Available online at http://scik.org
Adv. Fixed Point Theory, 4 (2014), No. 2, 199-206
ISSN: 1927-6303

A FIXED POINT THEOREM IN MODULAR METRIC SPACES

EMINE KILINÇ[1], CIHANGIR ALACA[2,*]

[1]Department of Mathematics, Institute of Natural and Applied Sciences,

Celal Bayar University, Muradiye Campus 45140 Manisa, Turkey

[2]Department of Mathematics, Faculty of Science and Arts,

Celal Bayar University, Muradiye Campus 45140 Manisa, Turkey

Abstract. In this paper, we define $(\varepsilon, k)-$uniformly locally contractive mappings and η-chainable concept and prove a fixed point theorem for these concepts in a complete modular metric spaces.

Keywords: modular metric spaces; fixed point; $(\varepsilon, k)-$uniformly locally contractive.

2010 AMS Subject Classification: 46A80, 47H10, 54E35.

1. Introduction

The beginning of fixed point theory in metric spaces is related to Banach Contraction Principle, published in 1922 [5], which in particular situation was already obtained by Liouville, Picard and Goursat. Since its simplicity and usefullness, it has become a very popular tool in solving existence problems in many branches of mathematical analiyisis. Following the Banach Contraction Principle many authors introduced various concepts of locally contraction mappings, or of weakly contraction mappings. In 1961, Edelstein [13] defined (ε, λ)-uniformly

*Corresponding author

E-mail addresses: eklnc07@gmail.com(E. Kılınç), cihangiralaca@yahoo.com.tr (C. Alaca)

Received December 7, 2013

locally contractive mappings and proved a fixed point theorem in a complete metric ε-chainable spaces.

The notion of modular space was introduced by Nakano [17] and was intensively developed by Koshi, Shimogaki, Yamamuro (see [14, 19]) and others. A lot of mathematicians are interested fixed point of modular space. In 2008, Chistyakov introduced the notion of modular metric space generated by F-modular and developed the theory of this space [8], on the same idea was defined the notion of a modular on an arbitrary set and developed the theory of metric space generated by modular such that called the modular metric spaces in 2010 [9]. Afrah A. N. Abdou [1] studied and proved some new fixed points theorems for pointwise and asymptotic pointwise contraction mappings in modular metric spaces. Azadifer *et. al.* [2] introduced the notion of modular G-metric spaces and proved some fixed point theorems of contractive and Azadifer *et. al.* [4] proved the existence and uniqueness of a common fixed point of compatible mappings of integral type in this space. Recently, many authors studied on different fixed point results for modular metric spaces; see [3, 6, 7, 12, 10, 16].

In this article, we study and prove some fixed point theorems for extensions and generalizations of contraction mappings in modular metric spaces.

2. Preliminaries

In this section, we will give some basic concepts and facts in modular metric spaces.

Definition 2.1. Let X be a vector space over $\mathbb{R}$ (or $\mathbb{C}$). A functional $\rho : X \to [0,\infty]$ is called a modular if for arbitrary x and y, elements of X satisfies the following three conditions:

(A1) $\rho(x) = 0$ iff $x = 0$;

(A2) $\rho(\alpha x) = \rho(x)$ for all scalar α with $|\alpha| = 1$;

(A3) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$, whenever $\alpha,\ \beta\ \geq\ 0$ and $\alpha + \beta = 1$.

Let X be nonempty set, $\lambda \in (0,\infty)$ and due to the disparity of the arguments, function $w : (0,\infty) \times X \times X \to [0,\infty]$ will be written as $w_\lambda(x,y) = w(\lambda,x,y)$ for all $\lambda > 0$ and $x,y \in X$.

Definition 2.2. [9] Let X be a nonempty set, a function $w : (0,\infty) \times X \times X \to [0,\infty]$ is said to be a metric modular on X if satisfying, for all $x,y,z \in X$ the following condition holds:

(i) $w_\lambda(x,y)=0$ for all $\lambda>0 \Leftrightarrow x=y$;

(ii) $w_\lambda(x,y)=w_\lambda(y,x)$ for all $\lambda>0$;

(iii) $w_{\lambda+\mu}(x,y)\leq w_\lambda(x,z)+w_\mu(z,y)$ for all $\lambda,\mu>0$.

If instead of (i), we have only the condition

(i) $w_\lambda(x,x)=0$ for all $\lambda>0$, then w is said to be a (metric) pseudomodular on X.

Definition 2.3 [16] Let X_w be a modular metric space. Then following definitions exist:

(1) The sequence $(x_n)_{n\in\mathbb{N}}$ in X_w is said to be convergent to $x\in X_w$ if $w_\lambda(x_n,x)\to 0$, as $n\to\infty$ for all $\lambda>0$.

(2) The sequence $(x_n)_{n\in\mathbb{N}}$ in X_w is said to be Cauchy if $w_\lambda(x_m,x_n)\to 0$, as $m,n\to\infty$ for all $\lambda>0$.

(3) A subset C of X_w is said to be closed if the limit of a convergent sequence of C always belong to C.

(4) A subset C of X_w is said to be complate if any Cauchy sequence in C is a convergent sequence and its limit is in C.

(5) A subset C of X_w is said to be bounded if for all $\lambda>0$,

$$\delta_w(C)=\sup\{w_\lambda(x,y);\ x,y\in C\}<\infty.$$

3. Main results

In this section, we will give an extension and a generalization of Banach contraction mapping in modular metric space.

Definition 3.1. Let w be a metric modular on X and X_w be a modular metric space induced by w and $T:X_w\to X_w$ is an arbitrary mapping. A mapping T is called a contraction if for each $x,y\in X_w$ and for all $\lambda>0$ there exists $0<k<1$ such that

$$w_\lambda(Tx,Ty)\leq kw_\lambda(x,y). \tag{3.1}$$

Definition 3.2. A mapping T of X_w into itself is said to be locally contractive if for every $x \in X_w$ there exist ε and k $(\varepsilon > o,\ 0 \leq k < 1)$, which may depend on x, such that:

$$p, q \in S(x, \varepsilon) = \{y : w_\lambda(x, y) < \varepsilon\}. \tag{3.2}$$

T is said to be $(\varepsilon, k)-$uniformly locally contractive if it is locally contractive and both ε and k do not depend on x.

Definition 3.3. (Extended Contraction principle) A modular metric space X_w is said to be η-chainable if for every $a,\ b \in X_w$ there exists an η-chain, that is a finite set of points $a = x_0,\ x_1,\ \ldots,\ x_n = b$ (n may depend on both a and b) such that $w_\lambda(x_{i-1}, x_i) < \eta$.

Theorem 3.1. *Let w be a metric modular on X and X_w be a complete modular metric $\varepsilon-$chainable space, T a mapping of X_w into itself which is $(\varepsilon, k)-$uniformly locally contractive, then there exists a uniuqe $x \in X_w$ such that $Tx = x$.*

Proof. Let x be an arbitrary pointof X_w. Consider the $\varepsilon-$chain:

$x = x_0,\ x_1,\ldots,\ x_n = Tx$; by the triangle inequality

$$w_\lambda(x, Tx) \leq \sum_{i=1}^{n} w_{\frac{\lambda}{n}}(x_{i-1},\ x_i) < n \cdot \varepsilon.$$

Hence, denoting $T(T^m x) = T^{m+1}x$, $(m = 1, 2, \ldots)$ we have

$$w_\lambda(Tx_{i-1},\ Tx_i) \leq k w_\lambda(x_{i-1},\ x_i) < k \cdot \varepsilon.$$

By induction, we find that

$$w_\lambda(T^m x_{i-1},\ T^m x_i) < k \cdot w_\lambda\left(T^{m-1}x_{i-1},\ T^{m-1}x_i\right) < \ldots < k^m w_\lambda(x_{i-1},\ x_i) < k^m \varepsilon. \tag{3.3}$$

By the last inequality we obtain

$$w_\lambda\left(T^m x, T^{m+1}x\right) \leq \sum_{i=1}^{m} w_{\frac{\lambda}{m}}(T^m x_{i-1},\ T^m x_i) < k^m \cdot n \cdot \varepsilon.$$

It follows that the sequence of iterates $(T^m x)$ is a Cauchy sequence. Indeed if j and t $(j<t)$ are positive integers,

$$\begin{aligned} w_\lambda\left(T^j x, T^t x\right) &\leq \sum_{i=j}^{t-1} w_{\frac{\lambda}{t-j}}\left(T^i x,\ T^{i+1} x\right) \\ &< n\cdot\varepsilon\cdot\left(k^j+\cdots+k^{t-1}\right) \\ &\leq n\cdot\varepsilon\frac{k^j}{1-t}\to 0, \qquad j\to\infty. \end{aligned}$$

By the completeness of X_w there exists $\lim\limits_{i\to\infty} T^i x$. From the contiunity of T it follows that

$$T\left(\lim_{i\to\infty} T^i x.\right)=\lim_{i\to\infty} T^{i+1} x=\lim_{i\to\infty} T^i x.$$

Thus $\lim\limits_{i\to\infty} T^i x$ is a fixed point of T.

Now, we are in a position to show $u=\lim\limits_{i\to\infty} T^i x$ is unique. Suppose that there exists $v\neq u$ with the property $v=Tv$ and let $u=x_0,\ x_1,\ \ldots,x_k=v$ be an $\varepsilon-$chain. Using (3.3) we obtain

$$w_\lambda\left(Tu,Tv\right)\leq\sum_{i=1}^{r} w_{\frac{\lambda}{r}}\left(T^i x_{i-1},\ T^i x_i\right)<k^i r\varepsilon\ ,\ \lim_{i\to\infty} k^i r\varepsilon=0.$$

In view of (i), we find the contradiction. Hence $u=v$ and this completes the proof.

Remark 3.4 *The theorem above does not guarantee the existence of fixed point. Indeed let* $T: X_w\to X_w,\ Tx=x+\frac{1}{x}$ *with the modular metric* $w_\lambda(x,y)=\max\{|x_i-y_i|\}$ *for* $X_w=\mathbb{R}^n$, *where* $x,\ y\in\mathbb{R}^n$ *then we obtain*

$$\begin{aligned} w_\lambda\left(Tx,Ty\right) &= \max_{1\leq i\leq n}\left\{\left|x_i+\frac{1}{x_i}-y_i-\frac{1}{y_i}\right|\right\} \\ &= \max_{1\leq i\leq n}\left\{\left|(x_i y_i)-\left(\frac{x_i-y_i}{x_i\cdot y_i}\right)\right|\right\} \\ &= \max_{1\leq i\leq n}\left\{|x_i-y_i|\cdot\left|1-\frac{1}{x_i\cdot y_i}\right|\right\} \\ &\leq \underbrace{\max_{1\leq i\leq n}\{|x_i-y_i|\}}_{w_\lambda(x,y)}\cdot\underbrace{\max_{1\leq i\leq n}\left\{\left|1-\frac{1}{x_i\cdot y_i}\right|\right\}}_{k(x,y)},\quad k(x,y)<1. \end{aligned}$$

But there is not any $x\in X_w$ *such that* $Tx=x$.

From this reason we will give a new generalization of contraction mappings in modular metric spaces which is given by Rakotch for metric spaces in 1962; see [18] and the references therein.

Definition 3.5 If there exists a mapping $k:(0,\infty)\to[0,1)$, $\sup\{k(r):0<p\leq r\leq q\}<1$ and

$$w_\lambda(Tx,Ty)\leq k[w_\lambda(x,y)]\cdot w_\lambda(x,y). \tag{3.4}$$

$T:X_w\to X_w$ is said to be a weak contraction mapping.

Theorem 3.2 *Let w be a metric modular on X and X_w be a complete modular metric space. Let $T:X_w\to X_w$ be a weak contraction mapping. Then, T has a unique fixed point in X_w.*

Proof. Let $x\in X_w$ be arbitrary. Consider (T^nx) iteration sequence. If $w_\lambda\left(T^nx,T^{n+1}x\right)=0$ for some $n\in\mathbb{N}$, then $TT^nx=T^nx$. Hence T^nx is a fixed point of T. Now consider $w_\lambda\left(T^nx,T^{n+1}x\right)>0$ for all $n\in\mathbb{N}$. Then since $k(r)<r$, T is contraction for $r>0$ by (3.4). Hence

$$\begin{aligned} w_\lambda\left(T^nx,T^{n+1}x\right) &= w_\lambda\left(TT^{n-1}x,TT^nx\right)\\ &\leq k\left[w_\lambda\left(T^{n-1}x,T^nx\right)\right]\cdot w_\lambda\left(T^{n-1}x,T^nx\right)\\ &< w_\lambda\left(T^{n-1}x,T^nx\right). \end{aligned}$$

Hence the sequence $\left\{w_\lambda\left(T^nx,T^{n+1}x\right)\right\}$ is monotone decreasing and it is bounded below with 0. Hence it is convergent. Let $\lim\limits_{n\to\infty}w_\lambda\left(T^nx,T^{n+1}x\right)=a$. Then we obtain

$$a<w_\lambda\left(T^nx,T^{n+1}x\right)\leq w_\lambda(x,Tx).$$

Now we show that $a=0$. Suppose that $a>0$. $k=\sup\{k(r):0<a\leq r\leq w_\lambda(x,Tx)\}$. Then

$$k\left(w_\lambda\left(T^nx,T^{n+1}x\right)\right)\leq k$$

$$0<a<w_\lambda\left(T^nx,T^{n+1}x\right)\leq k\left(w_\lambda\left(T^{n-1}x,T^nx\right)\right)\leq\ldots\leq k^nw_\lambda(x,Tx),$$

$\lim\limits_{n\to\infty}k^nw_\lambda(x,Tx)=0$. But it is impossible. Thus $a=0$. Now we will show (T^nx) is a Cauchy sequence. Let $\varepsilon>0$ and $0<k(\varepsilon)=\sup\left\{k(r):\frac{\varepsilon}{2}\leq r\leq\varepsilon\right\}$. Since $\lim\limits_{n\to\infty}w_\lambda\left(T^nx,T^{n+1}x\right)=0$ and $1-k(\varepsilon)>0$ there exists $n_0\in\mathbb{N}$ such that for all $n\geq n_0$

$$w_{\frac{\lambda}{2}}\left(T^nx,T^{n+1}x\right)<\frac{1-k(\varepsilon)}{2}\cdot\varepsilon \tag{3.5}$$

$n\geq n_0$ be a positive integer. We will show for all $m>n\geq n_0$

$$w_\lambda(T^nx,T^mx)<\varepsilon \tag{3.6}$$

with induction method. For $m=n+1$, (3.6) is satisfied from (3.5). Suppose that (3.5) is satisfied for $m>n+1$. If $w_\lambda\left(T^n x, T^m x\right) \geq \frac{\varepsilon}{2}$, then by (3.4) we get

$$
\begin{aligned}
& w_{\frac{\lambda}{2}}\left(T T^n x, T T^m x\right) \\
\leq \; & k\left[w_{\frac{\lambda}{2}}\left(T^n x, T^m x\right)\right] \cdot w \frac{\lambda}{2}\left(T^n x, T^m x\right)<k(\varepsilon) \cdot \varepsilon .
\end{aligned}
$$

Using triangle inequality and (3.5), we find that

$$
\begin{aligned}
w_\lambda\left(T^n x, T^{m+1} x\right) & \leq w_{\frac{\lambda}{2}}\left(T^n x, T T^n x\right)+w_{\frac{\lambda}{2}}\left(T T^n x, T^{m+1} x\right) \\
& <\frac{1-k(\varepsilon)}{2} \cdot \varepsilon+k(\varepsilon) \cdot \varepsilon<\varepsilon .
\end{aligned}
$$

If $w_\lambda\left(T^n x, T^m x\right) \leq \frac{\varepsilon}{2}$, then by triangle inequality and (3.5) we get

$$
\begin{aligned}
w_\lambda\left(T^n x, T^{m+1} x\right) & \leq w \frac{\lambda}{2}\left(T^n x, T^m x\right)+w_{\frac{\lambda}{2}}\left(T^m x, T^{m+1} x\right) \\
& <\frac{\varepsilon}{2}+\frac{1-k(\varepsilon)}{2} \cdot \varepsilon<\varepsilon .
\end{aligned}
$$

Thus we get $w_\lambda\left(T^n x, T^{m+1} x\right)<\varepsilon$. By (3.6), $\left(T^n x\right)$ is a Cauchy sequence. Since X_w is complete there exists $u \in X_w$ such that $\lim\limits_{n \rightarrow \infty} T^n x=u$. By the contuinity of T we get

$$
T u=T\left(\lim _{n \rightarrow \infty} T^n x\right)=\lim _{n \rightarrow \infty} T^{n+1} x=\lim _{n \rightarrow \infty} T^n x=u .
$$

Thus u is a fixed point of T. Now we show the uniqueness of u. Suppose that v is another fixed point of T such that $u \neq v$. Then $w_\lambda(u, v) \neq 0$, thus we get

$$
0<w_\lambda(u, v)=w_\lambda(T u, T v) \leq k\left[w_\lambda(u, v)\right] \cdot w_\lambda(u, v)
$$

$$
\left(1-k\left[w_\lambda(u, v)\right]\right) \cdot w_\lambda(u, v) \leq 0
$$

$$
w_\lambda(u, v)=0 .
$$

From (i), we see that it is impossible. Thus $u=v$. This completes the proof.

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] Afrah A. N. Abdou, On asymptotic pointwise contractions in modular metric spaces, Abst. Appl. Anal. 2013 (2013), Article ID 501631.

[2] B. Azadifar, M. Maramaei, Gh. Sadeghi, On the modular G-metric spaces and fixed point theorems, J. Nonlinear Sci. Appl. 6 (2013), 293-304.

[3] B. Azadifar, M. Maramaei, Gh. Sadeghi, Common fixed point theorems in modular G-metric spaces, J. Nonlinear Anal. Appl. 2013 (2013), 1-9.

[4] B. Azadifar, Gh. Sadeghi, R. Saadati, C. Park, Integral type contractions in modular metric spaces, J. Inequal. Appl. 2013 (2013), Article ID 483.

[5] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, Fund. Math. 3 (1922), 133-181.

[6] P. Chaipunya, Y. J. Cho, P. Kumam, Geraghty-type theorems in modular metric spaces with an application to partial differential equation, Adv. Diffence. Equ. 2012 (2012), Artice ID 83.

[7] Y. J. Cho, R. Saadati, Gh. Sadeghi, Quasi-Contractive Mappings in Modular Metric Spaces, J. Appl. Math. 2012 (2012), Article ID 907951.

[8] V. V. Chistyakov, Modular metric spaces generated by F-modulars, Folia Math. 14 (2008), 3-25.

[9] V. V. Chistyakov, Modular metric spaces I. basic conceps, Nonlinear Anal. 72 (2010), 1-14.

[10] V. V. Chistyakov, Fixed points of modular contractive maps, Dokl. Math. 86 (2012), 515-518.

[11] L. B. Ćirić, A new fixed point theorem forcontractive mappings, Publ. Inst. Math. 30 (1981), 25-27.

[12] H. Dehghan, M. Eshaghi Gordji, A. Ebadian, Comment on 'Fixed point theorems for contraction mappings in modular metric spaces, Fixed Point Theory Appl. 2012 (2012) Article ID 144.

[13] M. Edelstein, An extension of Banach's contraction principle, Proc. Amer. Math. Soc. 12 (1961), 7-10.

[14] S. Koshi, T. Shimogaki, On F-norms of quasi-modular space, J. Fac. Sci. Hokkaido Univ. Ser 1. 15 (1961), 202-218.

[15] A. Meir, E. Keeler, A theorem on contraction mappings, J. Math. Anal. Appl. 28 (1969), 326-329.

[16] C. Mongkolkeha, W. Sintunavarat, P.Kumam, Fixed point theorems for contraction mappings in modular metric spaces, Fixed Point Theory Appl. 2011 (2011), Article ID 93,

[17] H. Nakano, Modulared semi-ordered linearspace, In Tokyo Math. Book Ser, vol.1, Maruzen Co, Tokyo 1950.

[18] E. Rakotch, A note on contractive mappings, Proc. Amer. Math. Soc. 3 (1962), 459-465.

[19] S. Yamammuro, On conjugate space of Nakano space, Trans. Amer. Math. Soc. 90 (1959), 291-311.