K \cap (-K) = \{\theta\}$. We shall always assume that the cone $K$ has a nonempty interior int $K$ (such cones are called *solid*).

Each cone $K$ induces a partial order $\preceq$ on $E$ by $x \preceq y \Leftrightarrow y - x \in K$. $x \prec y$ will stand for ($x \preceq y$ and $x \neq y$), while $x \ll y$ will stand for $y - x \in \operatorname{int} K$. The pair $(E, K)$ is an *ordered topological vector space.*

For a pair of elements $x, y$ in $E$ such that $x \preceq y$, put $[x, y] = \{z \in E : x \preceq z \preceq y\}$. A subset $A$ of $E$ is said to be *order-convex* if $[x, y] \subset A$, whenever $x, y \in A$ and $x \preceq y$.

Ordered topological vector space $(E, K)$ is *order-convex* if it has a base of neighborhoods of $\theta$ consisting of order-convex subsets. In this case, the cone $K$ is said to be *normal.* If $E$ is a normed space, this condition means that the unit ball is order-convex, which is equivalent to the condition that there is a number $k$ such that $x, y \in E$ and $0 \preceq x \preceq y$ implies that $\|x\| \leq k\|y\|$. A proof of the following assertion can be found, e.g., in [2].

**Theorem 2.1.** *If the underlying cone of an ordered tvs is solid and normal, then such tvs must be an ordered normed space.*

We will call a sequence $\{u_n\}$ in $E$ a *c-sequence* if for each $c \gg \theta$ there exists $n_0 \in \mathbb{N}$ such that $u_n \ll c$ for $n \geq n_0$. It is easy to show that if $\{u_n\}$ and $\{v_n\}$ are $c$-sequences in $E$ and $\alpha, \beta > 0$, then $\{\alpha u_n + \beta v_n\}$ is a $c$-sequence.

If $V$ is an absolutely convex and absorbing subset of $E$, its Minkowski functional $q_V$ is defined by

$$E \ni x \mapsto q_V(x) = \inf\{\, \lambda > 0 : x \in \lambda V \,\}.$$

It is a semi-norm on $E$ (i.e., $q_V(x+y) \leq q_V(x) + q_V(y)$ for $x, y \in E$ and $q_V(\lambda x) = |\lambda| q_V(x)$ for $x \in E$, $\lambda$ a scalar). If $V$ is an absolutely convex neighborhood of $\theta$ in $E$, then $q_V$ is continuous and

$$\{x \in E : q_V(x) < 1\} = \operatorname{int} V \subset V \subset \overline{V} = \{x \in E : q_V(x) \leq 1\}.$$

Note also that $q_V(\operatorname{co} A) \subset \operatorname{co}(q_V(A))$ for arbitrary $A \subset E$.

Let now $e \in \operatorname{int} K$. Then $[-e, e] = (K - e) \cap (e - K) = \{z \in E : -e \preceq z \preceq e\}$ is an absolutely convex neighborhood of $\theta$; its Minkowski functional $q_{[-e,e]}$ will be denoted by $q_e$. It is an increasing function on $K$. Indeed, let $\theta \preceq x_1 \preceq x_2$; then $\{\lambda : x_1 \in \lambda[-e, e]\} \supset \{\lambda : x_2 \in \lambda[-e, e]\}$ and it follows that $q_e(x_1) \leq q_e(x_2)$. If the cone $K$ is solid and normal, $q_e$ is a norm in $E$.

We note that $\{u_n\}$ is a $c$-sequence in $E$ if and only if $q_e(u_n) \to 0$ as $n \to \infty$.

From [1, 4, 5, 7], we give the following

**Definition 2.2.** Let $\mathcal{X}$ be a nonempty set and $(E, K)$ an ordered *tvs*. A function $d : \mathcal{X} \times \mathcal{X} \to E$ is called a *tvs-cone metric* and $(\mathcal{X}, d)$ is called a *tvs-cone metric space* if the following conditions hold:

(c1) $\theta \preceq d(x, y)$ for all $x, y \in \mathcal{X}$ and $d(x, y) = \theta$ if and only if $x = y$;

(c2) $d(x, y) = d(y, x)$ for all $x, y \in \mathcal{X}$;

(c3) $d(x, z) \preceq d(x, y) + d(y, z)$ for all $x, y, z \in \mathcal{X}$.

Taking into account Theorem 2.1, proper generalizations when passing from norm-valued cone metric spaces of [1] to tvs-cone metric spaces can be obtained only in the case of nonnormal cones.

It is said that a sequence $\{x_n\}$ in $(\mathcal{X}, d)$ *converges* to $x \in \mathcal{X}$ (denoted as $x_n \to x$, $n \to \infty$) if $d(x_n, x)$ is a $c$-sequence in $E$. Similarly, Cauchy sequences and the completeness of the space are defined, see [1, 7, 11] and references therein.

The following properties are easy to prove:

($p_1$) If $u, v, w \in E, u \preceq v$ and $v \ll w$ then $u \ll w$.

($p_2$) If $u \in E$ and $\theta \preceq u \ll c$ for each $c \in \operatorname{int} K$ then $u = \theta$.

($p_3$) If $u_n, v_n, u, v \in E, \theta \preceq u_n \preceq v_n$ for each $n \in \mathbb{N}$, and $u_n \to u, v_n \to v$ $(n \to \infty)$, then $\theta \preceq u \preceq v$.

($p_4$) If $x_n, x \in \mathcal{X}, u_n \in E, d(x_n, x) \preceq u_n$ and $u_n \to \theta$ $(n \to \infty)$, then $x_n \to x$ $(n \to \infty)$.

($p_5$) If $u \preceq \lambda u$, where $u \in K$ and $0 \leq \lambda < 1$, then $u = \theta$.

($p_6$) If $c \gg \theta$ and $u_n \in E, u_n \to \theta$ $(n \to \infty)$, then there exists $n_0$ such that $u_n \ll c$ for all $n \geq n_0$.

By ($p_6$), each $\theta$-sequence in $K$ is a $c$-sequence. The converse is true if the cone $K$ is normal, but in general, a $c$-sequence need not be a $\theta$-sequence (see [7, 11]).

Similar assertions hold for double sequences $\{u_{mn}\}$.

Also, from [7], we know that the cone metric $d$ need not be a continuous function.

In the sequel, $E$ will always denote a topological vector space, with the zero vector $\theta$ and with order relation $\preceq$, generated by a solid cone $K$.

The following result was proved in [10].

**Theorem 2.3.** *Let $(\mathcal{X}, d)$ be a tvs-cone metric space over a solid cone $K$ and let $e \in \operatorname{int} K$. Let $q_e$ be the corresponding Minkowski functional of $[-e, e]$. Then $d_q = q_e \circ d$ is a (real-valued) metric on $\mathcal{X}$. Moreover:*

$1^\circ$ $x_n \stackrel{d}{\to} x \Leftrightarrow x_n \stackrel{d_q}{\to} x$.

$2^\circ$ *$\{x_n\}$ is a $d$-Cauchy sequence if and only if it is a $d_q$-Cauchy sequence.*

$3^\circ$ *$(\mathcal{X}, d)$ is complete if and only if $(\mathcal{X}, d_q)$ is complete.*

Hence, topologies induced on $\mathcal{X}$ by $d$ and $d_q$ are equivalent, i.e., these spaces have the same collections of closed, resp. open sets, and the same continuous functions (provided the underlying cone is solid).

### 2.2. w-cone-distance

Kada et al. [12] introduced the notion of $w$-distance in metric spaces as follows.

**Definition 2.4.** [12] Let $(\mathcal{X}, d)$ be a metric space. A function $p : \mathcal{X} \times \mathcal{X} \to [0, +\infty)$ is called a *$w$-distance* in $\mathcal{X}$ if:

(1) $p(x, z) \leq p(x, y) + p(y, z)$ for all $x, y, z \in \mathcal{X}$;

(2) $p(x, \cdot) : \mathcal{X} \to [0, +\infty)$ is lower-semicontinuous for each $x \in \mathcal{X}$;

(3) For each $\varepsilon > 0$, there exists $\delta > 0$, such that $p(z, x) < \delta$ and $p(z, y) < \delta$ implies $d(x, y) < \varepsilon$.

They proved some fixed point results using this notion (see also [13, 14]). The notion was transferred to the setting of (tvs)-cone metric spaces in two ways. Cho et al. [15] introduced what they called a *c-distance* and this was also used in several other papers (see, e.g., [16]–[18]) to obtain fixed point results.

The following variant of this definition was given in [19].

**Definition 2.5.** [19] Let $(\mathcal{X}, d)$ be a tvs-cone metric space. A function $p : \mathcal{X} \times \mathcal{X} \to K$ is called a *w-cone-distance* in $\mathcal{X}$ if:

(w1) $p(x,z) \preceq p(x,y) + p(y,z)$ for all $x,y,z \in \mathcal{X}$;

(w2) $p(x,\cdot) : \mathcal{X} \to K$ is lower-semicontinuous for each $x \in \mathcal{X}$;

(w3) For each $c \in E$ with $\theta \ll c$, there exists $e \in E$ with $\theta \ll e$, such that $p(z,x) \ll e$ and $p(z,y) \ll e$ implies $d(x,y) \ll c$.

Using the notion of $c$-sequences, condition (w2) of the previous definition can be formulated in the following equivalent way (see [19]):

(w2') If $y_n, y \in \mathcal{X}$, $y_n \to y$ as $n \to \infty$ and $g(y) = p(x,y)$, then $g(y) - g(y_n)$ is a $c$-sequence.

**Remark 2.6.** Definition of $c$-distance from [15] differs from Definition 2.5 of $w$-cone-distance in the way that the following condition is used instead of (w2):

If a sequence $\{y_n\}$ in $X$ converges to a point $y \in X$, and for some $x \in X$ and $u = u_x \in K$, $p(x,y_n) \preceq u$ holds for each $n \in \mathbb{N}$, then $p(x,y) \preceq u$.

By [15, Remark 2.6], it is clear that each $w$-cone distance is a $c$-distance, but the converse does not hold.

Several examples of $w$-cone distances and $c$-distances can be found in [15]–[19]. See also further Examples 4.2 and 4.4.

Similarly to Theorem 2.3 we obtain the following

**Lemma 2.7.** *Let $(\mathcal{X},d)$ be a tvs-cone metric space over a solid cone $K$ and let $p$ be a $w$-cone-distance on $\mathcal{X}$. Let $e \in \operatorname{int} K$ and let $q_e$ be the corresponding Minkowski functional of $[-e,e]$. Then $p_q = q_e \circ p$ is a (real-valued) $w$-distance on $\mathcal{X}$.*

**Proof.** We check the conditions of Definition 2.4. (1) is straightforward.

(2) Let $x \in \mathcal{X}$ and denote $g_q(y) = p_q(x,y)$. In order to prove that $g_q$ is lower semicontinuous, take $y, y_n \in \mathcal{X}$ such that $y_n \to y$ in $d_q$, which is equivalent to $y_n \to y$ in $d$. Since $p$ is lower semicontinuous, this means that, for arbitrary $\varepsilon > 0$, $p(x,y) \preceq p(x,y_n) + \varepsilon \cdot e$ for $n$ large enough. But then $g_q(y) \leq g_q(y_n) + \varepsilon$ and $g_q = p_q(x,\cdot)$ is also lower semicontinuous.

(3) For the given $\varepsilon > 0$ denote $c = \varepsilon \cdot e$. Then $c \in \operatorname{int} K$. By (w3), there exists $c_1 \in \operatorname{int} K$ such that

$$p(z,x) \ll c_1 \text{ and } p(z,y) \ll c_1 \text{ implies } d(x,y) \ll c. \tag{2.1}$$

Denote $\delta = q_e(c_1)$ and suppose that $p_q(z,x) < \delta$ and $p_q(z,y) < \delta$. This means that $q_e(p(z,x)) < \delta = q_e(c_1)$ and $q_e(p(z,y)) < \delta = q_e(c_1)$, which implies that $p(z,x) \ll c_1$ and $p(z,y) \ll c_1$. By (2.1), we obtain that $d(x,y) \ll c = \varepsilon \cdot e$ and hence $d_q(x,y) < \varepsilon$. This completes the proof.

Using the previous lemma, as well as Theorem 2.3, most of the results from papers [15]–[19] can be reduced to their metric counterparts. Moreover, an answer to the Question at the end of paper [19] can be deduced, since it reduces to Ume's result [35, Theorem 3.1].

**Theorem 2.8.** *Let $p$ be a w-cone-distance on a tvs-cone metric space $(\mathcal{X}, d)$. Suppose that $f, g : \mathcal{X} \to \mathcal{X}$ are self-maps such that for some constant $\lambda \in [0,1)$ and for every $x, y \in \mathcal{X}$ there exists*

$$v \in \operatorname{co}\{p(gx,gy), p(gx,fx), p(gy,fy), p(gx,fy), p(gy,fx)\}$$

*such that $p(fx,fy) \preceq \lambda v$. Suppose that $fX \subset gX$ and that one of these two subsets of $X$ is complete. Assume further that for each $y \in \mathcal{X}$ with $fy \neq gy$ there exists $c \gg \theta$ satisfying*

$$c \ll p(gx,gy) + p(gx,fx), \quad \text{for each } x \in \mathcal{X}.$$

*Then $f$ and $g$ have a unique point of coincidence. Moreover, if $f$ and $g$ are weakly compatible, then they have a unique common fixed point $z$, and $p(z,z) = \theta$ holds.*

**Proof.** Let $e \in \operatorname{int} K$ and let $q_e$, $d_q = q_e \circ d$ and $p_q = q_e \circ p$ be defined as in Theorem 2.3 and Lemma 2.7. Then the contractive condition implies that there exists

$$q_e v \in \operatorname{co}\{p_q(gx,gy), p_q(gx,fx), p_q(gy,fy), p_q(gx,fy), p_q(gy,fx)\}$$

such that $p_q(fx,fy) \leq \lambda q_e v$. But the last set is a set of real numbers, so this can be written as

$$p_q(fx,fy) \leq \lambda \max\{p_q(gx,gy), p_q(gx,fx), p_q(gy,fy), p_q(gx,fy), p_q(gy,fx)\}.$$

Moreover, the given condition implies that for each $y \in \mathcal{X}$ with $fy \neq gy$,

$$p_q(gx,gy) + p_q(gx,fx) \geq q_e c > 0 \quad \text{for each } x \in \mathcal{X}.$$

In other words, $\inf\{\, p_q(gx, gy) + p_q(gx, fx) : x \in \mathcal{X}\,\} > 0$, and hence [35, Theorem 3.1] can be applied to reach the conclusions.

We will make use of the following basic lemma.

**Lemma 2.9.** *Let $p$ be a $w$-cone-distance on a tvs-cone metric space $(\mathcal{X}, d)$. Define $D : \mathcal{X}^2 \times \mathcal{X}^2 \to K$ and $P : \mathcal{X}^2 \times \mathcal{X}^2 \to K$ by*

$$D(X,U) = d(x,u) + d(y,v), \quad P(X,U) = p(x,u) + p(y,v), \quad X = (x,y),\, U = (u,v). \tag{2.2}$$

*Then $D$ is a tvs-cone metric on $\mathcal{X}^2$ and $P$ is a $w$-cone-distance on $(\mathcal{X}^2, D)$. If the space $(\mathcal{X}, d)$ is complete, then so is $(\mathcal{X}^2, D)$.*

**Proof.** The axioms of a tvs-cone metric for the function $D$ can be checked directly. Let us prove that $(\mathcal{X}^2, D)$ is complete if $(\mathcal{X}, d)$ is such.

Let $\{X_n\} = \{(x_n, y_n)\}$ be a Cauchy sequence in $(\mathcal{X}^2, D)$. This means that $\{D(X_n, X_m)\}$ is a (double) $c$-sequence in $K$. Since $d(x_n, x_m) \preceq D(X_n, X_m)$ and $d(y_n, y_m) \preceq D(x_n, x_m)$, it follows that $\{d(x_n, x_m)\}$ and $\{d(y_n, y_m)\}$ are double $c$-sequences in $K$. Hence, $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in $(\mathcal{X}, d)$. By supposition, they converge to, say, $x$ and $y$, respectively. It easily follows that $\{X_n\}$ converges to $X = (x, y)$.

In order to prove the assertion about $P$, first note that the triangular inequality follows directly.

(w2) Since, by definition, the mapping $g(y) = p(x, y)$, $y \in \mathcal{X}$ (for fixed $x \in \mathcal{X}$) is lower semicontinuous, it can be easily checked that the mapping $G(U) = P(X, U) = p(x,u) + p(y,v)$, $U = (u,v) \in \mathcal{X}^2$ (for fixed $X = (x,y) \in \mathcal{X}^2$) is also lower semicontinuous.

(w3) For the given $c \gg \theta$, let $e \gg \theta$ be an element satisfying condition (w3) for the $w$-cone distance $p$, but for the vector $\frac{c}{2}$. Then the following implications hold:

$$\begin{cases} p(x,u) \ll e \wedge p(x,z) \ll e \ \Rightarrow\ d(u,z) \ll \dfrac{c}{2}, \\[2ex] p(y,v) \ll e \wedge p(y,w) \ll e \ \Rightarrow\ d(v,w) \ll \dfrac{c}{2}. \end{cases} \tag{2.3}$$

Now, if

$$P((x,y),(u,v)) = p(x,u) + p(y,v) \ll e,$$

then $p(x,u) \ll e$ and $p(y,v) \ll e$, and, similarly,

$$P((x,y),(z,w)) = p(x,z) + p(y,w) \ll e,$$

implies $p(x,z) \ll e$ and $p(y,w) \ll e$. Using (2.3), it follows that

$$D((u,v),(z,w)) = d(u,z) + d(v,w) \ll \frac{c}{2} + \frac{c}{2} = c.$$

This completes the proof.

## 3. Coupled fixed point results in tvs-cone metric spaces

First we recall some definitions.

**Definition 3.1.** [34] Let $\mathcal{X}$ be a non-empty set and let $f,g : \mathcal{X}^2 \to \mathcal{X}$ be two mappings. A point $(x,y) \in \mathcal{X}^2$ is called:

(1) a b-coupled coincidence point of $f$ and $g$ if $f(x,y) = g(x,y)$ and $f(y,x) = g(y,x)$; then $(g(x,y), g(y,x))$ is called a b-coupled point of coincidence;
(2) a b-common coupled fixed point of $f$ and $g$ if $f(x,y) = g(x,y) = x$ and $f(y,x) = g(y,x) = y$.
(3) The mappings $f$ and $g$ are called $\tilde{w}$-compatible if $f(x,y) = g(x,y)$ and $f(y,x) = g(y,x)$ imply that $f(g(x,y), g(y,x)) = g(f(x,y), f(y,x))$.

**Theorem 3.2.** *Let $(\mathcal{X}, d)$ be a tvs-cone metric space and let $f,g : \mathcal{X}^2 \to \mathcal{X}$ be two mappings satisfying:*

(i) *for any $(x,y) \in \mathcal{X}^2$ there exists $(u,v) \in \mathcal{X}^2$ such that $f(x,y) = g(u,v)$ and $f(y,x) = g(v,u)$;*
(ii) *$\{(g(x,y), g(y,x)) : x,y \in \mathcal{X}\}$ is a complete subspace of $\mathcal{X}^2$ (taken with the product cone metric, as in Lemma 2.9).*

*Suppose that for some nonnegative constants $\alpha, \beta, \gamma, \delta, \varepsilon$ such that $\alpha + \beta + \gamma + \delta + \varepsilon < 1$, and for all $x, y, u, v \in \mathcal{X}$, the following inequality holds:*

$$\begin{aligned} d(f(x,y), f(u,v)) + d(f(y,x), f(v,u)) &\preceq \alpha[d(g(x,y), g(u,v)) + d(g(y,x), g(v,u))] \\ &+ \beta[d(g(x,y), f(x,y)) + d(g(y,x), f(y,x))] + \gamma[d(g(u,v), f(u,v)) + d(g(v,u), f(v,u))] \\ &+ \delta[d(g(x,y), f(u,v)) + d(g(y,x), f(v,u))] + \varepsilon[d(g(u,v), f(x,y)) + d(g(v,u), f(y,x))] \end{aligned} \tag{3.1}$$

*Then, there exists a b-coupled coincidence point of $f$ and $g$ in $\mathcal{X}^2$.*

**Proof**. We will use the procedure similar as in [23]–[26], using Lemma 2.9.

So, consider in the set $\mathcal{X}^2$ the cone metric $D$ defined by (2.2). Then, $(\mathcal{X}^2, D)$ is a tvs-cone metric space. Define the mappings $F, G : \mathcal{X}^2 \to \mathcal{X}^2$ by

$$F(x,y) = (f(x,y), f(y,x)), \quad G(x,y) = (g(x,y), g(y,x)).$$

Then, it is easy to see that:

(1) $F(\mathcal{X}^2) \subset G(\mathcal{X}^2)$;

(2) subset $G(\mathcal{X}^2)$ of $\mathcal{X}^2$ is complete.

Moreover, condition (3.1) imply the following one:

$$D(FX, FU) \preceq \alpha D(GX, GU) + \beta D(GX, FX) + \gamma D(GU, FU) + \delta D(GX, FU) + \varepsilon D(GU, FX),$$

for all $X = (x,y)$ and $U = (u,v) \in \mathcal{X}^2$. This means that conditions of Hardy-Rogers type for two self-mappings in a tvs-cone metric space are fulfilled. Using Theorem 2.3 this can be further reduced to the classical Hardy-Rogers type condition in the associated metric space. It follows that $F$ and $G$ have a coincidence point $Z = (z,w) \in \mathcal{X}^2$. It is easy to see that this point is a b-coupled coincidence point of $f$ and $g$. This completes the proof.

**Remark 3.3.** Condition (3.1) of Theorem 3.2 is obviously a consequence of the contractive condition (h3) of [34, Corollary 1]. Hence, Theorem 3.2 is an improvement of that corollary, proved in a much easier way. We show in the next example that this improvement is proper.

It is also clear that several other known coupled fixed point results obtained recently, such as [27, Theorems 3.1, 3.2], [28, Theorem 16], [30, Theorem 2.2], [33, Theorem 2.2], and others, can be proved in a much easier way and/or improved using the presented method.

**Example 3.4.** Let $E = C^1_{\mathbb{R}}[0,1]$ with $\|\varphi\| = \|\varphi\|_\infty + \|\varphi'\|_\infty$ and $K = \{\varphi \in E : \varphi(t) \geq 0\}$. It is well known that this cone is nonnormal (see e.g. [2]). Let $\tau$ be the strongest vector (locally convex) topology on $E$. Then $(E, \tau)$ is a topological vector space which is not normable and is not even metrizable. Let $\mathcal{X} = \mathbb{R}$ and $d : \mathcal{X} \times \mathcal{X} \to E$ be defined by $d(x,y)(t) = |x-y| \cdot e^t$. Then $(\mathcal{X}, d)$ is a tvs-cone metric space over a nonnormal solid cone.

Consider the mappings $f, g : \mathcal{X}^2 \to \mathcal{X}$ given by

$$f(x,y) = \frac{x-4y}{3}, \qquad g(x,y) = 2x,$$

and take $\alpha = \frac{5}{6}$ and $\beta = \gamma = \delta = \varepsilon = 0$. Then condition (3.1) of Theorem 3.2 holds for all $x, y, u, v \in \mathcal{X}$ since

$$\begin{aligned}
&d(f(x,y), f(u,v)) + d(f(y,x), f(v,u)) \\
&\qquad = \left| \frac{x-4y}{3} - \frac{u-4v}{3} \right| \cdot e^t + \left| \frac{y-4x}{3} - \frac{v-4u}{3} \right| \cdot e^t \\
&\qquad \preceq \frac{5}{3}(|x-u| + |y-v|) \cdot e^t \\
&\qquad = \frac{5}{6}(|2x-2u| + |2y-2v|) \cdot e^t \\
&\qquad = \frac{5}{6}\left[d(g(x,y), g(u,v)) + d(g(y,x), g(v,u))\right],
\end{aligned}$$

holds for all $x, y, u, v \in \mathcal{X}$ and all $t \in [0,1]$. By Theorem 3.2, $f$ and $g$ have a b-coupled coincidence point (which is $(0,0)$).

On the other hand, suppose that condition (h3) of [34, Corollary 1] holds with $\alpha_1 = \cdots = \alpha_4 = 0$, i.e.,

$$\left| \frac{x-4y}{3} - \frac{u-4v}{3} \right| \cdot e^t \preceq \alpha_5 \cdot (|2x-2u| + |2y-2v|) \cdot e^t$$

holds for all $x, y, u, v \in \mathcal{X}$ and all $t \in [0,1]$. Put $x = u$, and the previous inequality reduces to

$$\frac{4}{3}|y-v| \leq 2\alpha_5 |y-v|,$$

which, for $y \neq v$, gives $\alpha_5 \geq \frac{2}{3}$, which is excluded by supposition $\alpha_5 < \frac{1}{2}$. Hence, the existence of a b-coupled coincidence point cannot be obtained using [34, Corollary 1] (with $\alpha_1 = \cdots = \alpha_4 = 0$).

**Remark 3.5.** As in the most results of this kind, if we add to the conditions of Theorem 3.2 that $f$ and $g$ are $\tilde{w}$-compatible, then it can be easily shown that $f$ and $g$ have a unique b-common coupled fixed point (see details in [34]).

## 4. Results on coupled fixed points under w-cone-distance contractive conditions

The first result of this section is the following

**Theorem 4.1.** *Let $(\mathcal{X}, d)$ be a complete tvs-cone metric space and let $p$ be a w-cone-distance on $\mathcal{X}$. Suppose that a continuous map $f : \mathcal{X}^2 \to \mathcal{X}$ satisfies the following two conditions:*

$$
\begin{aligned}
&p(f(x,y), f(u,v)) + p(f(y,x), f(v,u)) \\
&\qquad \preceq \alpha[p(x,u) + p(y,v)] + \beta[p(x, f(x,y)) + p(y, f(y,x))] \\
&\qquad\quad + \gamma[p(u, f(u,v)) + p(v, f(v,u))] + \delta[p(x, f(u,v)) + p(y, f(v,u))] \\
&\qquad\quad + \varepsilon[p(u, f(x,y)) + p(v, f(y,x))], \\
&p(f(u,v), f(x,y)) + p(f(v,u), f(y,x)) \\
&\qquad \preceq \alpha[p(u,x) + p(v,y)] + \beta[p(f(x,y), x) + p(f(y,x), y)] \\
&\qquad\quad + \gamma[p(f(u,v), u) + p(f(v,u), v)] + \delta[p(f(u,v), x) + p(f(v,u), y)] \\
&\qquad\quad + \varepsilon[p(f(x,u), u) + p(f(y,x), v)],
\end{aligned}
\tag{4.1}
$$

*for all $x, y, u, v \in \mathcal{X}$, where $\alpha, \beta, \gamma, \delta, \varepsilon$ are nonnegative constants such that $\alpha + \beta + \gamma + 2\delta + 2\varepsilon < 1$. Then $f$ has a coupled fixed point $(z, w)$ in $\mathcal{X}^2$. Moreover, it is $p(z,z) = p(w,w) = \theta$.*

Recall (e.g., [20]) that the point $(z,w)\in\mathcal{X}^2$ is called a coupled fixed point of a mapping $f:\mathcal{X}^2\to\mathcal{X}$ if $f(z,w)=z$ and $f(w,z)=w$.

**Proof.** Consider in the set $\mathcal{X}^2$ the cone metric $D$ and $w$-cone distance $P$ defined by (2.2). Then, by Lemma 2.9, $(\mathcal{X}^2,D)$ is a complete tvs-cone metric space. Define the mapping $F:\mathcal{X}^2\to\mathcal{X}^2$ by

$$F(x,y)=(f(x,y),f(y,x)).$$

Then, it is easy to see that $F$ is continuous, and that conditions (4.1) imply the following ones:

$$P(FX,FU)\preceq \alpha P(X,U)+\beta P(X,FX)+\gamma P(U,FU)+\delta P(X,FU)+\varepsilon P(U,FX),$$
$$P(FU,FX)\preceq \alpha P(U,X)+\beta P(FX,X)+\gamma P(FU,U)+\delta P(FU,X)+\varepsilon P(FX,U),$$

for all $X=(x,y)$ and $U=(u,v)\in\mathcal{X}^2$. Moreover, by Remark 2.6, $w$-cone distance $P$ on $\mathcal{X}^2$ is also a $c$-distance. Hence, all the conditions of [16, Theorem 2] are fulfilled and we conclude that $F$ has a fixed point $(z,w)\in\mathcal{X}^2$ such that $P((z,w),(z,w))=(\theta,\theta)\in\mathcal{X}^2$. But then obviously $(z,w)$ satisfies what is needed. This completes the proof.

**Example 4.2.** Let $\mathcal{X}=[0,+\infty)$, $E=C[0,1]$, $K=\{\varphi\in E:\varphi(t)\geq 0, t\in[0,1]\}$, $d(x,y)=|x-y|\varphi$ and $p(x,y)=y\varphi$ for $x,y\in\mathcal{X}$ and fixed $\varphi\in K$. Then $(\mathcal{X},d)$ is a (tvs)-cone metric space and $p$ is a $w$-cone distance on it (see [15, Example 2.10]). Consider the mapping $f:\mathcal{X}^2\to\mathcal{X}$ given by

$$f(x,y)=\frac{x+2y}{4},$$

and take $\alpha=\frac{13}{20}$, $\beta=\gamma=\frac{2}{15}$ and $\delta=\varepsilon=0$. Then the contractive conditions (4.1) of Theorem 4.1 are satisfied, since:

$$\begin{aligned}
&p(f(x,y),f(u,v))+p(f(y,x),f(v,u))\\
&\quad=\frac{3}{4}(u+v)\varphi\preceq\left[\frac{3}{4}(u+v)+\frac{1}{10}(x+y)\right]\varphi\\
&\quad=\alpha[p(x,u)+p(y,v)]+\beta[p(x,f(x,y))+p(y,f(y,x))]+\gamma[p(u,f(u,v))+p(v,f(v,u))]
\end{aligned}$$

and

$$\begin{aligned}
&p(f(u,v),f(x,y))+p(f(v,u),f(y,x))\\
&\quad=\frac{3}{4}(x+y)\varphi\preceq\left[\frac{3}{4}(x+y)+\frac{1}{10}(u+v)\right]\varphi\\
&\quad=\alpha[p(u,x)+p(v,y)]+\beta[p(u,f(u,v))+p(v,f(v,u))]+\gamma[p(x,f(x,y))+p(y,f(y,x))]
\end{aligned}$$

hold for all $x,y,u,v\in\mathcal{X}$. All other conditions of Theorem 4.1 are also satisfied, hence $f$ has a (unique) coupled fixed point (which is $(0,0)$).

In a similar way, using [16, Theorem 3], one can obtain the following common coupled fixed point result.

**Theorem 4.3.** *Let $(\mathcal{X},d)$ be a complete tvs-cone metric space and let $p$ be a $w$-cone-distance on $\mathcal{X}$. Suppose that continuous maps $f:\mathcal{X}^2\to\mathcal{X}$ and $g:\mathcal{X}\to\mathcal{X}$ satisfy the following two conditions:*

$$\begin{aligned}
p(f(x,y),gu)+p(f(y,x),gv)&\preceq\alpha[p(x,u)+p(y,v)]\\
&\quad+\beta[p(x,f(x,y))+p(y,f(y,x))+p(u,gu)+p(v,gv)]\\
&\quad+\delta[p(x,gu)+p(y,gv)+p(u,f(x,y))+p(v,f(y,x))],\\
p(gu,f(x,y))+p(gv,f(y,x))&\preceq\alpha[p(u,x)+p(v,y)]\\
&\quad+\beta[p(f(x,y),x)+p(f(y,x),y)+p(gu,u)+p(gv,v)]\\
&\quad+\delta[p(gu,x)+p(gv,y)+p(f(x,y),u)+p(f(y,x),v)],
\end{aligned}\tag{4.2}$$

*for all $x,y,u,v\in\mathcal{X}$, where $\alpha,\beta,\delta$ are nonnegative constants such that $\alpha+2\beta+4\delta<1$. Then $f$ and $g$ have a common coupled fixed point $(z,w)$ in $\mathcal{X}^2$. Moreover, it is $p(z,z)=p(w,w)=\theta$.*

Recall (e.g., [21]) that a point $(z,w)\in\mathcal{X}^2$ is called a common coupled fixed point of mappings $f:\mathcal{X}^2\to\mathcal{X}$ and $g:\mathcal{X}\to\mathcal{X}$ if $f(z,w)=gz=z$ and $f(w,z)=gw=w$.

As a corollary, we obtain, e.g., the common coupled fixed point result for mappings $f:\mathcal{X}^2\to\mathcal{X}$ and $g:\mathcal{X}\to\mathcal{X}$ satisfying

$$\begin{aligned}
p(f(x,y),gu)+p(f(y,x),gv)&\preceq\alpha[p(x,u)+p(y,v)],\\
p(gu,f(x,y))+p(gv,f(y,x))&\preceq\alpha[p(u,x)+p(v,y)],
\end{aligned}\tag{4.3}$$

$0 \leq \alpha < 1$. This is an improvement of [13, Theorem 2.1] (in the case without order; of course, an "ordered" variant could be derived similarly, as we have said in the beginning).

Modifying an example from [16], we show that it may happen that conditions of the type (4.3), taken in the tvs-cone metric of the given space, may not be sufficient for concluding about the existence of a common coupled fixed point.

**Example 4.4.** Let $E$, $K$, $\mathcal{X}$ and $d$ be the same as in Example 4.2 and define $p(x,y) = x\varphi$, for fixed $\varphi \in K$. Consider the mappings $f : \mathcal{X}^2 \to \mathcal{X}$ and $g : \mathcal{X} \to \mathcal{X}$ defined by $f(x,y) = \frac{x}{4}$ and $gx = \frac{x}{2}$. If $x = y = 5$, $u = v = \frac{15}{2}$, then

$$d(f(x,y), gu) + d(f(y,x), gv) = \left|\frac{5}{4} - \frac{15}{4}\right|\varphi + \left|\frac{5}{4} - \frac{15}{4}\right|\varphi = 5\varphi,$$

and

$$d(x,u) + d(y,v) = \left|5 - \frac{15}{2}\right|\varphi + \left|5 - \frac{15}{2}\right|\varphi = 5\varphi.$$

Thus, there is no $\alpha \in (0,1)$ (and hence no triple $(\alpha, \beta, \delta)$) such that

$$d(f(x,y), gu) + d(f(y,x), gv) \preceq \alpha[d(x,u) + d(y,v)]$$

for all $x, y, u, v \in \mathcal{X}$, i.e., the existence of a common coupled fixed point of $f$ and $g$ cannot be deduced from the tvs-cone metric version of Theorem 4.3.

However, conditions of the $w$-cone-distance version (Theorem 4.3) are satisfied. Indeed, take arbitrary $\alpha$, $\frac{1}{2} \leq \alpha < 1$ and $\beta = \delta = 0$. Then, for all $x, y, u, v \in \mathcal{X}$,

$$\begin{aligned} p(f(x,y), gu) + p(f(y,x), gv) &= [f(x,y) + f(y,x)]\varphi = \frac{x+y}{4}\varphi \\ &\preceq \alpha(x+y)\varphi = \alpha[p(x,u) + p(y,v)], \end{aligned}$$

and

$$\begin{aligned} p(gu, f(x,y)) + p(gv, f(y,x)) &= [gu + gv]\varphi = \frac{u+v}{2}\varphi \\ &\preceq \alpha(u+v)\varphi = \alpha[p(u,x) + p(v,y)]. \end{aligned}$$

Hence, the existence of a common coupled fixed point of $f$ and $g$ (which is $(0,0)$) follows from Theorem 4.3.

## References


[1] L.G. Huang, X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl. 332(2), 1468–1476 (2007).

[2] J.S. Vandergraft, Newton's method for convex operators in partially ordered spaces. SIAM J. Num. Anal. 4, 406–432 (1967).

[3] P.P. Zabrejko, K-metric and K-normed linear spaces, survey, Collect. Math. 48, 825–859 (1997).

[4] I. Beg, A. Azam, M. Arshad, Common fixed points for maps on topological vector space valued cone metric spaces, Intern. J. Math. Math. Sci. 2009, Article ID 560264, 8 p. (2009).

[5] W-Sh. Du, A note on cone metric fixed point theory and its equivalence, Nonlinear Anal. TMA 72, 2259–2261 (2010).

[6] A. Azam, I. Beg, M. Arshad, Fixed points in topological vector space valued cone metric spaces, Fixed Point Theory Appl. 2010, Art. ID 604084, 9 p. (2010).

[7] Z. Kadelburg, S. Radenović, V. Rakočević, Topological vector space valued cone metric spaces and fixed point theorems, Fixed Point Theory Appl. 2010, Article ID 170253, 18 p. (2010).

[8] I.D. Aranđelović, D.J. Kečkić, On nonlinear quasi-contractions on tvs-cone metric spaces, Appl. Math. Lett. 24, 1209–1213 (2011).

[9] M. Abbas, Y.J. Cho, T. Nazir, Common fixed point theorems for four mappings in tvs-valued cone metric spaces, J. Math. Inequal. 5, 287–299 (2011).

[10] Z. Kadelburg, S. Radenović, V. Rakočević, A note on the equivalence of some metric and cone metric fixed point results, Appl. Math. Lett. 24, 370–374 (2011).

[11] S. Janković, Z. Kadelburg, S. Radenović, On cone metric spaces, a survey, Nonlinear Anal. TMA 74, 2591–2601 (2011).

[12] O. Kada, T. Suzuki, W. Takahashi, Nonconvex minimization theorems and fixed point theorems in complete metric spaces, Math. Japonica 44, 381–391 (1996).

[13] M. Abbas, D. Ilić, M. Ali Khan, Coupled coincidence point and coupled common fixed point theorems in partially ordered metric spaces with w-distance, Fixed Point Theory Appl. 2010, Article ID 134897, 11p. (2010).

[14] D. Ilić, V. Rakočević, Common fixed point for maps with w-distance, Appl. Math. Comput. 199, 599–610 (2008).

[15] Y.J. Cho, R. Saadati, Sh. Wang, Common fixed point theorems on generalized distance in ordered cone metric spaces, Comput. Math. Appl. 61, 1254–1260 (2011).

[16] M. Đorđević, D. Đorić, Z. Kadelburg, S. Radenović, D. Spasić, Fixed point results under $c$-distance in tvs-cone metric spaces, Fixed Point Theory Appl. 2011:29, doi:10.1186/1687-1812-2011-29 (2011).

[17] W. Sintunavarat, Y.J. Cho, P. Kumam, Common fixed point theorems for c-distance in ordered cone metric spaces, Comput. Math. Appl. 62, 1969–1978 (2011).

[18] Y.J. Cho, Z. Kadelburg, R. Saadati, W. Shatanawi, *Coupled fixed point theorems under weak contractions*, to appear in Discrete Dynamics Nature Soc. (2012).

[19] Lj. Ćirić, H. Lakzian, V. Rakočević, Fixed point theorems for w-cone distance contraction mappings in tvs-cone metric spaces, Fixed Point Theory Appl. 2012:3, doi:10.1186/1687-1812-2012-3 (2012).

[20] T.G. Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Anal. TMA 65, 1379–1393 (2006).

[21] V. Lakshmikantham, Lj. Ćirić, Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Anal. TMA 70, 4341–4349 (2009).

[22] N.V. Luong, N.X. Thuan, Coupled fixed point theorems for mixed monotone mappings and an application to integral equations, Comput. Math. Appl. 62, 4238–4248 (2011).

[23] V. Berinde, Generalized coupled fixed point theorems for mixed monotone mappings in partially ordered metric spaces, Nonlinear Anal. TMA, 74, 7347–7355 (2011).

[24] V. Berinde, Coupled fixed point theorems for $\phi$-contractive mixed monotone mappings in partially ordered metric spaces, Nonlinear Anal. TMA 75, 3218–3228 (2012).

[25] A. Amini-Harandi, Coupled and tripled fixed point theory in partially ordered metric spaces with application to initial value problem, Math. Comput. Modelling (2012), doi:10.1016/j.mcm.2011.12.006.

[26] Z. Golubović, Z. Kadelburg, S. Radenović, Coupled coincidence points of mappings in ordered partial metric spaces, to appear in Abstract Appl. Anal. (2012).

[27] W. Shatanawi, Partially ordered cone metric spaces and coupled fixed point results, Comput. Math. Appl. 60 (2010), 2508–2515.

[28] E. Karapinar, Couple fixed point theorems for nonlinear contractions in cone metric spaces, Comput. Math. Appl. 59, 3656–3668 (2010).

[29] E. Karapinar, D. Turkogolu, Best approximation theorems for a couple in cone Banach spaces, Fixed Point Theory Appl. 2010, Article ID 754578, 9p. (2010).

[30] F. Sabetghadam, H.P. Masiha, A.H. Sanatpour, Some coupled fixed point theorems in cone metric spaces, Fixed Point Theory Appl. 2009, Article ID 125426, 8p. (2009).

[31] W-Sh. Du, Nonlinear contractive conditions for coupled cone fixed point theorems, Fixed Point Theory Appl. 2010, Article ID 190606, 16p. (2010).

[32] W. Shatanawi, On $w$-compatible mappings and common coupled coincidence point in cone metric spaces, Appl. Math. Lett. (2011), doi:10.1016/j.aml.2011.10.037.

[33] H-Sh. Ding, L. Li, Coupled fixed point theorems in partially ordered cone metric spaces, Filomat 25:2, 137–149 (2011).

[34] H. Aydi, B. Samet, C. Vetro, Coupled fixed point results in cone metric spaces for $w$-compatible mappings, Fixed Point Theory Appl. 2011:27, doi:10.1186/1687-1812-2011-27 (2011).

[35] J.S. Ume, Fixed point theorems related to Ćirić contraction principle, J. Math. Anal. Appl. 225, 630–640 (1998).