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UNIQUE FIXED POINT THEOREMS FOR CONTRACTIVE MAPS TYPE IN T_0-QUASI-METRIC SPACES

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Abstract. In [2], Agyingi proved that every generalized contractive mapping defined in a q-spherically complete T_0-ultra-quasi-metric space has a unique fixed point. In this article, we give and prove a fixed point theorem for C-contractive and S-contractive mappings in a bicomplete di-metric space. The connection between q-spherically complete T_0-ultra-quasi-metric spaces and bicomplete di-metric spaces is pointed out in Proposition 3.1.
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1. Introduction

If we delete, in the usual definition of the pseudometric d on a set X, the symmetry condition $d(x,y) = d(y,x)$ whenever $x,y \in X$, we are led to the concept of quasi-pseudometric. In this context, we recall, as mentioned in [7] (see the Kullback-Leibler distance in information theory) that most of the distance functions considered in science are not necessarily symmetric. Hence, many results established for metric spaces have their equivalent formulation for quasi-pseudometric spaces, the technicality of the proof being completely different.

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Definition 1.1. (Compare [3]) Let (X,m) be a metric space. A map $T : X \to X$ is called a C-contraction if there exists $0 \leq k < \frac{1}{2}$ such that for all $x,y \in X$, the following inequality holds:

$$mTx,Ty) \leq k[m(x,Tx)+m(y,Ty)].$$

Definition 1.2. (Compare[3]) Let (X,m) be a metric space. A map $T : X \to X$ is called a S-contraction if there exists $0 \leq k < \frac{1}{3}$ such that for all $x,y \in X$, the following inequality holds:

$$m(Tx,Ty) \leq k[m(x,Tx)+m(y,Ty)+m(x,y)].$$

2. Preliminaries

In this section, we recall some elementary definitions and terminology from the asymmetric topology which are necessary for a good understanding of the work below. For recent results in the area of Asymmetric Topology, the reader is advised to consult [1, 2, 5, 6, 9].

Definition 2.1. Let X be a non empty set. A function $d : X \times X \to [0;\infty)$ is called a *quasi-pseudometric* on X if:

i) $d(x,x) = 0 \quad \forall\, x \in X$,

ii) $d(x,z) \leq d(x,y)+d(y,z) \quad \forall\, x,y,z \in X$.

Moreover, if $d(x,y) = 0 = d(y,x) \Longrightarrow x = y$, then d is said to be a it T_0-quasi-pseudometric or a *di-metric*. The latter condition is referred as the T_0-condition.

Example 2.1. [1] On $\mathbb{R}\times\mathbb{R}$, we define the real valued map d given by

$$d(a,b) = a \dot{-} b = \max\{a-b,0\}.$$

Then $(\mathbb{R},d)$ is a di-metric space.

Remark 2.1.

- Let d be a quasi-pseudometric on X, then the map d^{-1} defined by $d^{-1}(x,y) = d(y,x)$ whenever $x,y \in X$ is also a quasi-pseudometric on X, called the conjugate of d. (In the literature, it is also denoted by d^t or $\bar{d}$).

- It is easy to verify that the function d^s defined by $d^s := d \vee d^{-1}$, i.e.

$$d^s(x,y) = \max\{d(x,y), d(y,x)\}$$

defines a metric on X whenever d is a T_0-quasi-pseudometric.

- In some cases, we need to replace $[0,\infty)$ by $[0,\infty]$ (where for a d attaining the value ∞, the triangle inequality is interpreted in the obvious way). In such case, we speak of *extended quasi-pseudometric*.

Definition 2.2. The di-metric space (X,d) is said to be *bicomplete* if the metric space (X,d^s) is complete.

Example 2.2. (Compare [9, Example 3]) Let $X = [0;\infty)$. Define for each $x,y \in X$, $n(x,y) = x$ if $x > y$, and $n(x,y) = 0$ if $x \leq y$. It is not difficult to check that (X,n) is a T_0-quasi-pseudometric space.

Notice also that for $x,y \in [0;\infty)$, we have $n^s(x,y) = \max\{x,y\}$ if $x \neq y$ and $n^s(x,y) = 0$ if $x = y$. The metric n^s is complete on $[0,\infty)$.

Definition 2.3. Let (X,d) be a quasi-pseudometric space. For $x \in X$ and $\varepsilon > 0$,

$$B_d(x,\varepsilon) = \{y \in X : d(x,y) < \varepsilon\}$$

denotes the open ε-ball at x. The collection of all such balls is a base for a topology $\tau(d)$ induced by d on X. Similarly, for $x \in X$ and $\varepsilon \geq 0$,

$$C_d(x,\varepsilon) = \{y \in X : d(x,y) \leq \varepsilon\}$$

denotes the closed ε-ball at x.

Definition 2.4. (Compare[1, Definition 5]) Let (X,d) be a quasi-pseudometric space. Let $(x_i)_{i\in I}$ be a family of points of X and let $(r_i)_{i\in I}$ and $(s_i)_{i\in I}$ be families of non-negative real numbers. We say that the family $(C_d(x_i,r_i), C_{d^{-1}}(x_i,s_i))_{i\in I}$ has the *mixed binary intersection property* provided that

$$C_d(x_i,r_i) \cap C_{d^{-1}}(x_j,s_j) \neq \emptyset,$$

for all $i,j \in I$.

Definition 2.5. (Compare[1, Definition 6]) Let (X,d) be a quasi-pseudometric space. We say that (X,d) is *Isbell complete* provided that each family $(C_d(x_i,r_i),C_{d^{-1}}(x_i,s_i))_{i\in I}$ that has the mixed binary intersection property is such that

$$\bigcap_{i\in I}((C_d(x_i,r_i)\cap C_{d^{-1}}(x_i,s_i)))\neq\emptyset.$$

Proposition 2.1. (Compare[1, Proposition 2]) If (X,d) is an extended Isbell-complete quasi-pseudometric space, then (X,d^s) is hypercomplete.

An interesting class of quasi-pseudometric spaces for which we investing a type of completeness are the *ultra-quasi-pseudometric*.

Definition 2.6. (Compare [8, page 2]) Let X be a set and $d : X\times X\to[0;\infty)$ be a function mapping into the set $[0;\infty)$ of non-negative reals. Then d is an *ultra-quasi-pseudometric* on X if

i) $d(x,x)=0$ for all $x\in X$, and
ii) $d(x,z)\leq max\{d(x,y),d(y,z)\}$ whenever $x,y,z\in X$.

The conjugate d^{-1} of d where $d^{-1}(x,y)=d(y,x)$ whenever $x,y\in X$ is also an ultra-quasi-pseudometric on X..

If d also satisfies the T_0-condition, then d is called a T_0-*ultra-quasi-metric* on X. Notice that $d^s=\sup\{d,d^{-1}\}=d\vee d^{-1}$ is an *ultra metric* on X whenever d is a T_0-ultra-quasi-metric.

In the literature, T_0-ultra-quasi-metric spaces are also know as non Archimedean T_0-quasi-metric spaces.

3. q-spherical completeness

In this section we shall recall some results about q-spherical completeness belonging mainly to [8].

Definition 3.2. (Compare [8, Definition 2]) Let (X,d) be an ultra-quasi-pseudometric space. Let $(x_i)_{i\in I}$ be a family of points in X and let $(r_i)_{i\in I}$ and $(s_i)_{i\in I}$ be families of non-negative real numbers. We say that (X,d) is *q-spherically complete* provided that each family

$$(C_d(x_i,r_i),C_{d^{-1}}(x_i,s_i))_{i\in I}$$

satisfying

$$d(x_i,x_j)\leq \max\{r_i,s_j\}$$

whenever $i,j\in I$, is such that

$$\bigcap_{i\in I}(C_d(x_i,r_i)\cap C_{d^{-1}}(x_i,s_i))\neq \emptyset.$$

For an example of a q-spherically complete ultra-quasi-metric space, the reader is advised to check [8, Example 2].

Proposition 3.1. (Compare [8, Proposition 3]) Each q-spherically complete T_0-ultra-quasi-metric space (X,d) is bicomplete.

4. Main results

We recall the following interesting results respectively due to Chatterja in [3] and to Shukla in [4].

Theorem 4.a. (Compare [3]) *A C-contraction on a complete metric space has a unique fixed point.*

Theorem 4.b.(Compare [4]) *An S-contraction on a complete metric space has a unique fixed point.*

The following results generalize the above theorems to the setting of a bicomplete di-metric space.

Definition 4.1. Let (X,d) be a quasi-pseudometric space. A map $T:X\to X$ is called a C-pseudocontraction if there exists $0\leq k<\frac{1}{2}$ such that for all $x,y\in X$, the following inequality holds:

$$d(Tx,Ty)\leq k[d(Tx,x)+d(y,Ty)].$$

Definition 4.2. Let (X,d) be a quasi-pseudometric space. A map $T:X\to X$ is called an S-pseudocontraction if there exists $0\leq k<\frac{1}{3}$ such that for all $x,y\in X$, the following inequality holds:

$$d(Tx,Ty) \leq k[d(Tx,x)+d(y,Ty)+d(x,y)].$$

Theorem 4.1. *Let (X,d) be a bicomplete di-metric space and let $T : X \to X$ be a C-pseudocontraction. Then T has a unique fixed point.*

Proof. Since $T : X \to X$ is a C-pseudocontraction, then there exists $0 \leq k < \frac{1}{2}$ such that for all $x,y \in X$, the following inequality holds:

$$d(Tx,Ty) \leq k[d(Tx,x)+d(y,Ty)].$$

We shall first show that $T : (X,d^s) \to (X,d^s)$ is a C-contraction.

Since for any $x,y \in X$, we have

$$d^{-1}(Tx,Ty) = d(Ty,Tx) \leq k[d(Ty,y)+d(x,Tx)] \leq k[d^{-1}(y,Ty)+d^{-1}(Tx,x)],$$

i.e

$$d^{-1}(Tx,Ty) \leq k[d^{-1}(Tx,x)+d^{-1}(y,Ty)],$$

and we see that $T : (X,d^{-1}) \to (X,d^{-1})$ is a C-pseudocontraction.

Therefore

$$d(Tx,Ty) \leq k[d(Tx,x)+d(y,Ty)] \leq k[d^s(x,Tx)+d^s(y,Ty)],$$

and

$$d^{-1}(Tx,Ty) \leq k[d^{-1}(y,Ty)+d^{-1}(Tx,x)] \leq k[d^s(x,Tx)+d^s(y,Ty)],$$

for all $x,y \in X$. Hence

$$d^s(Tx,Ty) \leq k[d^s(x,Tx)+d^s(y,Ty)],$$

for all $x,y \in X$ and so, $T : (X,d^s) \to (X,d^s)$ is a C-contraction.

By assumption, (X,d) is bicomplete, hence (X,d^s) is complete. Therefore, by Theorem 4.a., T has a unique fixed point. This completes the proof.

Corollary 4.2. *Let (X,d) be a T_0-Isbell-complete quasi-pseudometric space and $T : X \to X$ be a C-pseudocontraction. Then T has a unique fixed point.*

Proof. The proof follows from Proposition 2.1.

Corollary 4.3. *Any C-pseudocontraction on a q-spherically complete T_0-ultra-quasi-metric space has a unique fixed point.*

Proof. The proof follows from Proposition 3.1.

Theorem 4.4. *Let (X,d) be a bicomplete di-metric space and $T : X \to X$ be an S-pseudocontraction. Then T has a unique fixed point.*

Proof. As in the previous proof, it is enough to prove that $T : (X,d^s) \to (X,d^s)$ is an S-contraction.

Since $T : X \to X$ is an S-pseudocontraction, then there exists $0 \leq k < \frac{1}{3}$ such that for all $x,y \in X$, the following inequality holds:

$$d(Tx,Ty) \leq k[d(Tx,x)+d(y,Ty)+d(x,y)].$$

We shall first show that $T : (X,d^s) \to (X,d^s)$ is a C-contraction. Since for any $x,y \in X$, we have

$$\begin{aligned} d^{-1}(Tx,Ty) = d(Ty,Tx) &\leq k[d(Ty,y)+d(x,Tx)+d(y,x)] \\ &\leq k[d^{-1}(y,Ty)+d^{-1}(Tx,x)+d^{-1}(x,y)], \end{aligned}$$

i.e

$$d^{-1}(Tx,Ty) \leq k[d^{-1}(Tx,x)+d^{-1}(y,Ty)+d^{-1}(x,y)],$$

and we see that $T : (X,d^{-1}) \to (X,d^{-1})$ is a C-pseudocontraction.

Therefore

$$d(Tx,Ty) \leq k[d(Tx,x)+d(y,Ty)+d(x,y)] \leq k[d^s(x,Tx)+d^s(y,Ty)+d^s(x,y)],$$

and

$$d^{-1}(Tx,Ty) \leq k[d^{-1}(y,Ty)+d^{-1}(Tx,x)+d^{-1}(x,y)] \leq k[d^s(x,Tx)+d^s(y,Ty)+d^s(x,y)],$$

for all $x,y \in X$. Hence

$$d^s(Tx,Ty) \leq k[d^s(x,Tx)+d^s(y,Ty)+d^s(x,y)],$$

for all $x,y \in X$ and so, $T:(X,d^s) \to (X,d^s)$ is an S-contraction. By assumption, (X,d) is bicomplete, hence (X,d^s) is complete. Therefore, by Theorem 4.b., T has a unique fixed point. This completes the proof.

Corollary 3.5. *Let (X,d) be a T_0-Isbell-complete quasi-pseudometric space and $T:X \to X$ be an S-pseudocontraction. Then T has a unique fixed point.*

Corollary 4.6. *Any S-pseudocontraction on a q-spherically complete T_0-ultra-quasi-metric space has a unique fixed point.*

Conflict of Interests

The authors declare that there is no conflict of interests.

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