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COMMON FIXED POINT THEOREMS VIA WEAKLY COMPATIBLE MAPPINGS IN COMPLETE G-METRIC SPACES: USING CONTROL FUNCTIONS

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Abstract. In this paper, we proved common fixed points for class of mappings using control functions and satisfying contractive conditions in G-metric spaces. We get some improved and extended versions of several fixed point theorems in complete G-metric spaces.

Keywords: contractive mappings; weakly compatible mappings; complete G-metric space.

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1. Introduction

Dhage introduced the concept of D-metric spaces as generalization of ordinary metric functions and went on to present several fixed point results for single and multivalued mappings; see [1-4] and the references therein. Mustafa and Sims [11] generalized the concept of a metric space. Based on the notion of generalized metric spaces, Mustafa *et al.* obtained some

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fixed point theorems for mappings which satisfy different contractive conditions; see [10-14] for more details. Abbas and Rhoades [6] initiated the study of a common fixed point theory in generalized metric spaces. While, Abbas *et al.* [7] and Chugh *et al.* [8] obtained some fixed point results for mappings satisfying property P in G-metric spaces. Recently, Shatanawi [9] further proved some fixed point results for self mappings in a complete G-metric space under some contractive conditions related to a nondecreasing map $\phi : R^+ \to R^+$ with $\lim_{n\to\infty} \phi^n(t) = 0$ for all $t \geq 0$; see [9] for more details.

2. Preliminaries

Now we give basic definitions and some basic results which are helpful for proving our main result.

In 2006, Mustafa and Sims [11] introduced the concept of G-metric spaces as follows.

Definition 2.1. Let X be a nonempty set, and let $G : X \times X \times X \to R^+$ be a function satisfying the following properties:

(G-1) $G(x,y,z) = 0$ if $x = y = z$;

(G-2) $0 < G(x,x,y)$, for all $x,y \in X$ with $x \neq y$;

(G-3) $G(x,x,y) \leq G(x,y,z)$ for all $x,y,z \in X$ with $y \neq z$;

(G-4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \ldots\ldots$, symmetry in all three variables;

(G-5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ for all $x,y,z,a \in X$. The function G is called a generalized or a G-metric on X and the pair (X,G) is called a G-metric space.

Definition 2.2. A G - metric space (X,G) is said to be G-complete if every G-Cauchy sequence in (X,G) is G-convergent in X.

Definition 2.3. Let (X,G) be a G-metric space and let $\{x_n\}$ be a sequence of points of X. A point $x \in X$ is said to be the limit of the sequence $\{x_n\}$, if $\lim_{n,m\to\infty} G(x,x_n,x_m) = 0$ and we say that the sequence $\{x_n\}$ is G-convergent to x or $\{x_n\}$ G-converges to x.

Thus, $x_n \to x$ in a G-metric space (X,G) if for any $\varepsilon > 0$, there exists $k \in N$ such that $G(x,x_n,x_m) < \varepsilon$ for all $m,n > k$.

Proposition 2.1. Let (X,G) be a G-metric space. Then the following are equivalent:

(1) $\{x_n\}$ is G-convergent to x;

(2) $G(x_n,x_n,x) \to 0$ as $n \to \infty$;

(3) $G(x_n,x,x) \to 0$ as $n \to \infty$;

(4) $G(x_n,x_m,x) \to 0$ as $n,m \to \infty$.

Definition 2.4. Let (X,G) be a G-metric space, a sequence $\{x_n\}$ is called G-Cauchy if for every $\varepsilon > 0$, there is $k \in N$ such that $G(x_n,x_m,x_l) < \varepsilon$ for all $n,m,l \geq k$; that is $G(x_n,x_m,x_l) \to 0$ as $n,m,l \to \infty$.

Proposition 2.2. Let (X,G) be a G-metric space. Then the following are equivalent:

(1) $\{x_n\}$ is G-cauchy;

(2) for every $\varepsilon > 0$, there is $k \in N$, $G(x_n,x_n,x_m) < \varepsilon$ for all $n,m \geq k$.

Definition 2.5. Let A and B be two mappings from a G-metric space (X,G). Then the pair (A,B) is said to be weakly compatible pair if they commute at their coincidence point, that is $Ax = Bx$ implies that $ABx = BAx$ for all $x \in X$.

Define $\Phi = \{\phi : R^+ \to R^+\}$, where $R^+ = [0,\infty)$ and for each $\phi \in \Phi$ satisfies the following conditions:

(ϕ-1) ϕ is strict increasing;

(ϕ-2) ϕ is upper semi continuous from the right;

(ϕ-3) $\sum_{n=0}^{\infty} \phi(t) < \infty$ for all $t > 0$;

(ϕ-4) $\phi(0) = 0$.

3. Main results

Theorem 3.1. *Let A,B,C,S,R and T be self mappings of a complete G-metric space (X,G) and*

(i) *$A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$, $C(X) \subseteq R(X)$ and $A(X)$ or $B(X)$ or $C(X)$ is a closed subset of X.*

(ii)

$$G(Ax,By,Cz) \leq \phi\Big\{max\Big\{\alpha[G(Rx,Ty,Sz)+G(Rx,By,Cz)], \beta[G(Rx,Ax,By)+G(Ty,By,Cz) + G(Sz,Cz,Ax)+G(Ax,Rx,Ty)+G(By,Ty,Sz)+G(Cz,Rx,Sz)], \gamma[G(Rx,By,Ty)+G(Ty,Cz,Sz) + G(Sz,Ax,Rx)+G(Sz,Cz,Ax)+G(Ty,Ax,By)]\Big\}\Big\},$$

where $\alpha,\beta,\gamma \geq 0$ *and* $3\alpha+7\beta+6\gamma<1$.

(iii) $\phi : R^+ \to R^+$ *is increasing function such that* $\phi(t)<t$ *for all* $t>0$ *and* $\sum\phi(t)<\infty$ *as* $t\to\infty$.

(iv) *The pairs* (A,R), (B,T) *and* (C,S) *are weakly compatible pairs.*

Then the mappings A,B,C,S,T *and* R *have a unique common fixed point in* X.

Proof. Let $x_0 \in X$ be an arbitrary point. By (i) there exist $x_1,x_2,x_3 \in X$ such that $Ax_0 = Tx_1 = y_0$, $Bx_1 = Sx_2 = y_1$ and $Cx_2 = Rx_3 = y_2$. Inductively construct a sequence $\{y_n\}$ in X such that $Ax_{3n} = Tx_{3n+1} = y_{3n}$, $Bx_{3n+1} = Sx_{3n+2} = y_{3n+1}$ and $Cx_{3n+2} = Rx_{3n+3} = y_{3n+2}$ for $n = 0,1,2,3....$

We prove the sequence is a Cauchy sequence. Let $d_m = G(y_m, y_{m+1}, y_{m+2})$. Then we have

$$\begin{aligned}
d_{3n} &= G(y_{3n}, y_{3n+1}, y_{3n+2}) \\
&= G(Ax_{3n}, Bx_{3n+1}, Cx_{3n+2}) \\
&\leq \phi\Big\{max\Big\{\alpha[G(Rx_{3n}, Tx_{3n+1}, Sx_{3n+2}) + G(Rx_{3n}, Bx_{3n+1}, Cx_{3n+2})], \\
&\beta[G(Rx_{3n}, Ax_{3n}, Bx_{3n+1}) + G(Tx_{3n+1}, Bx_{3n+1}, Cx_{3n+2}) + G(Sx_{3n+2}, Cx_{3n+2}, Ax_{3n}) \\
&+ G(Ax_{3n}, Rx_{3n}, Tx_{3n+1}) + G(Bx_{3n+1}, Tx_{3n+1}, Sx_{3n+2}) + G(Cx_{3n+1}, Rx_{3n}, Sx_{3n+2})], \\
&\gamma[G(Rx_{3n}, Bx_{3n+1}, Tx_{3n+1}) + G(Tx_{3n+1}, Cx_{3n+2}, Sx_{3n+2}) + G(Sx_{3n+2}, Ax_{3n}, Rx_{3n}) \\
&+ G(Sx_{3n+2}, Cx_{3n+2}, Ax_{3n}) + G(Tx_{3n+1}, Ax_{3n}, Bx_{3n+1})]\Big\}\Big\} \\
&\leq \phi\Big\{max\Big\{\alpha[G(y_{3n-1}, y_{3n}, y_{3n+1}) + G(y_{3n-1}, y_{3n+1}, y_{3n+2})], \beta[G(y_{3n-1}, y_{3n}, y_{3n+1}) \\
&+ G(y_{3n}, y_{3n+1}, y_{3n+2}) + G(y_{3n+1}, y_{3n+2}, y_{3n}) + G(y_{3n}, y_{3n-1}, y_{3n}) + G(y_{3n+1}, y_{3n}, y_{3n+1}) \\
&+ G(y_{3n+2}, y_{3n-1}, y_{3n+1})], \gamma[G(y_{3n-1}, y_{3n+1}, y_{3n}) + G(y_{3n}, y_{3n+2}, y_{3n+1}) \\
&+ G(y_{3n+1}, y_{3n}, y_{3n-1}) + G(y_{3n+1}, y_{3n+2}, y_{3n}) + G(y_{3n}, y_{3n}, y_{3n+1})]\Big\}\Big\} \\
&\leq \phi\Big\{max\Big\{\alpha[2d_{3n-1} + d_{3n}], \beta[d_{3n-1} + d_{3n} + d_{3n} + d_{3n-1} + d_{3n} + (d_{3n-1} + d_{3n})], \\
&\gamma[d_{3n-1} + d_{3n} + d_{3n-1} + d_{3n} + d_{3n}]\Big\}\Big\}.
\end{aligned}$$

In above inequality, there arises 3 case:

Case I. If $max = \alpha[2d_{3n-1} + d_{3n}]$, i.e. $d_{3n} = \phi(\alpha[2d_{3n-1} + d_{3n}])$, we prove that $d_{3n} \leq d_{3n-1}$ for every $n \in N$. If $d_{3n} > d_{3n-1}$ for some $n \in N$ by above inequality, we have $d_{3n} \leq \phi(3\alpha d_{3n})$; $d_{3n} < 3\alpha d_{3n}$ as $\phi(t) < t$; $d_{3n} < d_{3n}$ as $3\alpha + 7\beta + 6\gamma < 1$, which is contradiction. So we have $d_{3n} \leq d_{3n-1}$.

Case II. If $max = \beta[d_{3n-1} + d_{3n} + d_{3n} + d_{3n-1} + d_{3n} + (d_{3n-1} + d_{3n})]$, i.e. $d_{3n} = \phi(\beta[d_{3n-1} + d_{3n} + d_{3n} + d_{3n-1} + d_{3n} + (d_{3n-1} + d_{3n})])$, we prove that $d_{3n} \leq d_{3n-1}$ for every $n \in N$. If $d_{3n} > d_{3n-1}$ for some $n \in N$ by above inequality, we have $d_{3n} \leq \phi(7\beta d_{3n})$; $d_{3n} < 7\beta d_{3n}$ as $\phi(t) < t$; $d_{3n} < d_{3n}$ as $3\alpha + 7\beta + 6\gamma < 1$, which is contradiction. So we have $d_{3n} \leq d_{3n-1}$.

Case III: If $max = \gamma[d_{3n-1} + d_{3n} + d_{3n-1} + d_{3n} + d_{3n}]$, i.e. $d_{3n} = \phi(\gamma[d_{3n-1} + d_{3n} + d_{3n-1} + d_{3n} + d_{3n}])$, we prove that $d_{3n} \leq d_{3n-1}$ for every $n \in N$. If $d_{3n} > d_{3n-1}$ for some $n \in N$ by above inequality, we have $d_{3n} \leq \phi(5\gamma d_{3n})$; $d_{3n} < 5\gamma d_{3n}$ as $\phi(t) < t$; $d_{3n} < d_{3n}$ as $3\alpha + 7\beta + 6\gamma < 1$, which is contradiction. So we have $d_{3n} \leq d_{3n-1}$.

If $m = 3n+1$, then

$$\begin{aligned}
d_{3n+1} &= G(y_{3n+1}, y_{3n+2}, y_{3n+3}) \\
&= G(Ax_{3n+3}, Bx_{3n+1}, Cx_{3n+2}) \\
&\leq \phi\Big\{max\Big\{\alpha[G(Rx_{3n+3}, Tx_{3n+1}, Sx_{3n+2}) + G(Rx_{3n+3}, Bx_{3n+1}, Cx_{3n+2})], \\
&\beta[G(Rx_{3n+3}, Ax_{3n+3}, Bx_{3n+1}) + G(Tx_{3n+1}, Bx_{3n+1}, Cx_{3n+2}) + G(Sx_{3n+2}, Cx_{3n+2}, Ax_{3n+3}) \\
&+ G(Ax_{3n+3}, Rx_{3n+3}, Tx_{3n+1}) + G(Bx_{3n+1}, Tx_{3n+1}, Sx_{3n+2}) + G(Cx_{3n+2}, Rx_{3n+3}, Sx_{3n+2})], \\
&\gamma[G(Rx_{3n+3}, Bx_{3n+1}, Tx_{3n+1}) + G(Tx_{3n+1}, Cx_{3n+2}, Sx_{3n+2}) + G(Sx_{3n+2}, Ax_{3n+3}, Rx_{3n+3}) \\
&+ G(Sx_{3n+2}, Cx_{3n+2}, Ax_{3n+3}) + G(Tx_{3n+1}, Ax_{3n+3}, Bx_{3n+1})]\Big\}\Big\} \\
&\leq \phi\Big\{max\Big\{\alpha[G(y_{3n+2}, y_{3n}, y_{3n+1}) + G(y_{3n+2}, y_{3n+1}, y_{3n+2})], \\
&\beta[G(y_{3n+2}, y_{3n}, y_{3n+1}) + G(y_{3n}, y_{3n+1}, y_{3n+2}) + G(y_{3n+1}, y_{3n+2}, y_{3n+3}) \\
&+ G(y_{3n+3}, y_{3n+2}, y_{3n}) + G(y_{3n+1}, y_{3n}, y_{3n+1}) + G(y_{3n+2}, y_{3n+2}, y_{3n+1})], \\
&\gamma[G(y_{3n+2}, y_{3n+1}, y_{3n}) + G(y_{3n}, y_{3n+2}, y_{3n+1}) + G(y_{3n+1}, y_{3n+3}, y_{3n+2}) \\
&+ G(y_{3n+1}, y_{3n+2}, y_{3n+3}) + G(y_{3n}, y_{3n+3}, y_{3n+1})]\Big\}\Big\} \\
&\leq \phi\Big\{max\Big\{\alpha[d_{3n} + d_{3n+1}], \beta[d_{3n+1} + d_{3n} + d_{3n+1} + (d_{3n+1} + d_{3n}) + d_{3n} + d_{3n+1}], \\
&\gamma[d_{3n} + d_{3n} + d_{3n+1} + d_{3n+1} + (d_{3n} + d_{3n+1})]\Big\}\Big\}.
\end{aligned}$$

In the above inequality, there arises 3 case:

Case I. If $max = \alpha[d_{3n} + d_{3n+1}]$, we now prove that $d_{3n+1} \leq d_{3n}$ for every $n \in N$. If $d_{3n+1} > d_{3n}$ for some $n \in N$ by above inequality, we have $d_{3n} \leq \phi(2\alpha d_{3n})$; $d_{3n} < 2\alpha d_{3n}$ as $\phi(t) < t$; $d_{3n} < d_{3n}$ as $3\alpha + 7\beta + 6\gamma < 1$, which is contradiction. So we have $d_{3n+1} \leq d_{3n}$.

Case II. If $max = \beta[d_{3n+1} + d_{3n} + d_{3n+1} + (d_{3n+1} + d_{3n}) + d_{3n} + d_{3n+1}]$, we prove that $d_{3n+1} \leq d_{3n}$ for every $n \in N$. If $d_{3n+1} > d_{3n}$ for some $n \in N$ by above inequality, we have $d_{3n} \leq \phi(7\beta d_{3n})$; $d_{3n} < 7\beta d_{3n}$ as $\phi(t) < t$; $d_{3n} < d_{3n}$ as $3\alpha + 7\beta + 6\gamma < 1$, which is contradiction. So we have $d_{3n+1} \leq d_{3n}$.

Case III. If $max = \gamma[d_{3n} + d_{3n} + d_{3n+1} + d_{3n+1} + (d_{3n} + d_{3n+1})]$, we prove that $d_{3n+1} \leq d_{3n}$ for every $n \in N$. If $d_{3n+1} > d_{3n}$ for some $n \in N$ by above inequality, we have $d_{3n} \leq \phi(6\gamma d_{3n})$; $d_{3n} < 6\gamma d_{3n}$ as $\phi(t) < t$; $d_{3n} < d_{3n}$ as $3\alpha + 7\beta + 6\gamma < 1$, which is contradiction. So we have $d_{3n+1} \leq d_{3n}$.

Further if $m = 3n+2$, then

$$
\begin{aligned}
d_{3n+2} &= G(y_{3n+2}, y_{3n+3}, y_{3n+4}) \\
&= G(Ax_{3n+3}, Bx_{3n+4}, Cx_{3n+2}) \\
&\leq \phi\Big\{max\Big\{\alpha[G(Rx_{3n+3}, Tx_{3n+4}, Sx_{3n+2}) + G(Rx_{3n+3}, Bx_{3n+4}, Cx_{3n+2})], \\
&\beta[G(Rx_{3n+3}, Ax_{3n+3}, Bx_{3n+4}) + G(Tx_{3n+4}, Bx_{3n+4}, Cx_{3n+2}) + G(Sx_{3n+2}, Cx_{3n+2}, Ax_{3n+3}) \\
&+ G(Ax_{3n+3}, Rx_{3n+3}, Tx_{3n+4}) + G(Bx_{3n+4}, Tx_{3n+4}, Sx_{3n+2}) + G(Cx_{3n+2}, Rx_{3n+3}, Sx_{3n+2})], \\
&\gamma[G(Rx_{3n+3}, Bx_{3n+4}, Tx_{3n+4}) + G(Tx_{3n+4}, Cx_{3n+2}, Sx_{3n+2}) + G(Sx_{3n+2}, Ax_{3n+3}, Rx_{3n+3}) \\
&+ G(Sx_{3n+2}, Cx_{3n+2}, Ax_{3n+3}) + G(Tx_{3n+4}, Ax_{3n+3}, Bx_{3n+4})]\Big\}\Big\} \\
&\leq \phi\Big\{max\Big\{\alpha[G(y_{3n+2}, y_{3n}, y_{3n+1}) + G(y_{3n+2}, y_{3n+4}, y_{3n+2})], \\
&\beta[G(y_{3n+2}, y_{3n+3}, y_{3n+4}) + G(y_{3n+3}, y_{3n+4}, y_{3n+2}) + G(y_{3n+1}, y_{3n+2}, y_{3n+3}) \\
&+ G(y_{3n+3}, y_{3n+2}, y_{3n+3}) + G(y_{3n+4}, y_{3n+3}, y_{3n+1}) + G(y_{3n+2}, y_{3n+2}, y_{3n+1})], \\
&\gamma[G(y_{3n+2}, y_{3n+4}, y_{3n+3}) + G(y_{3n+3}, y_{3n+2}, y_{3n+1}) + G(y_{3n+1}, y_{3n+3}, y_{3n+2}) \\
&+ G(y_{3n+1}, y_{3n+2}, y_{3n+3}) + G(y_{3n+3}, y_{3n+3}, y_{3n+1})]\Big\}\Big\} \\
&\leq \phi\Big\{max\Big\{\alpha[d_{3n+1} + d_{3n+2}], \beta[d_{3n+2} + d_{3n+2} + d_{3n+1} + d_{3n+2} + (d_{3n+1} + d_{3n+2}) + d_{3n+1}], \\
&\gamma[d_{3n+2} + d_{3n+1} + d_{3n+1} + d_{3n+1} + d_{3n+2}]\Big\}\Big\}.
\end{aligned}
$$

In the above inequality, there arises 3 case:

Case I. If $max = \alpha[d_{3n+1} + d_{3n+2}]$, we now prove that $d_{3n+2} \leq d_{3n+1}$ for every $n \in N$. If $d_{3n+2} > d_{3n+1}$ for some $n \in N$ by above inequality, we have $d_{3n+2} \leq \phi(2\alpha d_{3n+2})$; $d_{3n+2} < 2\alpha d_{3n+2}$ as $\phi(t) < t$; $d_{3n+2} < d_{3n+2}$ as $3\alpha + 7\beta + 6\gamma < 1$, which is contradiction. So we have $d_{3n+2} \leq d_{3n+1}$.

Case II. If $max = \beta[d_{3n+2} + d_{3n+2} + d_{3n+1} + d_{3n+2} + (d_{3n+1} + d_{3n+2}) + d_{3n+1}]$, we prove that $d_{3n+2} \leq d_{3n+1}$ for every $n \in N$. If $d_{3n+2} > d_{3n+1}$ for some $n \in N$ by above inequality, we have

$d_{3n+2} \leq \phi(7\beta d_{3n+2})$; $d_{3n+2} < 7\beta d_{3n+2}$ as $\phi(t) < t$; $d_{3n+2} < d_{3n+2}$ as $3\alpha + 7\beta + 6\gamma < 1$, which is contradiction. So we have $d_{3n+2} \leq d_{3n+1}$.

Case III. If $max = \gamma[d_{3n+2} + d_{3n+1} + d_{3n+1} + d_{3n+1} + d_{3n+2}]$, we prove that $d_{3n+2} \leq d_{3n+1}$ for every $n \in N$. If $d_{3n+2} > d_{3n+1}$ for some $n \in N$ by above inequality, we have $d_{3n+2} \leq \phi(5\gamma d_{3n+2})$; $d_{3n+2} < 5\gamma d_{3n+2}$ as $\phi(t) < t$; $d_{3n+2} < d_{3n+2}$ as $3\alpha + 7\beta + 6\gamma < 1$, which is contradiction. So we have $d_{3n+2} \leq d_{3n+1}$. Hence for every $n \in N$ we have $d_n \leq d_{n-1}$. Thus by above inequality we have $d_n \leq q d_{n-1}$m, where $q = 3\alpha + 7\beta + 6\gamma < 1$, i.e. $d_n = G(y_n, y_{n+1}, y_{n+2}) \leq qG(y_{n-1}, y_n, y_{n+1}) \leq q^n G(y_0, y_1, y_2)$. Now we have $G(x,x,y) \leq G(x,y,z)$. Therefore we have

$$G(y_n, y_n, y_{n+1}) \leq q^n G(y_0, y_1, y_2)$$

and

$$G(y_n, y_n, y_m) \leq G(y_n, y_n, y_{n+1}) + G(y_{n+1}, y_{n+1}, y_{n+2}) + ... + G(y_{m-1}, y_{m-1}, y_m).$$

Hence, we have

$$\begin{aligned} G(y_n, y_n, y_m) &\leq q^n G(y_0, y_1, y_2) + q^{n+1} G(y_0, y_1, y_2) + ... + q^{m-1} G(y_0, y_1, y_2) \\ &\leq \frac{q^n - q^m}{1-q} G(y_0, y_1, y_2) \\ &\leq \frac{q^n}{1-q} G(y_0, y_1, y_2) \to 0. \end{aligned}$$

So the sequence $\{y_n\}$ is Cauchy in X and $\{y_n\}$ converges to y in X, *i.e.,* $\lim_{n\to\infty} y_n = y$

$$\begin{aligned} \lim_{n,m\to\infty} y_n = \lim_{n,m\to\infty} Ax_{3n} = \lim_{n,m\to\infty} Bx_{3n+1} = \lim_{n,m\to\infty} Cx_{3n+2} \\ = \lim_{n,m\to\infty} Tx_{3n+1} = \lim_{n,m\to\infty} Sx_{3n+2} = \lim_{n,m\to\infty} Rx_{3n+3} = y. \end{aligned}$$

Let $C(X)$ be a closed subset of $R(X)$. Then there exist $u \in X$ such that $Ru = y$. Notice that

$$\begin{aligned} G(Au, Bx_{3n+1}, Cx_{3n+2}) \leq \phi\Big\{max\Big\{ &\alpha[G(Ru, Tx_{3n+1}, Sx_{3n+2}) + G(Ru, Bx_{3n+1}, Cx_{3n+2})], \\ &\beta[G(Ru, Au, Bx_{3n+1}) + G(Tx_{3n+1}, Bx_{3n+1}, Cx_{3n+2}) + G(Sx_{3n+2}, Cx_{3n+2}, Au) \\ &+ G(Au, Ru, Tx_{3n+1}) + G(Bx_{3n+1}, Tx_{3n+1}, Sx_{3n+2}) + G(Cx_{3n+2}, Ru, Sx_{3n+2})], \\ &\gamma[G(Ru, Bx_{3n+1}, Tx_{3n+1}) + G(Tx_{3n+1}, Cx_{3n+2}, Sx_{3n+2}) + G(Sx_{3n+2}, Au, Ru) \\ &+ G(Sx_{3n+2}, Cx_{3n+2}, Au) + G(Tx_{3n+1}, Au, Bx_{3n+1})]\Big\}\Big\}. \end{aligned}$$

Letting $n \to \infty$, we get

$$G(Au, Bx_{3n+1}, Cx_{3n+2}) = G(Au, y, y) \leq \phi\Big\{max\Big\{\alpha[G(Ru, y, y) + G(Ru, y, y)], \beta[G(Ru, Au, y) + G(y, y, y) + G(y, y, Au) + G(Au, Ru, y) + G(y, y, y) + G(y, Ru, y)], \gamma[G(Ru, y, y) + G(y, y, y) + G(y, Au, Ru) + G(y, y, Au) + G(y, Au, y)]\Big\}\Big\}.$$

This implies that

$$G(Au, y, y) \leq \phi(max\{2\alpha G(y, y, y), 3\beta G(y, Au, y), 3\gamma G(y, Au, y)\}).$$

In the above inequality, following case arise:

Case I. If max = $3\beta G(y, Au, y)$, $G(Au, y, y) \leq \phi(3\beta G(y, Au, y))$, $G(Au, y, y) < 3\beta G(y, Au, y)$, $G(Au, y, y) < G(y, Au, y)$ as $3\alpha + 7\beta + 6\gamma < 1$. This leads to contradiction. Thus $G(Au, y, y) = 0$ $\Rightarrow Au = y$.

Case II. If max = $3\gamma G(y, Au, y)$, $G(Au, y, y) \leq \phi(3\gamma G(y, Au, y))$, $G(Au, y, y) < 3\gamma G(y, Au, y)$, $G(Au, y, y) < G(y, Au, y)$ as $3\alpha + 7\beta + 6\gamma < 1$. This leads to contradiction. Thus $G(Au, y, y) = 0$ $\Rightarrow Au = y$. Therefore $Au = Ru = y$. By weak compatibility of the pair (R, A), we have $ARu = RAu$, hence $Ay = Ry$.

We prove that $Ay = y$. If $Ay \neq y$, then

$$G(Ay, Bx_{3n+1}, Cx_{3n+2}) \leq \phi\Big\{max\Big\{\alpha[G(Ry, Tx_{3n+1}, Sx_{3n+2}) + G(Ry, Bx_{3n+1}, Cx_{3n+2})], \beta[G(Ry, Ay, Bx_{3n+1}) + G(Tx_{3n+1}, Bx_{3n+1}, Cx_{3n+2}) + G(Sx_{3n+2}, Cx_{3n+2}, Ay) + G(Ay, Ry, Tx_{3n+1}) + G(Bx_{3n+1}, Tx_{3n+1}, Sx_{3n+2}) + G(Cx_{3n+2}, Ry, Sx_{3n+2})], \gamma[G(Ry, Bx_{3n+1}, Tx_{3n+1}) + G(Tx_{3n+1}, Cx_{3n+2}, Sx_{3n+2}) + G(Sx_{3n+2}, Ay, Ry) + G(Sx_{3n+2}, Cx_{3n+2}, Ay) + G(Tx_{3n+1}, Ay, Bx_{3n+1})]\Big\}\Big\}.$$

Letting $n \to \infty$, we get

$$\begin{aligned}G(Ay,Bx_{3n+1},Cx_{3n+2}) &= G(Ay,y,y)\\ &\leq \phi\Big\{max\Big\{\alpha[G(Ry,y,y)+G(Ry,y,y)],\beta[G(Ry,Ay,y)+G(y,y,y)\\ &+G(y,y,Ay)+G(Ay,Ry,y)+G(y,y,y)+G(y,Ry,y)],\\ &\gamma[G(Ry,y,y)+G(y,y,y)+G(y,Ay,Ry)+G(y,y,Ay)+G(y,Ay,y)]\Big\}\Big\}.\end{aligned}$$

This implies that

$$G(Ay,y,y) \leq \phi(max\{2\alpha G(Ay,y,y),4\beta G(y,Ay,y),4\gamma G(y,Ay,y)\}).$$

Now there arises 3 case:

Case I. If max = $2\alpha G(Ay,y,y)$, $G(Ay,y,y) \leq \phi(2\alpha G(Ay,y,y))$, $G(Ay,y,y) < 2\alpha G(Ay,y,y)$, $G(Ay,y,y) < G(Ay,y,y)$ as $3\alpha+7\beta+6\gamma < 1$. This leads to contradiction. Thus $G(Ay,y,y) = 0$ $\Rightarrow Ay = y$.

Case II. If max = $4\beta G(Ay,y,y)$, $G(Ay,y,y) \leq \phi(4\beta G(Ay,y,y))$, $G(Ay,y,y) < 4\beta G(Ay,y,y)$, $G(Ay,y,y) < G(Ay,y,y)$ as $3\alpha+7\beta+6\gamma < 1$. This leads to contradiction. Thus $G(Ay,y,y) = 0$ $\Rightarrow Ay = y$.

Case III. If max = $4\gamma G(Ay,y,y)$, $G(Ay,y,y) \leq \phi(4\gamma G(Ay,y,y))$, $G(Ay,y,y) < 4\gamma G(Ay,y,y)$, $G(Ay,y,y) < G(Ay,y,y)$ as $3\alpha+7\beta+6\gamma < 1$. This leads to contradiction. Thus $G(Ay,y,y) = 0$ $\Rightarrow Ay = y$. Hence $Ay = y$ and $Ry = Ay \Rightarrow Ay = Ry = y$. Hence$y$ is common fixed point of R and A. Since $y = Ay \in A(X) \subseteq T(X)$, there exists $v \in X$ such that $Tv = y$. We prove that $Bv = y$.

$$\begin{aligned}G(y,Bv,Cx_{3n+2}) &= G(Ay,Bv,Cx_{3n+2})\\ &\leq \phi\Big\{max\Big\{\alpha[G(Ry,Tv,Sx_{3n+2})+G(Ry,Bv,Cx_{3n+2})],\beta[G(Ry,Ay,Bv)\\ &+G(Tv,Bv,Cx_{3n+2})+G(Sx_{3n+2},Cx_{3n+2},Ay)+G(Ay,Ry,Tv)+G(Bv,Tv,Sx_{3n+2})\\ &+G(Cx_{3n+2},Ry,Sx_{3n+2})],\gamma[G(Ry,Bv,Tv)+G(Tv,Cx_{3n+2},Sx_{3n+2})\\ &+G(Sx_{3n+2},Ay,Ry)+G(Sx_{3n+2},Cx_{3n+2},Ay)+G(Tv,Ay,Bv)]\Big\}\Big\}.\end{aligned}$$

Letting $n \to \infty$, we get

$$\begin{aligned}G(y,Bv,y) &= G(y,Bv,y)\\ &\leq \phi\Big\{max\Big\{\alpha[G(y,y,y)+G(y,Bv,y)], \beta[G(y,y,Bv)+G(y,Bv,y)\\ &+G(y,y,y)+G(y,y,y)+G(Bv,y,y)+G(y,y,y)],\\ &\gamma[G(y,Bv,y)+G(y,y,y)+G(y,y,y)+G(y,y,y)+G(y,y,Bv)]\Big\}\Big\}.\end{aligned}$$

This implies that

$$G(y,Bv,y) \leq \phi(max\{\alpha G(y,Bv,y), 3\beta G(y,y,Bv), 2\gamma G(y,y,Bv)\}).$$

In above inequality, there arises 3 case:

Case I. If max = $\alpha G(y,Bv,y)$, $G(y,Bv,y) \leq \phi(\alpha G(y,Bv,y))$, $G(y,Bv,y) < \alpha G(y,Bv,y))$, $G(y,Bv,y) < G(y,Bv,y))$ as $3\alpha+7\beta+6\gamma < 1$. This leads to contradiction. Thus $G(y,Bv,y) = 0 \Rightarrow Bv = y$.

Case II. If max = $3\beta G(y,y,Bv)$, $G(y,Bv,y) \leq \phi(3\beta G(y,y,Bv))$, $G(y,Bv,y) < 3\beta G(y,y,Bv)$, $G(y,Bv,y) < G(y,y,Bv)$ as $3\alpha+7\beta+6\gamma < 1$. This leads to contadiction. Thus $G(y,Bv,y) = 0$ $\Rightarrow Bv = y$.

Case III. If max = $2\gamma G(y,y,Bv)$, $G(y,Bv,y) \leq \phi(2\gamma GG(y,y,Bv))$, $G(y,Bv,y) < 2\gamma G(y,y,Bv)$, $G(y,Bv,y) < G(y,y,Bv)$ as $3\alpha+7\beta+6\gamma < 1$. This leads to contradiction. Thus $G(y,Bv,y) = 0$ $\Rightarrow Bv = y$. Therefore $Bv = Ty = y$. By weak compatibility of (B,T) we have $BTv = TBv$. Hence $By = Ty$. We prove $By = y$. If $By \neq y$, then

$$\begin{aligned}G(Ay,By,Cx_{3n+2}) &\leq \phi\Big\{max\Big\{\alpha[G(Ry,Ty,Sx_{3n+2})+G(Ry,By,Cx_{3n+2})], \beta[G(Ry,Ay,By)\\ &+G(Ty,By,Cx_{3n+2})+G(Sx_{3n+2},Cx_{3n+2},Ay)+G(Ay,Ry,Ty)+G(By,Ty,Sx_{3n+2})\\ &+G(Cx_{3n+2},Ry,Sx_{3n+2})], \gamma[G(Ry,By,Ty)+G(Ty,Cx_{3n+2},Sx_{3n+2})\\ &+G(Sx_{3n+2},Ay,Ry)+G(Sx_{3n+2},Cx_{3n+2},Ay)+G(Ty,Ay,By)]\Big\}\Big\}.\end{aligned}$$

Letting $n \to \infty$, we find

$$\begin{aligned}G(y,By,y) &\leq \phi\Big\{max\Big\{\alpha[G(y,y,y)+G(y,By,y)], \beta[G(y,y,By)+G(y,By,y)+G(y,y,y)\\ &+G(y,y,y)+G(By,y,y)+G(y,y,y)], \gamma[G(y,By,y)+G(y,y,y)+G(y,y,y)\\ &+G(y,y,y)+G(y,y,By)]\Big\}\Big\}.\end{aligned}$$

This implies that

$$G(y,By,y) \leq \phi\Big\{max\Big\{\alpha G(y,By,y), 3\beta G(y,By,y), 2\gamma G(y,By,y)\Big\}\Big\}.$$

In above inequality, there arises 3 case:

Case I. If max = $\alpha G(y,By,y)$, $G(y,By,y) \leq \phi(\alpha G(y,By,y))$, $G(y,By,y) < \alpha G(y,By,y))$, $G(y,By,y) < G(y,By,y))$ as $3\alpha + 7\beta + 6\gamma < 1$. This leads to contadiction. Thus $G(y,By,y) = 0 \Rightarrow By = y$.

Case II. If max = $3\beta G(y,By,y)$, $G(y,By,y) \leq \phi(3\beta G(y,By,y))$, $G(y,By,y) < 3\beta G(y,By,y)$, $G(y,By,y) < G(y,By,y)$ as $3\alpha + 7\beta + 6\gamma < 1$. This leads to contradiction. Thus $G(y,Bv,y) = 0$ $\Rightarrow Bv = y$.

Case III: If max = $2\gamma G(y,By,y)$, $G(y,By,y) \leq \phi(2\gamma G(y,By,y)$, $G(y,By,y) < 2\gamma G(y,By,y)$, $G(y,By,y) < G(y,By,y)$ as $3\alpha + 7\beta + 6\gamma < 1$. This leads to contadiction. Thus $G(y,By,y) = 0$ $\Rightarrow By = y$. Also $Ty = y \Rightarrow By = Ty = y$, i.e. y is a common fixed point of B and T. similarly since $y = By \in B(X) \subseteq S(X)$ there exist $w \in X$ such that $Sw = y$. We prove that $Cw = y$. If $Cw \neq y$, we have

$$\begin{aligned}
G(y,y,Cw) &= G(Ay,By,Cw) \\
&\leq \phi\Big\{max\Big\{\alpha[G(Ry,Ty,Sw) + G(Ry,By,Cw)], \beta[G(Ry,Ay,By) + G(Ty,By,Cw) \\
&\quad + G(Sw,Cw,Ay) + G(Ay,Ry,Ty) + G(By,Ty,Sw) + G(Cw,Ry,Sw)], \gamma[G(Ry,By,Ty) \\
&\quad + G(Ty,Cw,Sw) + G(Sw,Ay,Ry) + G(Sw,Cw,Ay) + G(Ty,Ay,By)]\Big\}\Big\},
\end{aligned}$$

$$\begin{aligned}
G(y,y,Cw) \leq \phi\Big\{max\Big\{&\alpha[G(y,y,y) + G(y,y,Cw)], \beta[G(y,y,y) + G(y,y,Cw) + G(y,Cw,y) \\
&+ G(y,y,y) + G(y,y,y) + G(Cw,y,y)], \gamma[G(y,y,y) + G(y,Cw,y) + G(y,y,y) \\
&+ G(y,Cw,y) + G(y,y,y)]\Big\}\Big\}.
\end{aligned}$$

This implies that

$$G(y,y,Cw) \leq \phi(max\{\alpha G(y,y,Cw), 3\beta G(y,y,Cw), 2\gamma G(y,y,Cw)\}).$$

In the above inequality, there arises 3 case:

Case I. If max = $\alpha G(y,y,Cw)$, $G(y,y,Cw) \leq \phi(\alpha G(y,y,Cw))$, $G(y,y,Cw) < \alpha G(y,y,Cw))$, $G(y,y,Cw) < G(y,y,Cw)$ as $3\alpha+7\beta+6\gamma < 1$. This leads to contradiction. Thus $G(y,y,Cw) = 0$ $\Rightarrow cw = y$.

Case II. If max = $3\beta G(y,y,Cw)$, $G(y,y,Cw) \leq \phi(3\beta G(y,y,Cw))$, $G(y,y,Cw) < 3\beta G(y,y,Cw)$, $G(y,y,Cw) < G(y,y,Cw)$ as $3\alpha+7\beta+6\gamma < 1$. This leads to contadiction. Thus $G(y,y,Cw) = 0$ $\Rightarrow Cw = y$.

Case III. If max = $2\gamma G(y,y,Cw)$, $G(y,y,Cw) \leq \phi(2\gamma G(y,y,Cw)$, $G(y,y,Cw) < 2\gamma G(y,y,Cw)$, $G(y,y,Cw) < G(y,y,Cw)$ as $3\alpha+7\beta+6\gamma < 1$. This leads to contradiction. Thus $G(y,y,Cw) = 0$ $\Rightarrow Cw = y = Sy$. Therefore $Cw = Sw = y$. By weak compatibility of (C,S) we have $CSw = SCw$. Hence $Cy = Sy$. We prove that $Cy = y$. If $Cy \neq y$, then

$$\begin{aligned}
G(y,y,Cy) &= G(Ay,By,Cy) \\
&\leq \phi\Big\{max\Big\{\alpha[G(Ry,Ty,Sy)+G(Ry,By,Cy)], \beta[G(Ry,Ay,By)+G(Ty,By,Cy) \\
&+G(Sy,Cy,Ay)+G(Ay,Ry,Ty)+G(By,Ty,Sy)+G(Cy,Ry,Sy)], \gamma[G(Ry,By,Ty) \\
&+G(Ty,Cy,Sy)+G(Sy,Ay,Ry)+G(Sy,Cy,Ay)+G(Ty,Ay,By)]\Big\}\Big\}.
\end{aligned}$$

This implies that

$$G(y,y,Cy) \leq \phi\Big\{max\Big\{\alpha G(y,y,Cy), 3\beta G(y,y,Cy), 2\gamma G(y,y,Cy)\Big\}\Big\}.$$

In the above inequality, there arises 3 case:

Case I. If max = $\alpha G(y,y,Cy)$, $G(y,y,Cy) \leq \phi(\alpha G(y,y,Cy))$, $G(y,y,Cy) < \alpha G(y,y,Cy))$, $G(y,y,Cy) < G(y,y,Cy)$ as $3\alpha+7\beta+6\gamma < 1$. This leads to contadiction. Thus $G(y,y,Cy) = 0 \Rightarrow cy = y$.

Case II. If max = $3\beta G(y,y,Cy)$, $G(y,y,Cy) \leq \phi(3\beta G(y,y,Cy))$, $G(y,y,Cy) < 3\beta G(y,y,Cy)$, $G(y,y,Cy) < G(y,y,Cy)$ as $3\alpha+7\beta+6\gamma < 1$. This leads to contradiction. Thus $G(y,y,Cy) = 0 \Rightarrow Cy = y$.

Case III. If max = $2\gamma G(y,y,Cy)$, $G(y,y,Cy) \leq \phi(2\gamma G(y,y,Cy)$, $G(y,y,Cy) < 2\gamma G(y,y,Cy)$, $G(y,y,Cy) < G(y,y,Cy)$ as $3\alpha+7\beta+6\gamma < 1$. This leads to contradiction. Thus $G(y,y,Cy) = 0$ $\Rightarrow Cy = y$. Also $Sy = y \Rightarrow Sy = Cy = y$ i.e y is a common fixed point of S and C. Thus y is Common fixed point of A,B,C,S,T and R. i.e. $Ay = Sy = By = Ty = Cy = Ry = y$. Next uniqueness is established. Let v be another fixed point of A,B,C,S,T and R. If $G(y,y,v)>0$,

$$G(y,y,Cv) \leq \phi\Big\{max\Big\{\alpha[G(Ry,Ty,Sv)+G(Ry,By,Cv)],\beta[G(Ry,Ay,By)+G(Ty,By,Cv)+G(Sv,Cv,Ay)+G(Ay,Ry,Ty)+G(By,Ty,Sv)+G(Cv,Ry,Sv)],\gamma[G(Ry,By,Ty)+G(Ty,Cv,Sv)+G(Sv,Ay,Ry)+G(Sv,Cv,Ay)+G(Ty,Ay,By)]\Big\}\Big\}.$$

This implies that $G(y,y,Cv) \leq \phi\Big\{max\Big\{2\alpha G(y,y,Cv), 4\beta G(y,y,Cv), 3\gamma G(y,y,Cv)\Big\}\Big\}$. In above inequality, there arises 3 case:

Case I. If max = $2\alpha G(y,y,Cv)$, $G(y,y,Cv) \leq \phi(2\alpha G(y,y,Cv))$, $G(y,y,Cv) < 2\alpha G(y,y,Cv))$, $G(y,y,Cv) < G(y,y,Cv)$ as $3\alpha+7\beta+6\gamma<1$. This leads to contradiction. Thus $G(y,y,Cv)=0$ $\Rightarrow Cv=y$.

Case II. If max = $4\beta G(y,y,Cv)$, $G(y,y,Cv) \leq \phi(4\beta G(y,y,Cv))$, $G(y,y,Cv) < 4\beta G(y,y,Cv)$, $G(y,y,Cv) < G(y,y,Cv)$ as $3\alpha+7\beta+6\gamma<1$. This leads to contadiction. Thus $G(y,y,Cv)=0$ $\Rightarrow Cv=y$.

Case III: If max = $3\gamma G(y,y,Cv)$, $G(y,y,Cv) \leq \phi(3\gamma G(y,y,Cv)$, $G(y,y,Cv) < 3\gamma G(y,y,Cv)$, $G(y,y,Cv) < G(y,y,Cv)$ as $3\alpha+7\beta+6\gamma<1$. This leads to contadiction. Thus $G(y,y,Cv)=0$ $\Rightarrow Cv=y$. Hence $y=v$ is unique common fixed point of A,B,C,S,T and R. This completes the proof.

If we put $R=S$ and $C=B$ in Theorem (3.1), then we obtain the following corollary.

Corollary 3.2. *Let A,B,S and T be self mappings of a complete G-metric space (X,G) and*

(i) *$A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$ and $A(X)$ or $B(X)$ is a closed subset of X.*

(ii)

$$G(Ax,By,Bz) \leq \phi\Big\{max\Big\{\alpha[G(Sx,Ty,Sz)+G(Sx,By,Bz)],\beta[G(Sx,Ax,By)+G(Ty,By,Bz)+G(Sz,Bz,Ax)+G(Ax,Sx,Ty)+G(By,Ty,Sz)+G(Bz,Sx,Sz)],\gamma[G(Sx,By,Ty)+G(Ty,Bz,Sz)+G(Sz,Ax,Sx)+G(Sz,Bz,Ax)+G(Ty,Ax,By)]\Big\}\Big\},$$

where $\alpha,\beta,\gamma \geq 0$ and $3\alpha+7\beta+6\gamma<1$.

(iii) *$\phi : R^+ \to R^+$ is increasing function such that $\phi(t)<t$ for all $t>0$ and $\sum\phi(t)<\infty$ as $t\to\infty$.*

(iv) *The pairs* (A,S), (B,T) *are weakly commuting pairs.*

Then the mapping A,B,S *and* T *have a unique common fixed point in* X.

If we put $S=T$ and $B=A$ in Corollary 3.2, then we obtain the following corollary.

Corollary 3.3. *Let* A *and* T *be self mappings of a complete G-metric space* (X,G) *and*

(i) $A(X)\subseteq T(X)$ *and* $A(X)$ *is a closed subset of* X.

(ii)

$$G(Ax,Ay,Az)\le\phi\Big\{max\{\alpha[G(Tx,Ty,Tz)+G(Tx,Ay,Az)],\beta[G(Tx,Ax,Ay)+G(Ty,Ay,Az)$$
$$+G(Tz,Az,Ax)+G(Ax,Tx,Ty)+G(Ay,Ty,Tz)+G(Az,Tx,Tz)],$$
$$\gamma[G(Tx,Ay,Ty)+G(Ty,Az,Tz)+G(Tz,Ax,Tx)+G(Tz,Az,Ax)$$
$$+G(Ty,Ax,Ay)]\Big\}\Big\},$$

where $\alpha,\beta,\gamma\ge 0$ *and* $3\alpha+7\beta+6\gamma<1$.

(iii) $\phi:R^+\to R^+$ *is increasing function such that* $\phi(t)<t$ *for all* $t>0$ *and* $\sum\phi(t)<\infty$ *as* $n\to\infty$.

(iv) *The pairs* (A,T) *is weakly commuting pair.*

Then the mapping A *and* T *have a unique common fixed point in* X.

If we put $T=I$ (identity map) in Corollary 3.3, then we obtain the following corollary.

Corollary 3.4. *Let* A *and* T *be self mappings of a complete G-metric space* (X,G) *and*

(i) $A(X)\subseteq I(X)$ *and* $A(X)$ *is a closed subset of* X.

(ii)

$$G(Ax,Ay,Az)\le\phi\Big\{max\{\alpha[G(x,y,z)+G(x,Ay,Az)],\beta[G(x,Ax,Ay)+G(y,Ay,Az)$$
$$+G(z,Az,Ax)+G(Ax,x,y)+G(Ay,y,z)+G(Az,x,z)],$$
$$\gamma[G(x,Ay,y)+G(y,Az,z)+G(z,Ax,x)+G(z,Az,Ax)$$
$$+G(y,Ax,Ay)]\Big\}\Big\},$$

where $\alpha,\beta,\gamma\ge 0$ *and* $3\alpha+7\beta+6\gamma<1$.

(iii) *$\phi : R^+ \to R^+$ is increasing function such that $\phi(t) < t$ for all $t > 0$ and $\sum \phi(t) < \infty$ as $n \to \infty$.*

(iv) *The pairs (A,I) is weakly commuting pair.*

Then the mapping A and I have a unique common fixed point in X.

Theorem 3.4. *Let S,R,T, $\{A_i\}_{i\in I}$, $\{B_j\}_{j\in J}$ and $\{C_k\}_{k\in K}$ be the set of self mappings of a complete G-metric space (X,G) and*

(i) *There exists $i_0 \in I$, $j_0 \in J$ and $k_0 \in K$ such that $A_{i_0}(X) \subseteq T(X)$, $B_{j_0}(X) \subseteq S(X)$, $C_{k_0}(X) \subseteq R(X)$ and $A_{i_0}(X)$ or $B_{j_0}(X)$ or $C_{k_0}(X)$ is a closed subset of X.*

(ii)

$$G(A_ix,B_jy,C_kz) \le \phi\Big\{max\Big\{\alpha[G(Rx,Ty,Sz)+G(Rx,B_jy,C_kz)],\beta[G(Rx,A_ix,B_jy)+G(Ty,B_jy,C_kz)$$

$$+G(Sz,C_kz,A_ix)+G(A_ix,Rx,Ty)+G(B_jy,Ty,Sz)+G(C_kz,Rx,Sz)],$$

$$\gamma[G(Rx,B_jy,Ty)+G(Ty,C_kz,Sz)+G(Sz,A_ix,Rx)+G(Sz,C_kz,A_ix)$$

$$+G(Ty,A_ix,B_jy)]\Big\}\Big\},$$

where $\alpha,\beta,\gamma \ge 0$ and $3\alpha+7\beta+6\gamma<1$. For every $x,y,z \in X$ and for every $i \in I$, $j \in J$, $k \in K$.

(iii) *$\phi : R^+ \to R^+$ is increasing function such that $\phi(t) < t$ for all $t > 0$ and $\sum \phi(t) < \infty$ as $t \to \infty$.*

(iv) *The pairs (A_{i_0},R), (B_{j_0},T) and (C_{k_0},S) are weakly commuting pairs.*

Then the mapping A_i,B_j,C_k,S,T and R have a unique common fixed point in X.

Proof. By Theorem 3.1, we can say that S,R,T, A_{i_0}, B_{j_0} and C_{k_0} for some $i_0 \in I$, $j_0 \in J$, $k_0 \in K$ have a unique fixed point in X. That is there exist a unique $a \in X$ such that

$$R(a) = S(a) = T(a) = A_{i_0}(a) = B_{j_0}(a) = c_{k_0}(a) = a$$

. Let there exist $\lambda \in J$ such that $\lambda \neq j_0$ and $G(a, B_\lambda a, a) > 0$. Then we have

$$G(a,B_\lambda a,a) = G(A_{i_0}a,B_\lambda a,C_{k_0}a)$$
$$\leq \phi\Big\{max\Big\{\alpha[G(Ra,Ta,Sa)+G(Ra,B_ja,C_ka)],\beta[G(Ra,A_ia,B_ja)+G(Ta,B_ja,C_ka)$$
$$+G(Sa,C_ka,A_ia)+G(A_ia,Ra,Ta)+G(B_ja,Ta,Sa)+G(C_ka,Ra,Sa)],$$
$$\gamma[G(Ra,B_ja,Ta)+G(Ta,C_ka,Sa)+G(Sa,A_ia,Ra)+G(Sa,C_ka,A_ia)$$
$$+G(Ty,A_ia,B_ja)]\Big\}\Big\}.$$

This is a contradiction. Hence for every $\lambda \in J$ we have $B_\lambda(a) = a$. Similarily for every $\delta \in I$ and $\eta \in K$ we get $A_\delta(a) = C_\eta(a) = a$. Therefore for every $\delta \in I$, $\eta \in K$ and $\lambda \in J$, we get

$$A_\delta(a) = B_\lambda(a) = C_\eta(a) = S(a) = T(a) = R(a) = a.$$

Next we give an example to validate our Theorem 3.1.

Example 3.6. Let (X,G) be a G-metric space, where $X = [0,\infty]$ and

$$G(x,y,z) = |x-y| + |y-z| + |z-x|.$$

Define self maps A,B,C,S,R and T as follows

$$Ax = \frac{x}{8}, \qquad Bx = \frac{x}{16}, \qquad Cx = \frac{x}{32},$$
$$Tx = \frac{x}{2}, \qquad Sx = \frac{x}{4}, \qquad Rx = x,$$

and $\phi(t) = \frac{t}{k}$. Then $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$, $C(X) \subseteq R(X)$ and the pairs (A,R), (B,T) and (C,S) are weakly compatible. Also for x,y,z

$$G(Ax,By,Cz) \leq \phi\Big\{max\Big\{\alpha[G(Rx,Ty,Sz)+G(Rx,By,Cz)],\beta[G(Rx,Ax,By)+G(Ty,By,Cz)$$
$$+G(Sz,Cz,Ax)+G(Ax,Rx,Ty)+G(By,Ty,Sz)+G(Cz,Rx,Sz)],$$
$$\gamma[G(Rx,By,Ty)+G(Ty,Cz,Sz)+G(Sz,Ax,Rx)+G(Sz,Cz,Ax)$$
$$+G(Ty,Ax,By)]\Big\}\Big\}.$$

That is, all condition of Theorem (3.1) hold and 0 is the unique common fixed point of A,B,C,S,R and T.

Conflict of Interests

The authors declare that there is no conflict of interests.

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