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COMMON FIXED POINTS OF $(\psi-\varphi)$-WEAK AND GENERALIZED $(\psi-\varphi)$-WEAK CONTRACTION FOR TWO MAPPINGS

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Abstract. In this paper, we discuss some new fixed point theorems for two mappings which satisfy $(\psi-\varphi)$-weak contraction condition and generalized $(\psi-\varphi)$-weak contraction condition.

Keywords: complete Metric Spaces; common fixed point, $(\psi-\varphi)$-weak contraction condition and generalized $(\psi-\varphi)$-weak contraction condition.

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1. Introduction

In 1997, Alben and Cuerre-Delabriere [1] first introduced the concept of φ-weak contractions. Recently, Zhang and Song [2] further defined a new contractive which is generalized φ-weak in 2009. Very recently, Moradi and Farajzadeh [3] introduced the $(\psi-\varphi)$-weak contraction condition and generalized $(\psi-\varphi)$-weak contraction condition. In this paper, motivated by the above work, we prove two fixed point theorems for $(\psi-\varphi)$-weak contraction condition and

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generalized $(\psi-\varphi)$-weak contraction condition mappings. The results presented in this paper mainly extend of the corresponding results in Moradi and Farajzadeh [3].

2. Preliminaries

Let (X,d) be a metric space. A mapping $T : X \longrightarrow X$ is said to be φ-weak contraction, if there exists a map $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(0)=0$ and $\varphi(t)>0$ for all $t>0$ such that for all $x,y \in X$

$$d(Tx,Ty) \le d(x,y) - \varphi(d(x,y)). \tag{2.1}$$

The mapping $T : X \to X$ is said to be generalized φ-weak contraction, if there exist a map $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(0)=0$ and $\varphi(t)>0$ for all $t>0$ such that
for all $x,y \in X$

$$d(Tx,Ty) \le N(x,y) - \varphi(N(x,y)), \tag{2.2}$$

where, $N(x,y) = max\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}[d(x,Ty)+d(y,Tx)]\}$.

The mappings $T : X \longrightarrow X$ is said to be a $(\psi-\varphi)$-weak contraction, if there exist two maps $\psi,\varphi : [0,\infty) \to [0,\infty)$ with $\psi(0)=\varphi(0)=0$, $\psi(t)>0$ and $\varphi(t)>0$ for all $t>0$ such that for all $x,y \in X$

$$\psi(d(Tx,Ty)) \le \psi(d(x,y)) - \varphi(d(x,y)). \tag{2.3}$$

The mappings $T : X \to X$ is said to be generalized $(\psi-\varphi)$-weak contraction, if there exist two maps $\psi,\varphi : [0,\infty) \to [0,\infty)$ with $\psi(0)=\varphi(0)=0$, $\psi(t)>0$ and $\varphi(t)>0$ for all $t>0$ such that for all $x,y \in X$,

$$\psi(d(Tx,Ty)) \le \psi(N(x,y)) - \varphi(N(x,y)). \tag{2.4}$$

Rhoades [4] proved the following fixed point theorem for φ-weak contraction single-valued mappings.

Theorem 2.1. *Let (X,d) be a complete metric space and let $T : X \longrightarrow X$ be a mapping such that*

$$d(Tx,Ty) \le d(x,y) - \varphi(d(x,y)), \tag{2.5}$$

for all $x,y \in X$, where $\varphi : [0,\infty) \longrightarrow [0,\infty)$ is a continuous and nondecreasing function with $\varphi(0)=0$ and $\varphi(t)>0$ for all $t>0$. Then T has a unique fixed point.

Dutta and Choudhury [5] proved the following theorem on the existence of a fixed point for φ-weak contraction mappings and extended Theorem 2.1.

Theorem 2.2. *Let (X,d) be a complete metric space and let $T:X\longrightarrow X$ be a mapping satisfying the inequality*

$$\psi(d(Tx,Ty))\leq\psi(d(x,y))-\varphi(d(x,y)), \tag{2.6}$$

for all $x,y\in X$, where $\psi,\varphi:[0,\infty)\longrightarrow[0,\infty)$ are both continuous nondecreasing mappings with $\varphi(0)=\psi(0)=0$ if and only if $t=0$. Then T has a unique fixed point.

Moradi and Farajzadeh [3] extended Theorem 2.1 and Theorem 2.2 as the following:

Theorem 2.3. *Let (X,d) be a complete metric space and $T:X\longrightarrow X$ is a mapping that satisfies*

$$\psi(d(Tx,Ty))\leq\psi(d(x,y))-\varphi(d(x,y)) \tag{2.7}$$

for all $x,y\in X$ where, $\psi,\varphi:[0,\infty)\to[0,\infty)$ are two mappings with $\psi(0)=\varphi(0)=0$, $\varphi(t)>0$ and $\psi(t)>0$ for all $t>0$. Suppose also that either

(a) ψ is continuous and $\lim_{n\to\infty}t_n=0$ if $\underline{lim}_{n\to\infty}\varphi(t_n)=0$, or

(b) ψ is monotone nondecreasing and $\lim_{n\to\infty}t_n=0$ if $\{t_n\}$ is bounded and $\lim_{n\to\infty}\varphi(t_n)=0$.

Then T has a unique fixed point.

Doric [6] proved the following fixed point theorem for generalized φ-weak contraction single-valued mappings.

Theorem 2.4. *Let (X,d) be a complete metric space and let $T:X\longrightarrow X$ be a mapping satisfying the inequality*

$$\psi(d(Tx,Ty))\leq\psi(N(x,y))-\varphi(N(x,y)), \tag{2.8}$$

for all $x,y\in X$, and

(a) $\psi:[0,\infty)\to[0,\infty)$ is continuous monotone nondecreasing function with $\psi(t)=0$ if and only if $t=0$.

(b) $\varphi:[0,\infty)\to[0,\infty)$is a lower semi-continuous function with $\varphi(t)=0$ if and only if $t=0$.

Then T has a unique fixed point.

Popescu [7] proved the following theorem on the existence of a fixed point for generalized φ-weak contraction mappings and extended Theorem 2.3.

Theorem 2.5. *Let (X,d) be a complete metric space and $T : X \longrightarrow X$ is a mapping satisfying for all $x, y \in X$,*

$$\psi(d(Tx,Ty)) \leq \psi(N(x,y)) - \varphi(N(x,y)), \tag{2.9}$$

where,

(a) $\psi : [0,\infty) \to [0,\infty)$ is a monotone nondecreasing function with $\psi(t) = 0$ if and only if $t = 0$.

(b) $\varphi : [0,\infty) \to [0,\infty)$ is a function with $\varphi(t) = 0$ if and only if $t = 0$ and $\lim_{n\to\infty} \varphi(t_n) > 0$ if $\lim_{n\to\infty} t_n = t > 0$.

(c) $\varphi(a) > \psi(a) - \psi(a^-)$ for any $a > 0$, where $\psi(a^-)$ is the left limit of ψ at a.

Then T has a unique fixed point.

Moradi and Farajzadeh [3] extended the Theorem 2.4 and Theorem 2.5 as following:

Theorem 2.6. *Let (X,d) be a complete metric space and let $T : X \longrightarrow X$ be a mapping that satisfies,*

$$\psi(d(Tx,Ty)) \leq \psi(N(x,y)) - \varphi(N(x,y)), \tag{2.10}$$

for all $x, y \in X$, where, $\varphi : [0,\infty) \longrightarrow [0,\infty)$ is a mapping with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$ and $\lim_{n\to\infty} t_n = 0$, if $\{t_n\}$ is bounded and $\lim_{n\to\infty} \varphi(t_n) = 0$, and $\psi : [0,\infty) \longrightarrow [0,\infty)$ is a mapping with $\psi(0) = 0$ and $\psi(t) > 0$ for all $t > 0$. Also, suppose that either

(a) ψ is continuous, or

(b) ψ is monotone nondecreasing and for all $a > 0$, $\varphi(a) > \psi(a) - \psi(a^-)$, where $\psi(a^-)$ is the left limit of ψ at a.

Then T has a unique fixed point.

Recently, many authors have studied fixed point for $(\psi - \varphi)$-weak contraction conditions; see [6,8-11] and the references therein.

We introduce two types of contraction as follows:

Definition 2.7. Two mappings $S, T : X \longrightarrow X$ are said to be $(\psi - \varphi)$-weak contraction, if there exist two maps $\psi, \varphi : [0,\infty) \longrightarrow [0,\infty)$ with $\psi(0) = \varphi(0) = 0$ and $\psi(t) > 0$, $\varphi(t) > 0$ for all $t > 0$ such that $\psi(d(Sx,Ty)) \leq \psi(d(x,y)) - \varphi(d(x,y))$, for all $x, y \in X$.

Definition 2.8. Two mappings $S,T : X \longrightarrow X$ are said to be generalized $(\psi-\varphi)$-weak contraction, if there exist two maps $\psi,\varphi : [0,\infty) \longrightarrow [0,\infty)$ with $\psi(0)=\varphi(0)=0$ and $\psi(t)>0$, $\varphi(t)>0$ for all $t>0$ such that $\psi(d(Sx,Ty)) \leq \psi(M(x,y)) - \varphi(M(x,y))$, for all $x,y \in X$ where, $M(x,y) = max\{d(x,y), d(x,Sx), d(y,Ty), \frac{1}{2}[d(x,Ty)+d(y,Sx)]\}$.

3. Main results

The following theorem extends Moradi and Farajzadeh Theorem's (cf. [3] Theorem 3.1) to two mappings.

Theorem 3.1. *Let (X,d) be a complete metric space and $S,T : X \longrightarrow X$ be two continuous mappings that satisfy*

$$\psi(d(Sx,Ty)) \leq \psi(d(x,y)) - \varphi(d(x,y)), \tag{3.1}$$

for all $x,y \in X$ where, $\psi,\varphi : [0,\infty) \longrightarrow [0,\infty)$ are two mappings with $\psi(0)=\varphi(0)=0$, $\varphi(t)>0$ and $\psi(t)>0$ for all $t>0$. Suppose also that either
(a) ψ is continuous and $\lim_{n\to\infty} t_n = 0$ if $\lim_{n\to\infty}\varphi(t_n)=0$, or
(b) ψ is monotone nondecreasing and $\lim_{n\to\infty} t_n = 0$ if $\{t_n\}$ is bounded and $\lim_{n\to\infty}\varphi(t_n)=0$.
Then S and T have a common fixed point.

Proof. Let $x_0 \in X$. Define a sequence $\{x_n\}$ by $Tx_{2n} = x_{2n+1}$ and $Sx_{2n+1} = x_{2n+2}$ for all $n \in N \cup \{0\}$. Obviously, if $x_{2n} = x_{2n+1}$ and $x_{2n+1} = x_{2n+2}$ for some $n \in N \cup \{0\}$ then there is nothing to prove. So we may assume that $x_{2n} \neq x_{2n+1}$ and $x_{2n+1} \neq x_{2n+2}$ for all $n \in N \cup \{0\}$. From (3.1), we have

$$\psi(d(x_{2n+2},x_{2n+1})) \leq \psi(d(x_{2n+1},x_{2n})) - \varphi(d(x_{2n+1},x_{2n})), \tag{3.2}$$

for all $n \in N \cup \{0\}$ and hence the sequence $\{\psi(d(x_{m+1},x_m))\}$ is monotone decreasing and bounded below. Thus there exists $r \geq 0$ such that

$$\lim_{n\to\infty} \psi(d(x_{2n+2},x_{2n+1})) = r.$$

Using (3.2), we deduce

$$0 \leq \varphi(d(x_{2n+1},x_{2n})) \leq \psi(d(x_{2n+1},x_{2n})) - \psi(d(x_{2n+2},x_{2n+1})). \tag{3.3}$$

Letting $n \to \infty$ in the inequality (3.3), we get

$$\lim_{n\to\infty} \varphi(d(x_{2n+1}, x_{2n})) = 0.$$

If (a) holds, then by hypothesis

$$\lim_{n\to\infty} d(x_{2n+1}, x_{2n}) = 0. \tag{3.4}$$

We claim that $\{x_n\}$ is a Cauchy sequence. Indeed, if it is false, then there exist $\varepsilon > 0$ and the subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ such that $n(k)$ is minimal in the sense that $n(k) > m(k) > k$ and $d(x_{m(k)}, x_{n(k)}) > \varepsilon$. Therefore, $d(x_{m(k)}, x_{n(k)-1}) \leq \varepsilon$ and by using the triangle inequality, we obtain

$$\begin{aligned} \varepsilon &< d(x_{m(k)}, x_{n(k)}) \\ &\leq d(x_{m(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)}) \\ &\leq d(x_{m(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{m(k)}) + d(x_{m(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)}) \\ &\leq 2d(x_{m(k)}, x_{m(k)-1}) + \varepsilon + d(x_{n(k)-1}, x_{n(k)}). \end{aligned} \tag{3.5}$$

Letting $k \to \infty$ in the above inequality and using (3.4), we get

$$\lim_{n\to\infty} d(x_{m(k)}, x_{n(k)}) = \lim_{n\to\infty} d(x_{m(k)-1}, x_{n(k)-1}) = \varepsilon. \tag{3.6}$$

From (3.1), for all $k \in N$, we find that

$$\psi(d(x_{m(k)}, x_{n(k)})) \leq \psi(d(x_{m(k)-1}, x_{n(k)-1})) - \varphi(d(x_{m(k)-1}, x_{n(k)-1})). \tag{3.7}$$

If (a) holds, then

$$\lim_{n\to\infty} \psi(d(x_{m(k)-1}, x_{n(k)-1})) = \lim_{n\to\infty} \psi(d(x_{m(k)}, x_{n(k)})) = \psi(\varepsilon)$$

and hence from (3.7), we conclude that $\lim_{n\to\infty} \varphi(d(x_{m(k)-1}, x_{n(k)-1})) = 0$. By hypothesis, we have

$$\lim_{n\to\infty} d(x_{m(k)-1}, x_{n(k)-1}) = 0.$$

This a contradiction. If (b) holds, then from (3.7)

$$\varepsilon < d(x_{m(k)}, x_{n(k)}) < d(x_{m(k)-1}, x_{n(k)-1}),$$

and so $d(x_{m(k)},x_{n(k)}) \to \varepsilon^+$ and $d(x_{m(k)-1},x_{n(k)-1}) \to \varepsilon^+$ as $k \longrightarrow \infty$. Hence, we have

$$\lim_{n\to\infty} \psi(d(x_{m(k)-1},x_{n(k)-1})) = \lim_{n\to\infty} \psi(d(x_{m(k)},x_{n(k)})) = \psi(\varepsilon^+),$$

where $\psi(\varepsilon^+)$ is the right limit of ψ. Therefore from (3.7), $\lim_{n\to\infty} \varphi(d(x_{m(k)-1},x_{n(k)-1})) = 0$ by hypothesis, we find that

$$\lim_{n\to\infty} d(x_{m(k)-1},x_{n(k)-1}) = 0.$$

This is a contradiction. Thus $\{x_n\}$ is Cauchy. Since (X,d) is complete and $\{x_n\}$ is Cauchy, it follows that there exists $z \in X$ such that $\lim_{n\to\infty} x_n = z$. We now show that z is a common fixed point of S and T. If (a) is holds, then from (3.1), for all $n \in N \cup \{0\}$

$$\psi(d(x_{2n+2},Tz)) \leq \psi(d(x_{2n+1},z)) - \varphi(d(x_{2n+1},z)). \tag{3.8}$$

Letting $n \to \infty$ in (3.8), using condition (a) and $\lim_{n\to\infty} x_n = z$, we get that

$$\psi(d(z,Tz)) \leq \psi(d(z,z)) = \psi(0) = 0$$

and so $d(z,Tz) = 0$, (note that φ and ψ are nonnegative with $\psi(0) = \varphi(0) = 0$), which implies $z = Tz$. Similarly,

$$\psi(d(Sz,x_{2n+1})) \leq \psi(d(z,x_{2n})) - \varphi(d(z,x_{2n})). \tag{3.9}$$

Letting $n \to \infty$ in (3.9), using condition (a) and $\lim_{n\to\infty} x_n = z$, we get $\psi(d(Sz,z)) \leq \psi(d(z,z)) = \psi(0) = 0$ and so $d(Sz,z) = 0$, which implies $Sz = z$. Since S and T are continuous. Therefore $z = \lim_{n\to\infty} x_{2n+2} = \lim_{n\to\infty} Sx_{2n+1} = Sz$ and $z = \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} Tx_{2n} = Tz$. So, z is a common fixed point of S and T. Let z^* be another common fixed point of S and T (i.e., $Tz^* = z^*$ and $Sz^* = z^*$),

$$\psi(d(z,z^*)) = \psi(d(Sz,Tz^*)) \leq \psi(d(z,z^*)) - \varphi(d(z,z^*)),$$

which implies that $d(z,z^*) = 0$, that is $z = z^*$. Thus we have the uniqueness of the fixed point of S and T. This complete the prove.

The following theorem extends Moradi and Farajzadeh theorem's (cf. [6] Theorem 3.3) to two mappings as the following.

Theorem 3.2. *Let (X,d) be a complete metric space and $S,T : X \longrightarrow X$ be two continuous mappings that satisfy*

$$\psi(d(Sx,Ty)) \leq \psi(M(x,y)) - \varphi(M(x,y)), \tag{3.10}$$

where, $M(x,y) = max\{d(x,y), d(x,Sx), d(y,Ty), \frac{1}{2}[d(x,Ty)+d(y,Sx)]\}$. for all $x,y \in X$, $\varphi : [0,\infty) \to [0,\infty)$ is a mapping with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$ and $\lim_{n\to\infty} t_n = 0$, if $\{t_n\}$ is bounded and $\lim_{n\to\infty} \varphi(t_n) = 0$, and $\psi : [0,\infty) \to [0,\infty)$ is a mappings with $\psi(0) = 0$ and $\psi(t) > 0$ for all $t > 0$. Also, suppose that either

(a) ψ is continuous, or

(b) ψ is monotone nondecreasing and for all $a > 0$, $\varphi(a) > \psi(a) - \psi(a^-)$, where $\psi(a^-)$ is the left limit of ψ at a.

Then S and T have a common fixed point.

Proof. Let $x_0 \in X$. Define the sequence $\{x_n\}$ by $Tx_{2n} = x_{2n+1}$ and $Sx_{2n+1} = x_{2n+2}$ for all $n \in N \cup \{0\}$. Obviously, if $x_{2n} = x_{2n+1}$ and $x_{2n+1} = x_{2n+2}$ for some $n \in N \cup \{0\}$, then there is nothing to prove. So we may assume that $x_{2n} \neq x_{2n+1}$ and $x_{2n+1} \neq x_{2n+2}$ for all $n \in N \cup \{0\}$. From (3.10), we have

$$\psi(d(x_{2n+2}, x_{2n+1})) \leq \psi(M(x_{2n+1}, x_{2n})) - \varphi(M(x_{2n+1}, x_{2n})), \tag{3.11}$$

where

$$M(x_{2n+1}, x_{2n}) = \max\{d(x_{2n+1}, x_{2n}), d(x_{2n+1}, x_{2n+2}), d(x_{2n}, x_{2n+1}), \frac{1}{2}[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})]\}.$$

If $d(x_{2n+1}, x_{2n}) < d(x_{2n+2}, x_{2n+1})$, then from (3.11), we have

$$\begin{aligned} \psi(d(x_{2n+2}, x_{2n+1})) &\leq \psi(d(x_{2n+1}, x_{2n+2})) - \varphi(d(x_{2n+1}, x_{2n+2})) \\ &< \psi(d(x_{2n+1}, x_{2n+2})), \end{aligned} \tag{3.12}$$

and this is a contradiction, so $d(x_{2n+1}, x_{2n+2}) < d(x_{2n}, x_{2n+1})$ and hence, the sequence $\{d(x_{2n}, x_{2n+1})\}$ is monotone nondecreasing and hence bounded. Also, from (3.11) and (3.12), we have

$$\psi(d(x_{2n+2}, x_{2n+1})) \leq \psi(d(x_{2n+1}, x_{2n})) - \varphi(d(x_{2n+1}, x_{2n})). \tag{3.13}$$

Therefore the sequence $\{d(x_{2n+2},x_{2n+1})\}$ is monotone nondecreasing and bounded below. Thus there exists $r \geq 0$ such that $\lim_{n\to\infty} \psi(d(x_{2n},x_{2n+1})) = r$. It follows from (3.13) that

$$\lim_{n\to\infty} \varphi(d(x_{2n},x_{2n+1})) = 0.$$

Since $\{d(x_{2n},x_{2n+1})\}$ is bounded and $\lim_{n\to\infty} \varphi(d(x_{2n},x_{2n+1})) = 0$, we see that

$$\lim_{n\to\infty} d(x_{2n},x_{2n+1}) = 0. \tag{3.14}$$

We now prove that $\{x_n\}$ is a Cauchy sequence. Indeed, if the conclusion does not hold, then there exist $\varepsilon > 0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ such that $n(k)$ is minimal in the sense that $n(k) > m(k) > k$ and $d(x_{m(k)},x_{n(k)}) > \varepsilon$. Therefore, $d(x_{m(k)},x_{n(k)-1}) \leq \varepsilon$.Using the triangle inequality,

$$\begin{aligned}
\varepsilon &< d(x_{m(k)},x_{n(k)}) \\
&\leq d(x_{m(k)},x_{m(k)-1}) + d(x_{m(k)-1},x_{n(k)-1}) + d(x_{n(k)-1},x_{n(k)}) \\
&\leq d(x_{m(k)},x_{m(k)-1}) + d(x_{m(k)-1},x_{m(k)}) + d(x_{m(k)},x_{n(k)-1}) + d(x_{n(k)-1},x_{n(k)}) \\
&\leq 2d(x_{m(k)},x_{m(k)-1}) + \varepsilon + d(x_{n(k)-1},x_{n(k)}).
\end{aligned} \tag{3.15}$$

Letting $k \to \infty$ in the above inequality, we get

$$\lim_{n\to\infty} d(x_{m(k)},x_{n(k)}) = \lim_{n\to\infty} d(x_{m(k)-1},x_{n(k)-1}) = \varepsilon. \tag{3.16}$$

By use of (3.10), we find that

$$\psi(d(x_{m(k)},x_{n(k)})) \leq \psi(M(x_{m(k)-1},x_{n(k)-1})) - \varphi(M(x_{m(k)-1},x_{n(k)-1})), \tag{3.17}$$

where

$$\begin{aligned}
& d(x_{m(k)-1},x_{n(k)-1}) \\
& \leq M(x_{m(k)-1},x_{n(k)-1}) \\
& = \max\{d(x_{m(k)-1},x_{n(k)-1}), d(x_{m(k)-1},x_{m(k)}), d(x_{n(k)-1},x_{n(k)}), \\
& \frac{1}{2}[d(x_{m(k)-1},x_{n(k)}) + d(x_{n(k)-1},x_{m(k)})]\}.
\end{aligned} \tag{3.18}$$

Since (3.16) and (3.18) hold, we conclude that $\lim_{n\to\infty} M(x_{m(k)-1},x_{n(k)-1}) = \varepsilon$. If ψ is continuous, then

$$\lim_{n\to\infty} \psi(d(x_{m(k)},x_{n(k)})) = \lim_{n\to\infty} \psi(M(x_{m(k)-1},x_{n(k)-1})) = \psi(\varepsilon)$$

and hence from (3.17), we conclude that

$$\lim_{n\to\infty}\varphi(M(x_{m(k)-1},x_{n(k)-1}))=0.$$

Since $\{M(x_{m(k)-1},x_{n(k)-1})\}$ is bounded, we conclude that

$$\lim_{n\to\infty}M(x_{m(k)-1},x_{n(k)-1})=0.$$

This is a contradiction. If ψ is monotone nondecreasing, then from (3.17), we find

$$\varepsilon<d(x_{m(k)},x_{n(k)})<M(x_{m(k)-1},x_{n(k)-1}),$$

for all $k\in N\cup\{0\}$. Therefore $d(x_{m(k)},x_{n(k)})\to\varepsilon^{+}$ and $M(x_{m(k)-1},x_{n(k)-1})\to\varepsilon^{+}$ as $k\to\infty$. Hence $\lim_{n\to\infty}\psi(d(x_{m(k)},x_{n(k)}))=\lim_{n\to\infty}\psi(M(x_{m(k)-1},x_{n(k)-1}))=\psi(\varepsilon^{+})$. So from (3.17), $\lim_{n\to\infty}\varphi(M(x_{m(k)-1},x_{n(k)-1}))=0$. Since $\{M(x_{m(k)-1},x_{n(k)-1})\}$ is bounded, we find that

$$\lim_{n\to\infty}M(x_{m(k)-1},x_{n(k)-1})=0.$$

This is a contradiction. Thus $\{x_n\}$ is Cauchy. Since (X,d) is complete and $\{x_n\}$ is Cauchy, it follows that there exists $z\in X$ such that $\lim_{n\to\infty}x_n=z$. We now show that z is a common fixed point of S and T. For all $n\in N\cup\{0\}$, we have

$$M(x_{2n},z))=\max\{d(x_{2n},z),d(x_{2n},x_{2n+1}),d(z,Tz),\frac{1}{2}[d(x_{2n},Tz)+d(z,x_{2n+1})]\}. \tag{3.19}$$

If $Tz\neq z$, then from the above inequality, there exist $n^{*}\in N$ such that for all $n\geq n^{*}$, $M(x_{n},z))=d(z,Tz)$. So for all $n\geq n^{*}$, from (3.19)

$$M(x_{n},z))=d(z,Tz). \tag{3.20}$$

Hence from (3.10) and (3.20), for all $n\geq n^{*}$, we have

$$\psi(d(x_{2n+1},Tz))\leq\psi(d(z,Tz))-\varphi(d(z,Tz)). \tag{3.21}$$

If ψ is continuous, then

$$\begin{aligned}\psi(d(z,Tz))&\leq\psi(d(z,Tz))-\varphi(d(z,Tz))\\&<\psi(d(z,Tz))\end{aligned}$$

and this is a contradiction. If ψ is monotone, then from (3.21) we get, $d(x_{2n+1},Tz)<d(z,Tz)$ for all $n\geq n^{*}$. Letting $n\to\infty$ in (3.21), we get $\psi(a^{-})\leq\psi(a)-\varphi(a)$, where $a=d(z,Tz)$, and

this is a contradiction. Consequently, z is a fixed point of T. Similarly z is a fixed point of S. Let z^* be another common fixed point of S and T (i. e., $Tz^* = z^*$ and $Sz^* = z^*$),

$$\begin{aligned}\psi(d(z,z^*)) &= \psi(d(Sz,Tz^*))\\ &\leq \psi(M(z,z^*)) - \varphi(M(z,z^*))\\ &\leq \psi(d(z,z^*)) - \varphi(d(z,z^*))\\ &< \psi(d(z,z^*)),\end{aligned}$$

which implies that $d(z,z^*) = 0$, that is $z = z^*$. Thus we have the uniqueness of the fixed point of S and T. This completes the theorem.

Conflict of Interests

The authors declare that there is no conflict of interests.

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