Available online at http://scik.org
Adv. Fixed Point Theory, 4 (2014), No. 1, 140-148
ISSN: 1927-6303

T_F TYPE CONTRACTIVE CONDITIONS FOR KANNAN AND CHATTERJEA FIXED POINT THEOREMS

MEHMET KIR*, HUKMI KIZILTUNC

Department of Mathematics, Faculty of Science, Ataturk University, Erzurum, 25240, Turkey

Abstract. In this paper, the notation of T_F-contractive conditions are investigated for Kannan and Chatterjea type mappings. It is shown that these mappings have a unique fixed point in complete metric spaces.

Keywords: contraction mapping, fixed point, Kannan fixed point theorem, Chatterjea fixed point theorem.

2000 AMS Subject Classification: 47H10

1. Introduction and Preliminaries

It is well known that the first important result on fixed point theory is Banach Contraction Principle. Due to the importance, there exist many extension of it.

A mapping $T : X \to X$, where (X,d) is a metric space, is said to be a contraction if there exists $k \in [0,1)$ such that for all $x, y \in X$,

$$d(Tx,Ty) \leq kd(x,y). \tag{1}$$

If the metric space (X,d) is complete then the mapping satisfying (1) has a unique fixed point.

* Corresponding author

Received October 31, 2013

In 1968, Kannan [2] established a generalization of Banach Contraction Principle that need not continuity.

If a mapping $T : X \to X$ where (X,d) is a complete metric space, satisfies the inequality

$$d(Tx,Ty) \le a[d(x,Tx) + d(y,Ty)], \tag{2}$$

where $a \in \left[0, \frac{1}{2}\right)$ and $x,y \in X$, then T has a unique fixed point. The mappings satisfying (2) are called Kannan type mappings.

A similar contractive condition was introduced by Chatterjea [3] as following:

If a mapping $T : X \to X$ where (X,d) is a complete metric space, satisfies the inequality

$$d(Tx,Ty) \le b[d(x,Ty) + d(y,Tx)] \tag{3}$$

such that $b \in \left[0, \frac{1}{2}\right)$ and $x,y \in X$, then T has a unique fixed point. The mappings satisfying (3) are called Chatterjea type mappings.

In 2010, Moradi and Beiranvand introduced concept of the T_F -contraction mappings as follows:

Definition 1.1. [4] Let (X,d) be a metric space. A mapping $T : X \to X$ is said to be sequentially convergent if we have, for every sequence $\{y_n\}$, if $\{Ty_n\}$ is convergence then $\{y_n\}$ also is convergence. T is said to be subsequentially convergent if we have, for every sequence $\{y_n\}$, if $\{Ty_n\}$ is convergence then $\{y_n\}$ has a convergent subsequence.

Definition 1.2. [4] Let (X,d) be a metric spaces and $f,T : X \to X$ be two mappings. A mapping f is said to be a T_F-contraction if there exists $\alpha \in [0,1)$ such that for all $x,y \in X$

$$F(d(Tfx,Tfy)) \le \alpha F(d(Tx,Ty)), \tag{4}$$

where

1) $F : [0,\infty) \to [0,\infty)$, F is nondecreasing continuous from the right and $F^{-1}(0) = \{0\}$.

2) T is one to one and graph closed (or subsequentially convergent and continuous).

Moradi and Beiranvand proved that if f is a T_F-contraction mapping then, f has a unique fixed point in complete metric space (X,d).

In this study, we plan to introduce T_F -contractive conditions for Kannan and Chatterjea fixed point theorems.

2. Main Results

Theorem 2.1. (T_F Kannan Contractive Mapping Theorem) Let (X,d) be a complete metric space and $T,f : X \to X$ be mappings such that T is continuous, one to one and subsequentially convergent. If $\lambda \in \left[0,\frac{1}{2}\right)$ and $x,y \in X$

$$F\left(d\left(Tfx,Tfy\right)\right) \leq \lambda\left[F\left(d\left(Tx,Tfx\right)\right)+F\left(d\left(Ty,Tfy\right)\right)\right] \tag{5}$$

where; $F : [0,\infty) \to [0,\infty)$ is nondecreasing continuous from the right and $F^{-1}(0) = \{0\}$.

Then, f has a unique fixed point in X. Also, if T is sequentially convergent then for every $x_0 \in X$ the sequence of iterates $\{f^n x_0\}$ converges to the fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point and $x_n = fx_{n-1} = f^n x_0$

$$\begin{aligned} F\left(d\left(Tx_n,Tx_{n+1}\right)\right) &= F\left(d\left(Tfx_{n-1},Tfx_n\right)\right) \\ &\leq \lambda\left[F\left(d\left(Tx_{n-1},Tx_n\right)\right)+F\left(d\left(Tx_n,Tx_{n+1}\right)\right)\right] \end{aligned} \tag{6}$$

therefore we have

$$F\left(d\left(Tx_n,Tx_{n+1}\right)\right) \leq \frac{\lambda}{1-\lambda}F\left(d\left(Tx_{n-1},Tx_n\right)\right). \tag{7}$$

Also, by continuing the process (7), we obtain that

$$F\left(d\left(Tx_n,Tx_{n+1}\right)\right) \leq \left(\frac{\lambda}{1-\lambda}\right)^n F\left(d\left(Tx_0,Tx_1\right)\right). \tag{8}$$

Letting $n \to \infty$ in (8), we obtain that

$$F\left(d\left(Tx_n,Tx_{n+1}\right)\right) \to 0^+ \text{ as } n \to \infty. \tag{9}$$

Again using (8), for all $m,n \in \mathbb{N}$, taking $m > n$,we have

$$\begin{aligned} F\left(d\left(Tx_n,Tx_m\right)\right) &= F\left(d\left(Tf^n x_0,Tf^m x_0\right)\right) \\ &\leq \left(\frac{\lambda}{1-\lambda}\right)^n F\left(d\left(Tx_0,Tf^{m-n}x_0\right)\right). \end{aligned} \tag{10}$$

Letting $m,n \to \infty$, we have

$$F\left(d\left(Tx_n,Tx_m\right)\right) \to 0^+ \text{ as } m,n \to \infty. \tag{11}$$

So, we have $d(Tx_n, Tx_m) \to 0$ as, $m, n \to \infty$.

Thus, we hold that $\{Tx_n\}$ is Cauchy sequence in metric space (X, d). By taking in view of the completeness of X , we obtain that there exists $v \in X$ such that

$$\lim_{n\to\infty} Tx_n = v. \tag{12}$$

Note that T is subsequentially convergent, then $\{x_n\}$ has a convergent subsequence, so there is $u \in X$ such that

$$\lim_{k\to\infty} x_{n(k)} = u. \tag{13}$$

Also, T is continious and $x_{n(k)} \to u$, therefore

$$\lim_{k\to\infty} Tx_{n(k)} = Tu. \tag{14}$$

Note that $\{Tx_{n(k)}\}$ is a subsequence of $\{Tx_n\}$, so $Tu = v$.

Now, we will show that $u \in X$ is a fixed point of f. Indeed, we have

$$\begin{aligned} F(d(Tu, Tfu)) &\le F(d(Tu, Tx_{n(k)}) + d(Tx_{n(k)}, Tfu)) \\ &= F(d(Tu, Tx_{n(k)}) + d\left(Tf^{n(k)}x_0, Tfu\right)) \\ &\le F(d(Tu, Tx_{n(k)}) + d\left(Tf^{n(k)}x_0, Tf^{n(k)}x_1\right) + d\left(Tf^{n(k)}x_1, Tfu\right)) \\ &= F(d(Tu, Tx_{n(k)}) + d(Tfx_{n(k)}, Tfx_{n(k)+1}) + d(Tfx_{n(k)}, Tfu)). \end{aligned} \tag{15}$$

Letting $k \to \infty$ in (15), we have

$$F(d(Tu, Tfu)) \le 0. \tag{16}$$

Last inequality (16) is contradiction unless $d(Tu, Tfu) = 0$. Thus, we obtained $Tu = Tfu$. Also, T is one to one, we obtain $fu = u$. Thus, we provide $u \in X$ is a fixed point of f.

Now, we show that the fixed point is unique. Assume u' is an other fixed point of f then we have $fu' = u'$ and

$$
\begin{aligned}
F(d(Tu,Tu')) &= F(d(Tfu,Tfu')) \\
&\leq \lambda\left[F(d(Tu,Tfu))+F(d(Tu',Tfu'))\right] \\
&= \lambda\left[F(d(Tu,Tu))+F(d(Tu',Tu'))\right]. \qquad (17)
\end{aligned}
$$

The inequality (17) is contradiction unless $F(d(Tu,Tu'))=0$. Thus, we obtain $Tu=Tu'$ and take in view of one to one of T, we obtain $u=u'$. Thus, we obtain that the fixed point is unique.

Also, if we take T is sequentially convergent, by replacing $\{n\}$ with $\{n(k)\}$ we conclude that

$$\lim_{n\to\infty} x_n = u. \qquad (18)$$

Thus, the inequality (18) shows that $\{x_n\}$ converges to the fixed point of f. Thus, the proof is completed.

In 2011, Moradi and Davood [5] introduced a new extension of Kannan fixed point theorem as following:

Let (X,d) be a complete metric space and $T,S:X\to X$ be mappings such that T is continuous, one to one and subsequentially convergent. If $\lambda\in\left[0,\frac{1}{2}\right)$ and $x,y\in X$,

$$d(TSx,TSy)\leq\lambda\left[d(Tx,TSx)+d(Ty,TSy)\right], \qquad (19)$$

then, S has a unique fixed point. Also, if T is sequentially convergent then for every $x_0\in X$ the sequence of iterates $\{S^n x_0\}$ converges to the fixed point.

Now, we will give some important results of Theorem 1.1.

Corollary 2.1. If F is of the form $Fx=x$, we obtain the result of Moradi and Davood.

Corollary 2.2. If we take $Fx=Tx=x$ then, we obtain well-known Kannan fixed point theorem.

Theorem 2.2. (T_F Chatterjea Contractive Mapping Theorem) Let (X,d) be a complete metric space and $T,f:X\to X$ be mappings such that T is continuous, one to one and subsequentially convergent. If $\mu\in\left[0,\frac{1}{2}\right)$ and $x,y\in X$

$$F\left(d\left(Tfx,Tfy\right)\right)\leq\mu\left[F\left(d\left(Tx,Tfy\right)\right)+F\left(d\left(Ty,Tfx\right)\right)\right], \tag{20}$$

where $F:[0,\infty)\to[0,\infty)$ is nondecreasing continuous from the right and $F^{-1}(0)=\{0\}$ .Then, f has a unique fixed point in X. Also, if T is sequentially convergent then for every $x_0\in X$ the sequence of iterates $\{f^n x_0\}$ converges to the fixed point.

Proof. Let $x_0\in X$ be an arbitrary point and $x_n=fx_{n-1}=f^n x_0$

$$\begin{aligned}
F(d(Tx_n,Tx_{n+1})) &= F(d(Tfx_{n-1},Tfx_n))\\
&\leq \mu\left[F(d(Tx_{n-1},Tx_{n+1}))+F(d(Tx_n,Tx_n))\right]\\
&= \mu F(d(Tx_{n-1},Tx_n))\\
&\leq \mu F(d(Tx_{n-1},Tx_n))+\mu F(d(Tx_n,Tx_{n+1})). \qquad (21)
\end{aligned}$$

Therefore, we have

$$F(d(Tx_n,Tx_{n+1}))\leq\frac{\mu}{1-\mu}F(d(Tx_{n-1},Tx_n)). \tag{22}$$

Also, by continuing the process (22), we obtain that

$$F(d(Tx_n,Tx_{n+1})\leq\left(\frac{\mu}{1-\mu}\right)^n F(d(Tx_0,Tx_1)). \tag{23}$$

Letting $n\to\infty$ in (23), we obtaın that

$$F\left(d\left(Tx_n,Tx_{n+1}\right)\right)\to 0^+ \; as\; n\to\infty. \tag{24}$$

Again using (23), for all $m,n\in\mathbb{N}$, taking $m>n$,we have

$$
\begin{aligned}
F\left(d\left(Tx_n, Tx_m\right)\right) &= F\left(d\left(Tf^n x_0, Tf^m x_0\right)\right) \\
&\leq \left(\frac{\mu}{1-\mu}\right)^n F\left(d\left(Tx_0, Tf^{m-n}x_0\right)\right). \qquad (25)
\end{aligned}
$$

Letting $m, n \to \infty$, we have

$$
F\left(d\left(Tx_n, Tx_m\right)\right) \to 0^+ \text{ as } m, n \to \infty. \qquad (26)
$$

Thus, we hold that $\{Tx_n\}$ is Cauchy sequence in complete metric space (X, d). From completeness of X , we obtain that there exists $v \in X$ such that

$$
\lim_{n\to\infty} Tx_n = v. \qquad (27)
$$

Note that T is subsequentially convergent, then $\{x_n\}$ has a convergent subsequence, so there is $u \in X$ such that

$$
\lim_{k\to\infty} x_{n(k)} = u. \qquad (28)
$$

Also, T is continious and $x_{n(k)} \to u$, therefore

$$
\lim_{k\to\infty} Tx_{n(k)} = Tu. \qquad (29)
$$

Note that $\{Tx_{n(k)}\}$ is a subsequence of $\{Tx_n\}$, so $Tu = v$. Now, we will show that $u \in X$ is a fixed point of f.

$$
\begin{aligned}
F(d(Tu, Tfu) &\leq F(d\left(Tu, Tx_{n(k)}\right) + d\left(Tx_{n(k)}, Tfu\right)) \\
&= F(d\left(Tu, Tx_{n(k)}\right) + d\left(Tf^{n(k)}x_0, Tfu\right) \\
&\leq F\left(d\left(Tu, Tx_{n(k)}\right) + d\left(Tf^{n(k)}x_0, Tf^{n(k)}x_1\right) + d\left(Tf^{n(k)}x_1, Tfu\right)\right) \\
&= F\left(d\left(Tu, Tx_{n(k)}\right) + d\left(Tx_{n(k)}, Tx_{n(k)+1}\right) + d\left(Tfx_{n(k)}, Tfu\right)\right). \qquad (30)
\end{aligned}
$$

Since, F is subsequentially convergent and nondecreasing, if we let $k \to \infty$ in (30), we hold

$$F(d(Tu,Tfu)) \leq 0. \tag{31}$$

The inequality (31) is contradiction unless $F(d(Tu,Tfu)) = 0$. This implies that $d(Tu,Tfu) = 0$ so $Tu = Tfu$. Also, T is one to one, so $fu = u$. Thus, we provide $u \in X$ is a fixed point of f.

It is easy to see uniqueness of the fixed point. Now, we will give some important results of Theorem 2.2.

Corollary 2.3. Let (X,d) be a complete metric space and $T,S : X \to X$ be mappings such that T is continuous, one to one and subsequentially convergent. If $\mu \in \left[0,\frac{1}{2}\right)$ and $x,y \in X$,

$$d(TSx,TSy) \leq \mu\left[d(Tx,TSy) + d(Ty,TSx)\right], \tag{32}$$

then, S has a unique fixed point. Also, if T is sequentially convergent then for every $x_0 \in X$ the sequence of iterates $\{S^n x_0\}$ converges to the fixed point.

Corollary 2.4. If we take $Fx = Tx = x$, then we obtain well-known Chatterjea fixed point theorem.

Conflict of Interests

The authors declare that there is no conflict of interests.

References

[1] S. Banach, Sur les operations dans les ensembles abstraits et leur application aux equations integerales, Fundamenta Mathematicae, 3 (1922), 133-181.

[2] R. Kannan, Some results on fixed points, Bull. Calcutta Math. Soc. 60 (1968), 71-76.

[3] S. K. Chatterjea, Fixed point theorems, C.R. Acad. Bulgare Sci. 25 (1972), 727-730.

[4] S. Moradi, A. Beiranvand, Fixed point of T_F-contractive single-valued mappings, Iranian J. Math. Sic. Inf. 5 (2010), 25-32.

[5] S. Moradi, A. Davood, New extension of Kannan fixed point theorem on complete metric and generalized metric spaces, Int. J. Math. Anal. 5 (2011), 2313-1320.

[6] B. E. Rhoades, Some theorems on weakly contractive maps, Nonlinear Anal. 47 (2001) 2683-2693.

[7] B. S. Choudhury, Unique fixed point theorem for weakly c-contractive mappings, Kathmandu Univ. J.Sci, Eng and Tech, 5 (2009), 6-13.

[8] P. N Dutta, B. S. Choudhury, A generalization of contraction principle in metric spces, Fixed Point Theory Appl 2008 (2008), Article ID 406368.

[9] E. Karapınar, Fixed point theory $\phi-$contraction. Appl. Math. Lett. 24 (2011), 882-825.