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STRONG CONVERGENCE OF THE MANN ITERATIVE SEQUENCE FOR DEMICONTRACTIVE MAPS IN HILBERT SPACES

B.G. AKUCHU

Department of Mathematics, University of Nigeria, Nsukka

Abstract. In this paper, fixed points of demicontractive maps are investigated based on the Mann iterative scheme. Convergence theorems are established in the framework of real Hilbert spaces.

Keywords: fixed point; convergence theorem; demicontractive map; Hilbert space.

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1. Introduction

Let H be a real Hilbert space. Let T be a mapping on H. Recall that T is said to be demicontractive if there exists a constant $k > 0$ such that

$$\|Tx-p\|^2 \leq \|x-p\|^2 + k\|x-Tx\|^2 \tag{1.1}$$

for all $(x,p) \in H \times F(T)$, where $F(T) := \{x \in H : Tx = x\} \neq \emptyset$. More often than not, k is assumed to be in the interval $(0,1)$. However, this is a restriction of convenience. If $k = 1$, then T is called a hemicontractive map.

E-mail addresses: george.akuchu@unn.edu.ng, bgakuchu70@yahoo.com

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Recall that T is said to satisfy condition (A) if there exists $\lambda > 0$ such that

$$\langle x - Tx, x - p\rangle \geq \lambda \|x - Tx\|^2 \tag{1.2}$$

for all $(x, p) \in H \times F(T)$. It is easy to see that inequality (1.2) is equivalent to

$$\langle Tx - p, x - p\rangle \leq \|x - p\|^2 - \lambda \|x - Tx\|^2. \tag{1.3}$$

The above classes of maps were studied independently by Hicks and Kubicek [3] and Maruster [5]. In [1], It is shown that the two classes of maps coincide if $k \in (0,1)$ and $\lambda \in (0, \frac{1}{2})$. The class of demicontractive maps includes the class of quasi-nonexpansive and the class of strictly pseudocontractive maps. Any strictly pseudocontractive mapping with a nonempty fixed point set is demicontractive.

Mann iterative scheme, which was introduced in [2], generated a sequence in the following manner:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n,$$

where $\{\alpha_n\}$ is a real number sequence in $(0,1)$.

Several authors have studied the convergence of the Mann iteration for fixed points of nonlinear mappings in Banach spaces; see, *e.g.*, [2], [3], [4], [5], [6]. The Mann iteration is very efficient for the study of convergence to fixed points of demicontractive mappings. It is well known that demicontractivity alone is not sufficient for the convergence of the Mann iteration; see [1] and the references therein. Some additional smoothness properties of T such as continuity and demiclosedness are necessary.

Recall that T is said to be demiclosed at a point x_0 if whenever $\{x_n\}$ is a sequence in the domain of T such that $\{x_n\}$ converges weakly to $x_0 \in D(T)$ and $\{Tx_n\}$ converges strongly to y_0, then $Tx_0 = y_0$. In [7], Maruster studied the convergence of the Mann iteration for demicontractive maps in finite dimensional spaces with an application to the study of the so-called relaxation algorithm for the solutions of a particular convex feasibility problem. More precisely, he proved the following result.

Theorem 1.1. [7] *Let $T : \Re^m \to \Re^m$ be a nonlinear mapping, where $\Re^m$ is the m-Euclidean space. Suppose the following are satisfied:*

(i) $I-T$ *is demiclosed at* 0

(ii) T *is demicontractive with constant* k, *or equivalently* T *satisfies condition A with* $\lambda = \frac{1-k}{2}$.

Then the Mann iteration sequence converges to a point of $F(T)$ *for any starting* x_0.

Maruster [1] noted that in infinite dimensional spaces, demicontractivity and demiclosedness of T are not sufficient for strong convergence. However, the two conditions ensure weak convergence. More precisely, he proved the following result.

Theorem 1.2 [5] *Let* $T : C \to C$ *be a nonlinear mapping with* $F(T) \neq \emptyset$, *where* C *is a closed convex subset of a real Hilbert space* H. *Suppose the following conditions are satisfied:*

(i) $I-T$ *is demiclosed at* 0

(ii) T *is demicontractive with constant* k, *or equivalently* T *satisfies condition A with* $\lambda = \frac{1-k}{2}$

(iii) $0 < a \leq \alpha_n \leq b < 2\lambda = 1-k$

Then the Mann iteration sequence converges weakly to a fixed point of $F(T)$, *for any starting* x_0.

2. Strong Convergence Theorems

As noted above, demicontractivity and demiclosedness of T are not sufficient for strong convergence of the Mann iteration sequence in infinite dimensional spaces. Some additional conditions on T, or some modifications of the Mann iteration sequence are required for strong convergence to fixed points of demicontractive maps. Such additional conditions or modifications have been studied by several authors; see, *e.g.,* [3], [4], [6], [8], [9] and the references therein.

There is however an interesting connection between the strong convergence of the Mann iteration sequence to a fixed point of a demicontractive map T, and the existence of a non-zero solution of a certain variational inequality. This connection was observed by Maruster [5], and has been studied by several authors. More precisely, Maruster proved the following theorem.

Theorem 2.1. [5] *Suppose* T *satisfies the conditions of the Theorem 1.2. If in addition there exists* $0 \neq h \in H$ *such that*

$$\langle x - Tx, h \rangle \leq 0, \quad \forall x \in D(T), \tag{2.1}$$

then starting from a suitable x_0, *the Mann iteration sequence converges strongly to an element of* $F(T)$.

The conditions on the variational inequality in Theorem 2.1 have been used and generalized by several authors; see, *e.g.,* [8], [9] and the references therein. The existence of a non-zero solution to the variational inequality is sometimes gotten under very stringent conditions. In [1], Maruster and Maruster made the following observation ”It would therefore be interesting to study more closely the existence of a non-zero solution of the variational inequality”.

The purpose of this paper is to provide a monotonicity condition under which the Mann iteration sequence converges strongly to a fixed point of a demicontractive map. The convergence does not need to pass through the variational inequality (2.1). The condition is embodied in the following theorem.

Theorem 2.2. *Suppose* T *satisfies:*

(i) The conditions of Theorem 1.2.

(ii) $\langle Tx, x\rangle \geq ||x||^2 - \lambda||x - Tx||^2$ *for all* $x \in D(T)$, $\lambda > 0$.

Then starting from a suitable x_0, *the Mann iteration sequence converges strongly to an element of* $F(T)$.

Proof. From the demicontractivity of T and condition (ii) of Theorem 2.2, we have

$$\begin{aligned}
\langle x - Tx, x - p\rangle &\geq \lambda||x - Tx||^2 \ \forall x \in D(T), p \in F(T), \lambda > 0 \\
&\geq ||x||^2 - \langle Tx, x\rangle \\
\Leftrightarrow \langle Tx, x\rangle - \langle x, x\rangle + \langle x - Tx, x - p\rangle &\geq 0 \\
\Leftrightarrow \langle Tx - x, x\rangle + \langle x - Tx, x - p\rangle &\geq 0 \\
\Leftrightarrow -\langle x - Tx, x\rangle + \langle x - Tx, x - p\rangle &\geq 0 \\
\Leftrightarrow \langle x - Tx, -x\rangle + \langle x - Tx, x - p\rangle &\geq 0 \\
\Leftrightarrow \langle x - Tx, -p\rangle &\geq 0 \\
\Leftrightarrow \langle x - Tx, p\rangle \leq 0.
\end{aligned}$$

Demicontractive maps T abound for which $p \neq 0$ or for which $F(T)$ is not a singleton set. Thus from our theorem and for such maps, there exists $0 \neq h = p \in F(T) \subseteq H$ such that $\langle x - Tx, h\rangle \leq 0$ is satisfied as in Theorem 2.1. Thus by using Theorem 2.1, the Mann iteration sequence converges strongly to an element of $F(T)$, starting from a suitable $x_0 \in D(T)$.

Remark 2.3. In [1], Maruster and Maruster noted that if T satisfies the positivity type condition $\langle Tx, x\rangle \geq \|x\|^2$, then it is sufficient to find a non-zero solution of the variational inequality (2.1). Observe that any mapping that satisfies the positivity condition also satisfies our condition (ii) in Theorem 2.2. Hence, the class of maps that satisfy our condition is wider than the class of maps that satisfy the positivity condition and hence our condition is weaker than the positivity condition of Maruster and Maruster. Moreover, our condition $0 \neq h \in H$ is the more natural $0 \neq p \in F(T)$.

Conflict of Interests

The author declares that there is no conflict of interests.

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