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HYBRID PROJECTION ALGORITHMS FOR FIXED POINT AND EQUILIBRIUM PROBLEMS IN A BANACH SPACE

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Abstract. In this article, a common solution problem is investigated based on a hybrid projection algorithm. Strong convergence of the algorithm is obtained in a uniformly smooth and strictly convex Banach space which also enjoys the Kadec-Klee property.

Keywords: equilibrium problem; fixed point; generalized projection; quasi-ϕ-nonexpansive mapping; zero point.

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1. Introduction

The theory of fixed points as an important branch of functional analysis has been applied in the study of nonlinear phenomena. A Lot of problems arising in economics, engineering, and physics can be studied by fixed point techniques. Krasnoselskii-Mann iteration, which is also known as a one-step iteration is an classic algorithm to study fixed points of nonlinear operators. However, Krasnoselskii-Mann iteration only enjoys weak convergence for nonexpansive mappings only; see [1] and the reference therein.

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There are a lot of real world problems which exist in infinite dimension spaces. In such problems, strong convergence or norm convergence is often much more desirable than weak convergence. To guarantee the strong convergence of Krasnoselskii-Mann iteration, many authors use different regularization methods. The projection technique which was first introduced by Haugazeau [2] has been considered for the approximation of fixed points of nonexpansive mappings. The advantage of projection methods is that strong convergence of iterative sequences can be guaranteed without any compact assumptions.

In this paper, we study common zero points of a family of maximal monotone operators and common solutions of a system of equilibrium problems based on a projection algorithm. Strong convergence of the algorithm is obtained in a uniformly smooth and strictly convex Banach space which also enjoys the Kadec-Klee property.

2. Preliminaries

Let E be a real Banach space, E^* be the dual space of E and C be a nonempty subset of a E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. Recall that the following equilibrium problem. Find $\bar{x} \in C$ such that

$$f(\bar{x}\, y) \geq 0, \quad \forall y \in C. \tag{2.1}$$

We use $EP(f)$ to denote the solution set of the equilibrium problem (2.1). That is,

$$EP(f) = \{p \in C : f(p, y) \geq 0, \quad \forall y \in C\}.$$

Given a mapping $B : C \to E^*$, let

$$f(x, y) = \langle Bx, y - x\rangle, \quad \forall x, y \in C.$$

Then $\bar{x} \in EP(f)$ iff $\bar{x}$ is a solution of the following variational inequality. Find $\bar{x}$ such that

$$\langle B\bar{x}\, y - \bar{x}\rangle \geq 0, \quad \forall y \in C. \tag{2.2}$$

In order to study the solution of the equilibrium problem (2.1), we assume that f satisfies the following conditions:

(A1) $f(x,x)=0, \forall x \in C$;

(A2) f is monotone, i.e., $f(x,y)+f(y,x) \le 0, \forall x,y \in C$;

(A3)

$$\limsup_{t\downarrow 0} f(tz+(1-t)x,y) \le f(x,y), \forall x,y,z \in C;$$

(A4) for each $x \in C$, $y \mapsto f(x,y)$ is convex and weakly lower semi-continuous.

Recall that the normalized duality mapping J from E to 2^{E^*} is defined by

$$Jx = \{f^* \in E^* : \langle x, f^*\rangle = \|x\|^2 = \|f^*\|^2\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing.

A Banach space E is said to be strictly convex if $\|\frac{x+y}{2}\| < 1$ for all $x,y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is said to be uniformly convex if $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty} \|\frac{x_n+y_n}{2}\| = 1$. Let $U_E = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the Banach space E is said to be smooth provided

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists for each $x,y \in U_E$. It is also said to be uniformly smooth if the above limit is attained uniformly for $x,y \in U_E$. It is well known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E. It is also well known that if E is uniformly smooth if and only if E^* is uniformly convex.

Recall that a Banach space E enjoys Kadec-Klee property if for any sequence $\{x_n\} \subset E$, and $x \in E$ with $x_n \rightharpoonup x$, and $\|x_n\| \to \|x\|$, then $\|x_n - x\| \to 0$ as $n \to \infty$. For more details on Kadec-Klee property, the readers can refer to [3] and the reference therein. It is well known that if E is a uniformly convex Banach spaces, then E enjoys Kadec-Klee property.

Next, we assume that E is a smooth Banach space. Consider the functional defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \quad \forall x,y \in E.$$

Observe that, in a Hilbert space H, the equality is reduced to $\phi(x,y) = \|x-y\|^2$, $x,y \in H$.

As we all know if C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and consequently, it is not available in more general Banach spaces. In this connection, Alber [4] recently introduced a generalized projection operator Π_C in a Banach space E which is an analogue of the metric projection P_C in Hilbert spaces. Recall that the generalized projection $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x, y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x}, x) = \min_{y \in C} \phi(y, x).$$

Existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(x, y)$ and strict monotonicity of the mapping J; see, for example, [3] and [4]. In Hilbert spaces, $\Pi_C = P_C$. It is obvious from the definition of function ϕ that

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|y\| + \|x\|)^2, \quad \forall x, y \in E, \tag{2.3}$$

and

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy\rangle, \quad \forall x, y, z \in E. \tag{2.4}$$

Remark 2.1. If E is a reflexive, strictly convex and smooth Banach space, then $\phi(x, y) = 0$ if and only if $x = y$; for more details, for more details; see [3] for more details.

Let $T : C \to C$ be a mapping. In this paper, we use $F(T)$ to denote the fixed point set of T. T is said to be closed if for any sequence $\{x_n\} \subset C$ such that $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} Tx_n = y_0$, then $Tx_0 = y_0$. In this paper, we use $\to$ and $\rightharpoonup$ to denote the strong convergence and weak convergence, respectively.

A point p in C is said to be an asymptotic fixed point of T [5] iff C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\widetilde{F}(T)$. T is said to be relatively nonexpansive iff $\widetilde{F}(T) = F(T) \neq \emptyset$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. T is said to be quasi-ϕ-nonexpansive [6] iff $F(T) \neq \emptyset$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and

$p \in F(T)$. Recently, a number of projection methods have been investigated for the class of mappings; for more details, see [6-10] and the references therein.

Remark 2.2. The class of quasi-ϕ-nonexpansive mappings is more general than the class of relatively nonexpansive mappings which requires the restriction: $F(T) = \widetilde{F}(T)$.

Remark 2.3. The class of quasi-ϕ-nonexpansive mappings is a generalization of the class of quasi-nonexpansive mappings in Hilbert spaces.

Let A be a multivalued operator from E to E^* with domain $Dom(A) = \{z \in E : Az \neq \emptyset\}$ and range $Ran(A) = \cup\{Az : z \in Dom(A)\}$. An operator A is said to be monotone iff $\langle x_1 - x_2, y_1 - y_2\rangle \geq 0$ for each $x_i \in Dom(A)$ and $y_i \in Ax_i$, $i = 1, 2$. A monotone operator A issaid to be maximal if its graph $Grap(A) = \{(x, y) : y \in Ax\}$ is not properly contained in the graph of any other monotone operator. We know that if A is a maximal monotone operator, then $A^{-1}(0)$ is closed and convex.

Let E be a reflexive, strictly convex and smooth Banach space, and let A be a maximal monotone operator from E to E^*. From Rockafellar [11], we find that $s > 0$ and $x \in E$, there exists a unique $x_s \in D(A)$ such that $Jx \in Jx_s + sAx_s$. If $J_s x = x_s$, then we can define a single valued mapping $J_s : E \to Dom(A)$ by $J_s = (J + sA)^{-1}J$ and such a J_s is called the resolvent of A. We know that $A^{-1}(0) = F(J_s)$ for all $s > 0$.

Lemma 2.4. From [6], we know that $J_s : E \to Dom(A)$ is closed quasi-ϕ-nonexpansive with $A^{-1}(0) = F(J_s)$ for all $s > 0$.

In order to our main results, we also need the following lemmas.

Lemma 2.5 [4] *Let E be a reflexive, strictly convex, and smooth Banach space, C a nonempty, closed, and convex subset of E, and $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \quad \forall y \in C.$$

Lemma 2.6 [4] *Let C be a nonempty, closed, and convex subset of a smooth Banach space E, and $x \in E$. Then $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.7 [12] *Let E be a strictly convex, and smooth Banach space. Let C be a nonempty closed and convex subset of E. Let $T : C \to C$ be a closed quasi-ϕ-nonexpansive mapping. Then $F(T)$ is a closed convex subset of C.*

Lemma 2.8 [13] *Let E be a smooth and uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \to R$ such that $g(0) = 0$ and*

$$\|tx + (1-t)y\|^2 \leq t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)g(\|x - y\|)$$

for all $x, y \in B_r = \{x \in E : \|x\| \leq r\}$ and $t \in [0, 1]$.

Lemma 2.9. *Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4). Let $r > 0$ and $x \in E$. Then*

(a) *[14] There exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle \geq 0, \quad \forall y \in C.$$

(b) *([6], [15]) Define a mapping $T_r : E \to C$ by*

$$S_r x = \{z \in C : f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle, \quad \forall y \in C\}.$$

Then the following conclusions hold:

(1) *S_r is single-valued;*

(2) *S_r is a firmly nonexpansive-type mapping, i.e., for all $x, y \in E$,*

$$\langle S_r x - S_r y, JS_r x - JS_r y \rangle \leq \langle S_r x - S_r y, Jx - Jy \rangle$$

(3) *$F(S_r) = EP(f)$;*

(4) S_r *is quasi-ϕ-nonexpansive;*

(5)

$$\phi(q, S_r x) + \phi(S_r x, x) \leq \phi(q, x), \quad \forall q \in F(S_r);$$

(6) $EP(f)$ *is closed and convex.*

3. Main results

Theorem 3.1. *Let E be a uniformly smooth and strictly convex Banach space which also enjoys the Kadec-Klee property. Let C be a nonempty closed and convex subset of E. Let Λ be an index set. Let $\{s_i\}$ be a positive real number sequence. Let f_i be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and $A_i : E \to E^*$ be a maximal monotone operator such that $Dom(A_i) \subset C$ for every $i \in \Lambda$. Assume that the common solution set $CSS := \cap_{i\in\Lambda} A_i^{-1}(0) \bigcap \cap_{i\in\Lambda} EF(f_i)$ is nonempty. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily}, \\ C_{(1,i)} = C, \\ C_1 = \cap_{i\in\Lambda} C_{(1,i)}, \\ x_1 = \Pi_{C_1} x_0, \\ y_{(n,i)} = J^{-1}(\alpha_{(n,i)} J x_n + (1-\alpha_{(n,i)}) J J_{s_i}^{A_i} x_n), \\ u_{(n,i)} \in C \text{ such that } f_i(u_{(n,i)}, y) + \frac{1}{r_{(n,i)}} \langle y - u_{(n,i)}, J u_{(n,i)} - J y_{(n,i)} \rangle \geq 0, \quad \forall y \in C, \\ C_{(n+1,i)} = \{ z \in C_{(n,i)} : \phi(z, u_{(n,i)}) \leq \phi(z, x_n) \}, \\ C_{n+1} = \cap_{i\in\Lambda} C_{(n+1,i)}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \end{cases}$$

where $J_{s_i}^{A_i} = (J + s_i A_i)^{-1} J$, $\{\alpha_{(n,i)}\}$ is a real sequence in $[0,1]$ such that $\liminf_{n\to\infty} \alpha_{(n,i)}(1-\alpha_{(n,i)}) > 0$, and $\{r_{(n,i)}\}$ is a real sequence in $[a, \infty)$, where a is some positive real number, for every $i \in \Lambda$. Then the sequence $\{x_n\}$ converges strongly to $\Pi_{CSS} x_1$, where Π_{CSS} is the generalized projection from E onto CSS.

Proof. In the light of Remark 2.4, we see that $J_{s_i}^{A_i}$ is closed quasi-ϕ-nonexpansve with $F(J_{s_i}^{A_i}) = A_i^{-1}(0)$. We conclude from Lemma 2.7 and Lemma 2.9 that the common solution set CSS is closed and convex. Next, we prove that C_n is closed, and convex. It suffices to show that, for each fixed but arbitrary $i \in \Lambda$, $C_{(n,i)}$ is closed and convex. This can be proved by induction on n. It is obvious that $C_{(1,i)} = C$ is closed and convex. Assume that $C_{(k,i)}$ is closed and convex for some $k \geq 1$. Let For $z_1, z_2 \in C_{(k+1,i)}$, we see that $z_1, z_2 \in C_{(k,i)}$. It follows that $z = tz_1 + (1-t)z_2 \in C_{(k,i)}$, where $t \in (0,1)$. Notice that

$$\phi(z_1, u_{(k,i)}) \leq \phi(z_1, x_k),$$

and

$$\phi(z_2, u_{(k,i)}) \leq \phi(z_2, x_k),$$

The above inequalities are equivalent to

$$2\langle z_1, Jx_k - Ju_{(k,i)}\rangle \leq \|x_k\|^2 - \|u_{(k,i)}\|^2, \tag{3.1}$$

and

$$2\langle z_2, Jx_k - Ju_{(k,i)}\rangle \leq \|x_k\|^2 - \|u_{(k,i)}\|^2, \tag{3.2}$$

Multiplying t and $(1-t)$ on the both sides of (3.1) and (3.2), respectively yields that and

$$2\langle z, Jx_k - Ju_{(k,i)}\rangle \leq \|x_k\|^2 - \|u_{(k,i)}\|^2.$$

That is,

$$\phi(z, u_{(k,i)}) \leq \phi(z, x_k),$$

where $z \in C_{(k,i)}$. This finds that $C_{(k+1,i)}$ is closed and convex. We conclude that $C_{(n,i)}$ is closed and convex. This in turn implies that $C_n = \cap_{i\in\Lambda} C_{(n,i)}$ is closed, and convex. This implies that $\Pi_{C_{n+1}} x_1$ is well defined.

Next, we show that $CSS \subset C_n$. $CSS \subset C_1 = C$ is clear. Suppose that $CSS \subset C_{(k,i)}$ for some positive integer k. For any $w \in CSS \subset C_{(k,i)}$, we see that

$$\begin{aligned}
&\phi(w, u_{(k,i)}) \\
&= \phi(w, S_{r_{(k,i)}} y_{(k,i)}) \\
&\leq \phi(w, y_{(k,i)}) \\
&= \phi(w, J^{-1}(\alpha_{(k,i)} J x_k + (1-\alpha_{(k,i)}) J J_{s_i}^{A_i} x_k)) \\
&= \|w\|^2 - 2\langle w, \alpha_{(k,i)} J x_k + (1-\alpha_{(k,i)}) J J_{s_i}^{A_i} x_k \rangle \\
&\quad + \|\alpha_{(k,i)} J x_k + (1-\alpha_{(k,i)}) J J_{s_i}^{A_i} x_k\|^2 \qquad (3.3) \\
&\leq \|w\|^2 - 2\alpha_{(k,i)} \langle w, J x_k \rangle - 2(1-\alpha_{(k,i)}) \langle w, J J_{s_i}^{A_i} x_k \rangle \\
&\quad + \alpha_{(k,i)} \|x_k\|^2 + (1-\alpha_{(k,i)}) \|J_{s_i}^{A_i} x_k\|^2 \\
&= \alpha_{(k,i)} \phi(w, x_k) + (1-\alpha_{(k,i)}) \phi(w, J_{s_i}^{A_i} x_k) \\
&\leq \alpha_{(k,i)} \phi(w, x_k) + (1-\alpha_{(k,i)}) \phi(w, x_k) \\
&= \phi(w, x_k),
\end{aligned}$$

which shows that $w \in C_{(k+1,i)}$. This implies that $CSS \subset C_{(n,i)}$. This in turn implies that $CSS \subset \cap_{i\in\Lambda} C_{(n,i)}$. This is completes the proof that $CSS \subset C_n$.

Next, we show that the sequence $\{x_n\}$ is bounded. In view of $x_n = \Pi_{C_n} x_1$, we find from Lemma 2.6 that $\langle x_n - z, Jx_1 - Jx_n \rangle \geq 0$, for any $z \in C_n$. Since $CSS \subset C_n$, we find that

$$\langle x_n - w, Jx_1 - Jx_n \rangle \geq 0, \quad \forall w \in CSS. \tag{3.4}$$

It follows from Lemma 2.5 that

$$\begin{aligned}
\phi(x_n, x_1) &\leq \phi(\Pi_{CSS} x_1, x_1) - \phi(\Pi_{CSS} x_1, x_n) \\
&\leq \phi(\Pi_{CSS)} x_1, x_1).
\end{aligned}$$

This implies that the sequence $\{\phi(x_n, x_1)\}$ is bounded. It follows from (2.3) that the sequence $\{x_n\}$ is also bounded.

Since the space is reflexive, we may assume that $x_n \rightharpoonup \bar{x}$. Next, we show that $\bar{x} \in CSS$. Since C_n is closed, and convex, we find that $\bar{x} \in C_n$. On the other hand, we see from the weakly lower semicontinuity of the norm that

$$\begin{aligned}\phi(\bar{x}, x_1) &= \|\bar{x}\|^2 - 2\langle \bar{x}, Jx_1\rangle + \|x_1\|^2 \\ &\le \liminf_{n\to\infty}(\|x_n\|^2 - 2\langle x_n, Jx_1\rangle + \|x_1\|^2) \\ &= \liminf_{n\to\infty} \phi(x_n, x_1) \\ &\le \limsup_{n\to\infty} \phi(x_n, x_1) \\ &\le \phi(\bar{x}, x_1),\end{aligned}$$

which implies that $\lim_{n\to\infty} \phi(x_n, x_1) = \phi(\bar{x}, x_1)$. Hence, we have $\lim_{n\to\infty} \|x_n\| = \|\bar{x}\|$. In view of Kadec-Klee property of E, we find that $x_n \to \bar{x}$ as $n \to \infty$. Since $x_n = \Pi_{C_n} x_1$, and $x_{n+1} = \Pi_{C_{n+1}} x_1 \in C_{n+1} \subset C_n$, we find that $\phi(x_n, x_1) \le \phi(x_{n+1}, x_1)$. This shows that $\{\phi(x_n, x_1)\}$ is nondecreasing. It follows from its boundedness that $\lim_{n\to\infty} \phi(x_n, x_1)$ exists. In view of the construction of $x_{n+1} = \Pi_{C_{n+1}} x_1 \in C_{n+1} \subset C_n$, we arrive at

$$\begin{aligned}\phi(x_{n+1}, x_n) &= \phi(x_{n+1}, \Pi_{C_n} x_1) \\ &\le \phi(x_{n+1}, x_1) - \phi(\Pi_{C_n} x_1, x_1) \\ &= \phi(x_{n+1}, x_1) - \phi(x_n, x_1).\end{aligned}$$

This implies that

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{3.5}$$

In the light of $x_{n+1} = \Pi_{C_{n+1}} x_1 \in C_{n+1}$, we find that $\phi(x_{n+1}, u_{(n,i)}) \le \phi(x_{n+1}, x_n)$. This implies from (3.5) that

$$\lim_{n\to\infty} \phi(x_{n+1}, u_{(n,i)}) = 0. \tag{3.6}$$

In view of (2.3), we see that $\lim_{n\to\infty}(\|x_{n+1}\| - \|u_{(n,i)}\|) = 0$. It follows that $\lim_{n\to\infty} \|u_{(n,i)}\| = \|\bar{x}\|$. This is equivalent to

$$\lim_{n\to\infty} \|Ju_{(n,i)}\| = \|J\bar{x}\|. \tag{3.7}$$

This implies that $\{Ju_{(n,i)}\}$ is bounded. Note that both E and E^* are reflexive. We may assume that $Ju_{(n,i)} \rightharpoonup u^{(*,i)} \in E^*$. In view of the reflexivity of E, we see that $J(E) = E^*$.

This shows that there exists an element $u^i \in E$ such that $Ju^i = u^{(*,i)}$. It follows that

$$\begin{aligned}\phi(x_{n+1}, u_{(n,i)}) &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, Ju_{(n,i)}\rangle + \|u_{(n,i)}\|^2 \\ &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, Ju_{(n,i)}\rangle + \|Ju_{(n,i)}\|^2.\end{aligned}$$

Taking $\liminf_{n\to\infty}$ on the both sides of the equality above yields that

$$\begin{aligned}0 &\geq \|\bar{x}\|^2 - 2\langle \bar{x}, u^{(*,i)}\rangle + \|u^{(*,i)}\|^2 \\ &= \|\bar{x}\|^2 - 2\langle \bar{x}, Ju^i\rangle + \|Ju^i\|^2 \\ &= \|\bar{x}\|^2 - 2\langle \bar{x}, Ju^i\rangle + \|u^i\|^2 \\ &= \phi(\bar{x}, u^i).\end{aligned}$$

That is, $\bar{x} = u^i$, which in turn implies that $u^{(*,i)} = J\bar{x}$. It follows that $Ju_{(n,i)} \rightharpoonup J\bar{x} \in E^*$. Since E^* enjoys Kadec-Klee property, we obtain from (3.7) that $\lim_{n\to\infty} Ju_{(n,i)} = J\bar{x}$. Since E enjoys the Kadec-Klee property, we obtain that $u_{(n,i)} \to \bar{x}$, as $n \to \infty$. Note that $\|x_n - u_{(n,i)}\| \leq \|x_n - \bar{x}\| + \|\bar{x} - u_{(n,i)}\|$. It follows that

$$\lim_{n\to\infty} \|x_n - u_{(n,i)}\| = 0. \tag{3.8}$$

Notice that

$$\begin{aligned}\phi(w, x_n) - \phi(w, u_{(n,i)}) &= \|x_n\|^2 - \|u_{(n,i)}\|^2 - 2\langle w, Jx_n - Ju_{(n,i)}\rangle \\ &\leq \|x_n - u_{(n,i)}\|(\|x_n\| + \|u_{(n,i)}\|) + 2\|w\|\|Jx_n - Ju_{(n,i)}\|.\end{aligned}$$

In view of (3.8), we find that

$$\lim_{n\to\infty} (\phi(w, x_n) - \phi(w, u_{(n,i)})) = 0. \tag{3.9}$$

Since E is uniformly smooth, we know that E^* is uniformly convex. In view of Lemma 2.8, we find that

$$\begin{aligned}
&\phi(w, u_{(n,i)}) \\
&= \phi(w, S_{r_{(n,i)}} y_{(n,i)}) \\
&\leq \phi(w, y_{(n,i)}) \\
&= \phi(w, J^{-1}(\alpha_{(n,i)} Jx_n + (1-\alpha_{(n,i)}) JJ_{s_i}^{A_i} x_n)) \\
&= \|w\|^2 - 2\langle w, \alpha_{(n,i)} Jx_n + (1-\alpha_{(n,i)}) JJ_{s_i}^{A_i} x_n \rangle \\
&\quad + \|\alpha_{(n,i)} Jx_n + (1-\alpha_{(n,i)}) JJ_{s_i}^{A_i} x_n\|^2 \\
&\leq \|w\|^2 - 2\alpha_{(n,i)} \langle w, Jx_n \rangle - 2(1-\alpha_{(n,i)}) \langle w, JJ_{s_i}^{A_i} x_n \rangle \\
&\quad + \alpha_{(n,i)} \|x_n\|^2 + (1-\alpha_{(n,i)}) \|J_{s_i}^{A_i} x_n\|^2 - \alpha_{(n,i)}(1-\alpha_{(n,i)}) g(\|Jx_n - JJ_{s_i}^{A_i} x_k\|) \\
&= \alpha_{(n,i)} \phi(w, x_n) + (1-\alpha_{(n,i)}) \phi(w, J_{s_i}^{A_i} x_n) - \alpha_{(n,i)}(1-\alpha_{(n,i)}) g(\|Jx_n - JJ_{s_i}^{A_i} x_n\|) \\
&\leq \phi(w, x_n) - \alpha_{(n,i)}(1-\alpha_{(n,i)}) g(\|Jx_n - JJ_{s_i}^{A_i} x_n\|).
\end{aligned}$$

This implies that

$$\alpha_{(n,i)}(1-\alpha_{(n,i)}) g(\|Jx_n - JJ_{s_i}^{A_i} x_n\|) \leq \phi(w, x_n) - \phi(w, u_{(n,i)}).$$

In view of the restrictions on the sequence $\{\alpha_{(n,i)}\}$, we find from (3.9) that

$$\lim_{n\to\infty} \|Jx_n - JJ_{s_i}^{A_i} x_n\| = 0.$$

Notice that $\|JJ_{s_i}^{A_i} x_n - J\bar{x}\| \leq \|JJ_{s_i}^{A_i} x_n - Jx_n\| + \|Jx_n - J\bar{x}\|$. It follows that

$$\lim_{n\to\infty} \|JJ_{s_i}^{A_i} x_n - J\bar{x}\| = 0. \tag{3.10}$$

The demicontinuity of $J^{-1} : E^* \to E$ implies that $J_{s_i}^{A_i} x_n \rightharpoonup \bar{x}$. Note that

$$|\|J_{s_i}^{A_i} x_n\| - \|\bar{x}\|| = |\|JJ_{s_i}^{A_i} x_n\| - \|J\bar{x}\|| \leq \|JJ_{s_i}^{A_i} x_n - J\bar{x}\|.$$

This implies from (3.10) that $\lim_{n\to\infty} \|J_{s_i}^{A_i} x_n\| = \|\bar{x}\|$. Since E has Kadec-Klee property, we obtain that $\lim_{n\to\infty} \|J_{s_i}^{A_i} x_n - \bar{x}\| = 0$. It follows from the closedness of $J_{s_i}^{A_i}$ that $\bar{x} \in F(J_{s_i}^{A_i}) = A_i^{-1}(0)$ for every $i \in \Lambda$. This proves that $\bar{x} \in \cap_{i\in\Lambda} A_i^{-1}(0)$.

Next, we show that $\bar{x} \in \cap_{i\in\Lambda} EF(f_i)$. In view of Lemma 2.5, we find that

$$\begin{aligned}\phi(u_{(n,i)}, y_{(n,i)}) &\leq \phi(w, y_{(n,i)}) - \phi(w, u_{(n,i)})\\ &\leq \phi(w, x_n) - \phi(w, u_{(n,i)}).\end{aligned}$$

It follows from (3.9) that $\lim_{n\to\infty} \phi(u_{(n,i)}, y_{(n,i)}) = 0$. This implies that $\lim_{n\to\infty}(\|u_{(n,i)}\| - \|y_{(n,i)}\|) = 0$. In view of (3.8), we see that $u_{(n,i)} \to \bar{x}$ as $n \to \infty$. This implies that $\|y_{(n,i)}\| - \|\bar{x}\| \to 0$, as $n \to \infty$. It follows that $\lim_{n\to\infty} \|Jy_{(n,i)}\| = \|J\bar{x}\|$. Since E^* is reflexive, we may assume that $Jy_{(n,i)} \rightharpoonup p^{(*,i)} \in E^*$. In view of $J(E) = E^*$, we see that there exists $p^i \in E$ such that $Jp^i = p^{(*,i)}$. It follows that

$$\begin{aligned}\phi(u_{(n,i)}, y_{(n,i)}) &= \|u_{(n,i)}\|^2 - 2\langle u_{(n,i)}, Jy_{(n,i)}\rangle + \|y_{(n,i)}\|^2\\ &= \|u_{(n,i)}\|^2 - 2\langle u_{(n,i)}, Jy_{(n,i)}\rangle + \|Jy_{(n,i)}\|^2.\end{aligned}$$

Taking $\liminf_{n\to\infty}$ the both sides of equality above yields that

$$\begin{aligned}0 &\geq \|\bar{x}\|^2 - 2\langle \bar{x}, p^{(*,i)}\rangle + \|p^{(*,i)}\|^2\\ &= \|\bar{x}\|^2 - 2\langle \bar{x}, Jp^i\rangle + \|Jp^i\|^2\\ &= \|\bar{x}\|^2 - 2\langle \bar{x}, Jp^i\rangle + \|p^i\|^2\\ &= \phi(\bar{x}, p^i).\end{aligned}$$

That is, $\bar{x} = p^i$, which in turn implies that $p^{(*,i)} = J\bar{x}$. It follows that $Jy_{(n,i)} \rightharpoonup J\bar{x} \in E^*$. Since E^* enjoys the Kadec-Klee property, we obtain that $Jy_{(n,i)} - J\bar{x} \to 0$ as $n \to \infty$. Note that $J^{-1} : E^* \to E$ is demi-continuous. It follows that $y_{(n,i)} \rightharpoonup \bar{x}$. Since E enjoys the Kadec-Klee property, we obtain that $y_{(n,i)} \to \bar{x}$ as $n \to \infty$. Note that $\|u_{(n,i)} - y_{(n,i)}\| \leq \|u_{(n,i)} - \bar{x}\| + \|\bar{x} - y_{(n,i)}\|$. This implies that

$$\lim_{n\to\infty} \|u_{(n,i)} - y_{(n,i)}\| = 0. \tag{3.11}$$

Since J is uniformly norm-to-norm continuous on any bounded sets, we have $\lim_{n\to\infty} \|Ju_{(n,i)} - Jy_{(n,i)}\| = 0$. From the assumption $r_{(n,i)} \geq a$, we see that

$$\lim_{n\to\infty} \frac{\|Ju_{(n,i)} - Jy_{(n,i)}\|}{r_{(n,i)}} = 0. \tag{3.12}$$

In view of $u_{(n,i)} = S_{r_{(n,i)}} y_{(n,i)}$, we see that

$$f_i(u_{(n,i)}, y) + \frac{1}{r_{(n,i)}} \langle y - u_{(n,i)}, Ju_{(n,i)} - Jy_{(n,i)} \rangle \geq 0, \quad \forall y \in C.$$

It follows from $(A2)$ that

$$\|y - u_{(n,i)}\| \frac{\|Ju_{(n,i)} - Jy_{(n,i)}\|}{r_n} \geq \frac{1}{r_{(n,i)}} \langle y - u_{(n,i)}, Ju_{(n,i)} - Jy_{(n,i)} \rangle \geq f_i(y, u_{(n,i)}), \quad \forall y \in C.$$

In view of $(A4)$, we find from (3.12) that

$$f_i(y, \bar{x}) \leq 0, \quad \forall y \in C.$$

For $0 < t_i < 1$ and $y \in C$, define $y_{(t,i)} = t_i y + (1 - t_i)\bar{x}$. It follows that $y_{(t,i)} \in C$, which yields that $f(y_{(t,i)}, \bar{x}) \leq 0$. It follows from the $(A1)$ and $(A4)$ that

$$0 = f(y_{(t,i)}, y_{(t,i)}) \leq t_i f(y_{(t,i)}, y) + (1 - t_i) f(y_{(t,i)}, \bar{x}) \leq t_i f(y_{(t,i)}, y).$$

That is,

$$f(y_{(t,i)}, y) \geq 0.$$

Letting $t_i \downarrow 0$, we obtain from $(A3)$ that $f_i(\bar{x}, y) \geq 0, \forall y \in C$. This implies that $\bar{x} \in EP(f_i)$ for every $i \in \Lambda$. This shows that $\bar{x} \in CSS$.

Finally, we prove that $\bar{x} = \Pi_{CSS} x_1$. Letting $n \to \infty$ in (3.4), we see that

$$\langle \bar{x} - w, Jx_1 - J\bar{x} \rangle \geq 0, \quad \forall w \in CSS.$$

In view of Lemma 2.6, we find that that $\bar{x} = \Pi_{CSS} x_1$. This completes the proof.

Remark 3.2. Since every uniformly convex Banach space is a strictly convex Banach space which also enjoys the Kadec-Klee property, we see that Theorem 3.1 is still valid in uniformly smooth and uniformly convex Banach space. Theorem 3.1 improves the corresponding results in Qin, Cho and Kang [6].

For a single bifunction and maximal monotone operator, we find from Theorem 3.1 the following.

Corollary 3.3. *Let E be a uniformly smooth and strictly convex Banach space which also enjoys the Kadec-Klee property. Let C be a nonempty closed and convex subset of E. Let s be a positive real number. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* $(A1)$*-*$(A4)$

and $A : E \to E^*$ be a maximal monotone operator such that $Dom(A) \subset C$. Assume that the common solution set $CSS := A^{-1}(0) \cap EF(f)$ is nonempty. Let $\{x_n\}$ be a sequence generated in the following manner:

$$\begin{cases} x_0 \in E \quad \textit{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J J_s^A x_n), \\ u_n \in C \ \textit{such that} \ f(u_n, y) + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n \rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \end{cases}$$

where $J_s^A = (J+sA)^{-1}J$, $\{\alpha_n\}$ is a real sequence in $[0,1]$ such that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, and $\{r_n\}$ is a real sequence in $[a, \infty)$, where a is some positive real number. Then the sequence $\{x_n\}$ converges strongly to $\Pi_{CSS} x_1$, where Π_{CSS} is the generalized projection from E onto CSS.

Remark 3.4. Since J_s^A is closed quasi-ϕ-nonexpansive, we see that Corollary 3.3 mainly improves the corresponding results in Qin, Cho and Kang [6].

If A is a zero mapping, then we have from Theorem 3.1 the following.

Corollary 3.5. *Let E be a uniformly smooth and strictly convex Banach space which also enjoys the Kadec-Klee property. Let C be a nonempty closed and convex subset of E. Let Λ be an index set. Let f_i be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) for every $i \in \Lambda$. Assume that the common solution set $CSS := \cap_{i\in\Lambda} EF(f_i)$ is nonempty. Let $\{x_n\}$*

be a sequence generated in the following manner:

$$\begin{cases} x_0 \in E \quad \textit{chosen arbitrarily}, \\ C_{(1,i)} = C, \\ C_1 = \cap_{i\in\Lambda} C_{(1,i)}, \\ x_1 = \Pi_{C_1} x_0, \\ u_{(n,i)} \in C \ \textit{such that} \ f_i(u_{(n,i)}, y) + \frac{1}{r_{(n,i)}}\langle y - u_{(n,i)}, Ju_{(n,i)} - Jx_n\rangle \geq 0, \quad \forall y \in C, \\ C_{(n+1,i)} = \{z \in C_{(n,i)} : \phi(z, u_{(n,i)}) \leq \phi(z, x_n)\}, \\ C_{n+1} = \cap_{i\in\Lambda} C_{(n+1,i)}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \end{cases}$$

where $\{\alpha_{(n,i)}\}$ *is a real sequence in* $[0,1]$ *such that* $\liminf_{n\to\infty} \alpha_{(n,i)}(1-\alpha_{(n,i)}) > 0$*, and* $\{r_{(n,i)}\}$ *is a real sequence in* $[a,\infty)$*, where* a *is some positive real number, for every* $i \in \Lambda$*. Then the sequence* $\{x_n\}$ *converges strongly to* $\Pi_{CSS}x_1$*, where* Π_{CSS} *is the generalized projection from* E *onto* CSS*.*

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