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# CONVERGENCE OF ALGORITHMS FOR AN INFINITE FAMILY NONEXPANSIVE MAPPINGS AND RELAXED COCOERCIVE MAPPINGS IN HILBERT SPACES

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**Abstract.** In this article, the problem of finding a common element of the set of solutions of a variational problem and the set of fixed points of nonexpansive mappings. Our results improve and extend the recent ones announced by many others.

**Keywords**: solution; variational inequality; fixed point; nonexpansive mapping.

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## 1. Introduction

Optimization theory has emerged as a powerful and effective tool for studying a wide class of problems which arise in economics, finance, image reconstruction, ecology, transportation, network, elasticity and optimization; see [1-25] and the references therein. The computation of solutions of variational inequalities (fixed points of nonexpansive mappings) is important in the study of many real world problems. The aim of this paper is to investigate a solution problem

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of a family of nonexpansive mappings and relaxed cocoercive mappings. In Section 2, strong convergence theorems of common solutions are established in a Hilbert space.

## 2. Preliminaries

Let $H$ be a real Hilbert space, whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let $C$ be a closed convex subset of $H$ and let $A : C \to H$ be a nonlinear map. Let $P_C$ be the projection of $H$ onto the convex subset $C$. The classical variational inequality which denoted by $VI(C,A)$ is to find $u \in C$ such that $\langle Au, v-u\rangle \geq 0$, $v \in C$. One can see that the variational inequality is equivalent to a fixed point problem. The function $u \in C$ is a solution of the variational inequality if and only if $u \in C$ satisfies the relation $u = P_C(u-\lambda Au)$, where $\lambda > 0$ is a constant.

Recall that $A$ is said to be inverse-strongly monotone if there exists a constants $u > 0$ such that

$$\langle Ax-Ay, x-y\rangle \geq u\|Ax-Ay\|^2, \quad \forall x,y \in C.$$

A mapping $S : C \to C$ is said to be nonexpansive if $\|Sx-Sy\| \leq \|x-y\|$ for all $x,y \in C$. Next, we denote by $F(S)$ the set of fixed points of $S$. A mapping $f : C \to C$ is said to be a contraction if there exists a coefficient $\alpha$ $(0 < \alpha < 1)$ such that

$$\|f(x)-f(y)\| \leq \alpha\|x-y\|,$$

for $\forall x,y \in C$. A linear bounded operator $B$ is strongly positive if there exists a constant $\bar{\gamma} > 0$ with the property $\langle Bx, x\rangle \geq \bar{\gamma}\|x\|^2$, $x \in H$. A set-valued mapping $T : H \to 2^H$ is called monotone if for all $x,y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x-y, f-g\rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is maximal if the graph of $G(T)$ of $T$ is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping $T$ is maximal if and only if for $(x,f) \in H \times H$, $\langle x-y, f-g\rangle \geq 0$ for every $(y,g) \in G(T)$ implies $f \in Tx$. Let $A$ be a monotone map of $C$ into $H$ and let $N_C v$ be the normal cone to $C$ at $v \in C$, i.e., $N_C v = \{w \in H : \langle v-u, w\rangle \geq 0, \ \forall u \in C\}$ and define

$$Tv = \begin{cases} Av + N_C v, & v \in C \\ \emptyset, & v \notin C. \end{cases}$$

Then $T$ is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C,A)$; see [1] and the references therein.

Recently iterative methods for nonexpansive mappings have been applied to solve convex minimization problems. A typical problem is to minimize a quadratic function over the set of the fixed points a nonexpansive mapping on a real Hilbert space $H$:

$$\min_{x\in C} \frac{1}{2}\langle Bx,x\rangle - \langle x,b\rangle, \tag{2.1}$$

where $B$ is a linear bounded operator, $C$ is the fixed point set of a nonexpansive mapping $S$ and $b$ is a given point in $H$. In [11], it is proved that the sequence $\{x_n\}$ defined by the iterative method below, with the initial guess $x_0 \in H$ chosen arbitrarily,

$$x_{n+1} = (I - \alpha_n B)Sx_n + \alpha_n b, \quad n \geq 0,$$

converges strongly to the unique solution of the minimization problem (2.1) provided the sequence $\{\alpha_n\}$ satisfies certain conditions. More recently, Marino and Xu [12] introduced a new iterative scheme by the viscosity approximation method:

$$x_{n+1} = (I - \alpha_n B)Sx_n + \alpha_n \gamma f(x_n), \quad n \geq 0.$$

They proved the sequence $\{x_n\}$ generated by above iterative scheme converges strongly to the unique solution of the variational inequality

$$\langle (B - \gamma f)x^*, x - x^*\rangle \geq 0, \quad x \in C,$$

which is the optimality condition for the minimization problem $\min_{x\in C} \frac{1}{2}\langle Bx,x\rangle - h(x)$, where $C$ is the fixed point set of a nonexpansive mapping $S$, $h$ is a potential function for $\delta f$ (i.e., $h'(x) = \delta f(x)$ for $x \in H$.)

Concerning a family of nonexpansive mappings has been considered by many authors. The well-known convex feasibility problem reduces to finding a point in the intersection of the fixed point sets of a family of nonexpansive mappings. The problem of finding an optimal point that minimizes a given cost function over common set of fixed points of a family of nonexpansive mappings is of wide interdisciplinary interest and practical importance. A simple algorithmic solution to the problem of minimizing a quadratic function over common set of fixed points of a family of nonexpansive mappings is of extreme value in many applications including set theoretic signal estimation.

Recently Yao *et al.* [13] considered a general iterative algorithm for an infinite family of non-expansive mapping in the framework of Hilbert spaces. To be more precisely, they introduced the following general iterative algorithm.

$$x_{n+1} = \lambda_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \lambda_n A)W_n x_,$$

where $f$ is a contraction on $H$, $A$ is a stronglyy positive bounded linear operator, $W_n$ are nonexpansive mappings which are generated by an finite family of nonexpansive mapping $T_1, T_2, \ldots$. To be more precisely,

$$\begin{aligned}
&U_{n,n+1} = I,\\
&U_{n,n} = \gamma_n T_n U_{n,n+1} + (1-\gamma_n)I,\\
&\vdots\\
&U_{n,k} = \gamma_k T_k U_{n,k+1} + (1-\gamma_k)I,\\
&u_{n,k-1} = \gamma_{k-1} T_{k-1} U_{n,k} + (1-\gamma_{k-1})I,\\
&\vdots\\
&U_{n,2} = \gamma_2 T_2 U_{u,3} + (1-\gamma_2)I,\\
&W_n = U_{n,1} = \gamma_1 T_1 U_{n,2} + (1-\gamma_1)I,
\end{aligned} \tag{2.2}$$

where $\{\gamma_1\}, \{\gamma_2\}, \ldots$ are real numbers such that $0 \le \gamma \le 1$, $T_1, T_2, \ldots$ be an infinite family of mappings of $C$ into itself. Nonexpansivity of each $T_i$ ensures the nonexpansivity of $W_n$.

Concerning $W_n$ we have the following lemmas which are important to prove our main results.

**Lemma 2.1** [14] *Let $C$ be a nonempty closed convex subset of a strictly convex Banach space $E$. Let $T_1, T_2, \ldots$ be nonexpansive mappings of $C$ into itself such that $\cap_{n=1}^{\infty} F(T_n)$ is nonempty, and let $\gamma_1, \gamma_2, \ldots$ be real numbers such that $0 < \gamma_n \le \eta < 1$ for any $n \ge 1$. Then, for every $x \in C$ and $k \in N$, the limit $\lim_{n\to\infty} U_{n,k} x$ exists.*

Using Lemma 2.1, one can define the mapping $W$ of $C$ into itself as follows. $Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x$, for every $x \in C$. Such a $W$ is called the $W$-mapping generated by $T_1, T_2, \dots$ and $\gamma_1, \gamma_2, \dots$. Throughout this paper, we will assume that $0 < \gamma_n \le \eta < 1$ for all $n \ge 1$.

**Lemma 2.2** [14] *Let $C$ be a nonempty closed convex subset of a strictly convex Banach space $E$. Let $T_1, T_2, \dots$ be nonexpansive mappings of $C$ into itself such that $\cap_{n=1}^{\infty} F(T_n)$ is nonempty, and let $\gamma_1, \gamma_2, \dots$ be real numbers such that $0 < \gamma_n \le \eta < 1$ for any $n \ge 1$. Then, $F(W) = \cap_{n=1}^{\infty} F(T_n)$.*

In this paper, we introduce a composite iterative process as following:

$$\begin{cases} x_1 \in C \\ y_n = P_C\big(\beta_n \gamma f(x_n) + (I - \beta_n B) W_n P_C (I - r_n A) x_n\big), \\ x_{n+1} = P_C\big(\alpha_n x_n + (1-\alpha_n) y_n + e_n\big), \quad n \ge 1, \end{cases} \tag{2.3}$$

where $A$ is an inverse-strongly monotone mapping, $B$ is a strongly positive linear bounded operator, $f$ is a contraction on $C$ and $W_n$ is a mapping generated by (2.2).

We prove the sequence $\{x_n\}$ generated by the above iterative scheme converges strongly to a common element of the set of common fixed points of an infinite nonexpansive mappings and the set of solutions of the variational inequalities for an inverse-strongly mapping, which solves another variation inequality $\langle \gamma f(q) - Bq, q - p\rangle \le 0, \quad p \in \cap_{i=1}^{\infty} F(T_i) \cap VI(C,A)$ and is also the optimality condition for the minimization problem $\min_{x\in C} \frac{1}{2}\langle Bx, x\rangle - h(x)$, where $C$ is the intersection of the common fixed points set of a nonexpansive mappings and the set of solutions of the variational inequalities for relaxed $(\gamma, r)$-cocoercive maps, $h$ is a potential function for $\delta f$ (i.e., $h'(x) = \delta f(x)$ for $x \in H$.)

In order to prove our main results, we need the following lemmas.

**Lemma 2.3** [11] *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \le (1-\gamma_n)\alpha_n + \delta_n,$$

*where $\gamma_n$ is a sequence in (0,1) and $\{\delta_n\}$ is a sequence such that*

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \delta_n/\gamma_n \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

*Then* $\lim_{n\to\infty} \alpha_n = 0$.

**Lemma 2.4** [12] *Assume $B$ is a strongly positive linear bounded operator on a Hilbert space $H$ with coefficient $\bar{\gamma} > 0$ and $0 < \rho \le \|B\|^{-1}$. Then*

$$\|I - \rho B\| \le 1 - \rho\bar{\gamma}.$$

**Lemma 2.5** [15] *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space $X$ and let $\beta_n$ be a sequence in [0,1] with*

$$0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1.$$

*Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \ge 0$ and*

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0.$$

*Then* $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

**Lemma 2.6.** *In a real Hilbert space $H$, there holds the the following inequality*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle,$$

*for all $x, y \in H$.*

## 3. Main results

**Theorem 3.1.** *Let $H$ be a real Hilbert space, let $C$ be a nonempty closed convex subset of $H$ and let $A : C \to H$ be an u-inverse-strongly monotone mapping. Let $f : C \to C$ be a contraction with the coefficient $\alpha$ $(0 < \alpha < 1)$ and $\{T_i\}_{i=1}^{\infty}$ be an infinite nonexpansive mappings from $C$ into itself generated by (2.2) such that $F = \bigcap_{i=1}^{\infty} F(T_i) \cap VI(C,A) \neq \emptyset$. Let $B$ be a strongly positive linear bounded self-adjoint operator of $C$ into itself with coefficient $\bar{\gamma} > 0$ such that $\|B\| \le 1$. Assume that $0 < \gamma < \bar{\gamma}/\alpha$. Assume that $x_1 \in C$ and $\{x_n\}$ is generated by*

$$\begin{cases} x_1 \in C \\ y_n = P_C\big(\beta_n \gamma f(x_n) + (I - \beta_n B)W_n P_C(I - r_n A)x_n\big), \\ x_{n+1} = P_C\big(\alpha_n x_n + (1-\alpha)y_n + e_n\big), \quad n \ge 1, \end{cases}$$

*where $e_n$ is a bounded sequence in $H$, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$. If $\{\alpha_n\}$, $\{\beta_n\}$ and $\{r_n\}$ are chosen such that*

(a) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;

(b) $\lim_{n\to\infty} \beta_n = 0,\ \sum_{n=1}^{\infty} \beta_n = \infty$;

(c) $\lim_{n\to\infty} |r_{n+1} - r_n| = 0,\ \sum_{n=1}^{\infty} \|e_n\| < \infty$;

(d) $\{r_n\} \subset [a,b]$ *for some* $a,b$ *with* $0 < a < b < 2u$.

*Then $\{x_n\}$ converges strongly to $q \in F$, where $q = P_F(\gamma f + (I-B))(q)$, which solves the variation inequality $\langle \gamma f(q) - Bq, p - q\rangle \leq 0, \quad \forall p \in F$.*

**Proof.** From the definition of inverse-strongly monotone mappings, we find that $I - r_nA$ is nonexpansive. Since the condition (a), we may assume, with no loss of generality, that $\beta_n < \|B\|^{-1}$ for all $n$. From Lemma 2.4, we know that if $0 < \rho \leq \|B\|^{-1}$, then $\|I - \rho B\| \leq 1 - \rho\bar{\gamma}$. Letting $p \in F$, we have

$$\begin{aligned}
\|y_n - p\| &\leq \beta_n\|\gamma f(x_n) - Bp\| + (1-\beta_n\bar{\gamma})\|W_nP_C(I - r_nA)x_n - p\| \\
&\leq \beta_n\gamma\|f(x_n) - f(p)\| + \beta_n\|\gamma f(p) - Bp\| + (1-\beta_n\bar{\gamma})\|x_n - p\| \\
&= [1 - \beta_n(\bar{\gamma} - \gamma\alpha)]\|x_n - p\| + \beta_n\|\gamma f(p) - Bp\|.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\|x_{n+1} - p\| &\leq \alpha_n\|x_n - p\| + (1-\alpha_n)\|y_n - p\| + \|e_n\| \\
&\leq \alpha_n\|x_n - p\| + (1-\alpha_n)[(1 - \beta_n(\bar{\gamma} - \gamma\alpha))\|x_n - p\| \\
&\quad + \beta_n\|\gamma f(p) - Bp\|] + \|e_n\|.
\end{aligned}$$

By simple inductions, we have that the sequence $\{x_n\}$ is bounded. Putting $\rho_n = P_C(I - r_nA)x_n$, we have

$$\begin{aligned}
\|\rho_n - \rho_{n+1}\| &\leq \|(I - r_nA)x_n - (I - r_{n+1}A)x_{n+1}\| \\
&= \|(x_n - r_nAx_n) - (x_{n+1} - r_nAx_{n+1}) + (r_{n+1} - r_n)Ax_{n+1}\| \qquad (3.1) \\
&\leq \|x_n - x_{n+1}\| + |r_{n+1} - r_n|M_1,
\end{aligned}$$

where $M_1$ is an appropriate constant. It follows that

$$\begin{aligned}\|y_n - y_{n+1}\| \leq (1-\beta_{n+1}\bar{\gamma})(\|\rho_{n+1}-\rho_n\| + \|W_{n+1}\rho_n - W_n\rho_n\|) \\ + |\beta_{n+1}-\beta_n|M_2 + \gamma\beta_{n+1}\alpha\|x_{n+1}-x_n\|,\end{aligned} \tag{3.2}$$

where $M_2$ is an appropriate constant. Since both $T_i$ and $U_{n,i}$ are nonexpansive, we find from (2.2) that

$$\begin{aligned}\|W_{n+1}\rho_n - W_n\rho_n\| &\leq \gamma_1\|U_{n+1,2}\rho - U_{n,2}\rho_n\| \\ &= \gamma_1\|\gamma_2 T_2 U_{u+1,3}\rho_n - \gamma_2 T_2 U_{n,3}\rho_n\| \\ &\leq \gamma_1\gamma_2\|U_{u+1,3}\rho_n - U_{n,3}\rho_n\| \\ &\leq \cdots \\ &\leq \gamma_1\gamma_2\cdots\gamma_n\|U_{n+1,n+1}\rho_n - U_{n,n+1}\rho_n\| \\ &\leq M_3\prod_{i=1}^{n}\gamma_i,\end{aligned} \tag{3.3}$$

where $M_3 \geq 0$ is an appropriate. Therefore, we have

$$\begin{aligned}\|y_n - y_{n+1}\| \leq [1-\beta_{n+1}(\bar{\gamma}-\alpha\gamma)]\|x_{n+1}-x_n\| \\ + M_4(|r_{n+1}-r_n| + |\beta_{n+1}-\beta_n| + \prod_{i=1}^{n}\gamma_i),\end{aligned}$$

where $M_4$ is an appropriate appropriate constant. It follows that

$$\limsup_{n\to\infty}\{\|y_{n+1}-y_n\| - |x_{n+1}-x_n\|\} \leq 0. \tag{3.4}$$

Using Lemma 2.6, we obtain that

$$\lim_{n\to\infty}\|y_n - x_n\| = 0. \tag{3.5}$$

It follows that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0 \tag{3.6}$$

and

$$\lim_{n\to\infty}\|W_n\rho_n - y_n\| = 0. \tag{3.7}$$

For $p \in F$, we have

$$\begin{aligned}
&\|\rho_n - p\|^2 \\
&= \|P_C(I - r_nA)x_n - P_C(I - r_nA)p\|^2 \\
&\leq \|(x_n - p) - r_n(Ax_n - Ap)\|^2 \\
&= \|x_n - p\|^2 - 2r_n\langle x_n - p, Ax_n - Ap\rangle + r_n^2\|Ax_n - Ap\|^2 \\
&\leq \|x_n - p\|^2 - 2r_n[-\gamma\|Ax_n - Ap\|^2 + r\|x_n - p\|^2] + r_n^2\|Ax_n - Ap\|^2 \\
&\leq \|x_n - p\|^2 + 2r_n\gamma\|Ax_n - Ap\|^2 - 2r_nr\|x_n - p\|^2 + r_n^2\|Ax_n - Ap\|^2 \\
&\leq \|x_n - p\|^2 + (2r_n\gamma + r_n^2 - \frac{2r_nr}{\mu^2})\|Ax_n - Ap\|^2.
\end{aligned} \tag{3.8}$$

Since

$$\begin{aligned}
\|y_n - p\|^2 &= \|\beta_n(\gamma f(x_n) - Bp) + (I - \beta_nB)(W_n\rho_n - p)\|^2 \\
&\leq (\beta_n\|\gamma f(x_n) - Bp\| + (1 - \beta_n\bar{\gamma})\|\rho_n - p\|)^2 \\
&\leq \beta_n\|\gamma f(x_n) - Bp\|^2 + \|\rho_n - p\|^2 + 2\beta_n\|\gamma f(x_n) - Bp\|\|\rho_n - p\|,
\end{aligned} \tag{3.9}$$

we see that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq 2\|\alpha_n(x_n - p) + (1 - \alpha_n)(y_n - p)\|^2 + 2\|e_n\|^2 \\
&\leq 2\alpha_n\|x_n - p\|^2 + 2(1 - \alpha_n)\|y_n - p\|^2 + 2\|e_n\|^2 \\
&\leq 2\alpha_n\|x_n - p\|^2 + 2(1 - \alpha_n)[\beta_n\|\gamma f(x_n) - Bp\|^2 + 2\|\rho_n - p\|^2 \\
&\quad + 4\beta_n\|\gamma f(x_n) - Bp\|\|\rho_n - p\|] + 2\|e_n\|^2.
\end{aligned} \tag{3.10}$$

Substituting (3.8) into (3.10), we have

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\leq \|x_n - p\|^2 + \beta_n\|\gamma f(x_n) - Bp\|^2 + (2r_n\gamma + r_n^2 - \frac{2r_nr}{\mu^2})\|Ax_n - Ap\|^2 \\
&\quad + 2\beta_n\|\gamma f(x_n) - Bp\|\|\rho_n - p\| + \|e_n\|^2.
\end{aligned} \tag{3.11}$$

This obtains that

$$\lim_{n\to\infty}\|Ax_n - Ap\| = 0. \tag{3.12}$$

On the other hand, we have

$$\begin{aligned}\|\rho_n - p\|^2 =& \|P_C(I - r_nA)x_n - P_C(I - r_nA)p\|^2 \\ \le& \langle (I - r_nA)x_n - (I - r_nA)p, \rho_n - p\rangle \\ =& \frac{1}{2}\{\|(I - r_nA)x_n - (I - r_nA)p\|^2 + \|\rho_n - p\|^2 \\ &- \|(I - r_nA)x_n - (I - r_nA)p - (\rho_n - p)\|^2\} \\ \le& \frac{1}{2}\{\|x_n - p\|^2 + \|\rho_n - p\|^2 - \|(x_n - \rho_n) - r_n(Ax_n - Ap)\|^2\} \\ =& \frac{1}{2}\{\|x_n - p\|^2 + \|\rho_n - p\|^2 - \|x_n - \rho_n\|^2 - r_n^2\|Ax_n - Ap\|^2 \\ &+ 2r_n\langle x_n - \rho_n, Ax_n - Ap\rangle\},\end{aligned}$$

which yields that

$$\|\rho_n - p\|^2 \le \|x_n - p\|^2 - \|\rho_n - x_n\|^2 + 2r_n\|\rho_n - x_n\|\|Ax_n - Ap\|. \tag{3.13}$$

Therefore, we have

$$\begin{aligned}&(1 - \alpha_n)\|\rho_n - x_n\|^2 \\ &\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \beta_n\|\gamma f(x_n) - Bp\|^2 + 2r_n\|\rho_n - x_n\|\|Ax_n - Ap\| \\ &\quad + 2\beta_n\|\gamma f(x_n) - Bp\|\|\rho_n - p\| \\ &\le (\|x_n - p\| + \|x_{n+1} - p\|)\|x_n - x_{n+1}\| + \beta_n\|\gamma f(x_n) - Bp\|^2 \\ &\quad + 2r_n\|\rho_n - x_n\|\|Ax_n - Ap\| + 2\beta_n\|\gamma f(x_n) - Bp\|\|\rho_n - p\|.\end{aligned}$$

From the conditions (i), (ii), (3.6) and (3.12), we have

$$\lim_{n\to\infty}\|\rho_n - x_n\| = 0. \tag{3.14}$$

On the other hand, we have $\|\rho_n - W_n\rho_n\| \le \|x_n - \rho_n\| + \|x_n - y_n\| + \|y_n - W_n\rho_n\|$. Therefore, we have $\lim_{n\to\infty}\|W\rho_n - W_n\rho_n\| = 0$. Since

$$\|W\rho_n - \rho_n\| \le \|W_n\rho_n - \rho_n\| + \|W_n\rho_n - W\rho_n\|,$$

we have

$$\lim_{n\to\infty}\|W\rho_n - \rho_n\| = 0. \tag{3.15}$$

Since $P_F(\gamma f+(I-B))$ is a contraction, we find that $P_F(\gamma f+(I-B))$ has a unique fixed point, say $q\in H$. That is, $q=P_F(\gamma f+(I-B))(q)$. To show it, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\limsup_{n\to\infty}\langle\gamma f(q)-Bq,x_n-q\rangle=\lim_{i\to\infty}\langle\gamma f(q)-Bq,x_{n_i}-q\rangle$. As $\{x_{n_i}\}$ is bounded, we have that there is a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ converges weakly to $p$. We may assume that without loss of generality that $x_{n_i}\rightharpoonup p$. Hence we have $p\in F$. Indeed, let us first show that $p\in VI(C,A)$. Put

$$Tw_1=\begin{cases}Aw_1+N_Cw_1, & w_1\in C\\ \emptyset, & w_1\notin C.\end{cases}$$

Since $A$ is inverse-strongly monotone, we have $T$ is maximal monotone. Let $(w_1,w_2)\in G(T)$. Since $w_2-Aw_1\in N_Cw_1$ and $\rho_n\in C$, we have $\langle w_1-\rho_n,w_2-Aw_1\rangle\geq 0$. On the other hand, from $\rho_n=P_C(I-r_nA)x_n$, we have $\langle w_1-\rho_n,\rho_n-(I-r_nA)x_n\rangle\geq 0$ and hence $\langle w_1-\rho_{n_i},w_2\rangle\geq\langle w_1-\rho_{n_i},A\rho_{n_i}-Ax_{n_i}\rangle-\langle w_1-\rho_{n_i},\frac{\rho_{n_i}-x_{n_i}}{r_{n_i}}\rangle$, which implies that $\langle w_1-p,w_2\rangle\geq 0$. We have $p\in T^{-1}0$ and hence $p\in VI(C,A)$. Next, let us show $p\in\bigcap_{i=1}^{\infty}F(T_i)$. Since Hilbert spaces are Opial's spaces, from (3.15), we have

$$\begin{aligned}\liminf_{i\to\infty}\|\rho_{n_i}-p\|<&\liminf_{i\to\infty}\|\rho_{n_i}-Wp\|\\ =&\liminf_{i\to\infty}\|\rho_{n_i}-W\rho_{n_i}+W_n\rho_{n_i}-Wp\|\\ \leq&\liminf_{i\to\infty}\|W\rho_{n_i}-Wp\|\\ \leq&\liminf_{i\to\infty}\|\rho_{n_i}-p\|,\end{aligned}$$

which derives a contradiction. Thus, we have $p\in F(W)=\bigcap_{i=1}^{\infty}F(T_i)$. On the other hand, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle\gamma f(q)-Bq,x_n-q\rangle=&\lim_{n\to\infty}\langle\gamma f(q)-Bq,x_{n_i}-q\rangle\\ =&\langle\gamma f(q)-Bq,p-q\rangle\leq 0.\end{aligned}\tag{3.16}$$

It follows from Lemma 2.5 that

$$\begin{aligned}
\|y_n-q\|^2 \leq & \frac{(1-\beta_n\bar{\gamma})^2+\beta_n\gamma\alpha}{1-\beta_n\gamma\alpha}\|x_n-q\|^2+\frac{2\beta_n}{1-\alpha_n\gamma\alpha}\langle\gamma f(q)-Bq,y_n-q\rangle \\
= & \frac{(1-2\beta_n\bar{\gamma}+\beta_n\alpha\gamma)}{1-\beta_n\gamma\alpha}\|x_n-q\|^2+\frac{\beta_n^2\bar{\gamma}^2}{1-\beta_n\gamma\alpha}\|x_n-q\|^2 \\
& +\frac{2\beta_n}{1-\beta_n\gamma\alpha}\langle\gamma f(q)-Bq,y_n-q\rangle \\
\leq & [1-\frac{2\beta_n(\bar{\gamma}-\alpha\gamma)}{1-\beta_n\gamma\alpha}]\|x_n-q\|^2 \\
& +\frac{2\beta_n(\bar{\gamma}-\alpha\gamma)}{1-\beta_n\gamma\alpha}[\frac{1}{\bar{\gamma}-\alpha\gamma}\langle\gamma f(q)-Bq,y_n-q\rangle+\frac{\alpha_n\bar{\gamma}^2}{2(\bar{\gamma}-\alpha\gamma)}M_5],
\end{aligned}$$

where $M_5$ is an appropriate constant. On the other hand, we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\leq \|\alpha_n(x_n-p)+(1-\alpha_n)(y_n-p)\|^2+\|e_n\|^2 \\
&\leq 2\alpha_n\|x_n-p\|^2+2(1-\alpha_n)\|y_n-p\|^2+2\|e_n\|^2.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 \leq & [1-(1-\alpha_n)\frac{2\beta_n(\bar{\gamma}-\alpha\gamma)}{1-\beta_n\gamma\alpha}]\|x_n-q\|^2 \\
& +(1-\alpha_n)\frac{2\beta_n(\bar{\gamma}-\alpha\gamma)}{1-\beta_n\gamma\alpha}[\frac{1}{\bar{\gamma}-\alpha\gamma}\langle\gamma f(q)-Aq,y_n-q\rangle+\frac{\beta_n\bar{\gamma}^2}{2(\bar{\gamma}-\alpha\gamma)}M_5].
\end{aligned}$$

Using Lemma 2.3, we find that $x_n\to q$ as $n\to\infty$. This completes the proof.

**Corollary 3.2.** *Let $H$ be a real Hilbert space, let $C$ be a nonempty closed convex subset of $H$ and let $A:C\to H$ be an u-inverse-strongly monotone mapping. Let $f:C\to C$ be a contraction with the coefficient $\alpha$ $(0<\alpha<1)$ and $\{T_i\}_{i=1}^{\infty}$ be an infinite nonexpansive mappings from $C$ into itself generated by (2.2) such that $F=\bigcap_{i=1}^{\infty}F(T_i)\cap VI(C,A)\neq\emptyset$. Let $B$ be a strongly positive linear bounded self-adjoint operator of $C$ into itself with coefficient $\bar{\gamma}>0$ such that $\|B\|\leq 1$. Assume that $0<\gamma<\bar{\gamma}/\alpha$. Assume that $x_1\in C$ and $\{x_n\}$ is generated by*

$$\begin{cases}
x_1\in C \\
y_n=P_C\big(\beta_n\gamma f(x_n)+(I-\beta_nB)W_nP_C(I-r_nA)x_n\big), \\
x_{n+1}=\alpha_nx_n+(1-\alpha)y_n, \quad n\geq 1,
\end{cases}$$

*where , $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$. If $\{\alpha_n\}$, $\{\beta_n\}$ and $\{r_n\}$ are chosen such that*

(a) $0<\liminf_{n\to\infty}\alpha_n\leq\limsup_{n\to\infty}\alpha_n<1$;

(b) $\lim_{n\to\infty}\beta_n = 0,\ \sum_{n=1}^{\infty}\beta_n = \infty$;

(c) $\lim_{n\to\infty}|r_{n+1}-r_n| = 0$;

(d) $\{r_n\}\subset[a,b]$ *for some* $a,b$ *with* $0<a<b<2u$.

*Then* $\{x_n\}$ *converges strongly to* $q\in F$*, where* $q = P_F(\gamma f+(I-B))(q)$*, which solves the variation inequality* $\langle\gamma f(q)-Bq, p-q\rangle\le 0, \quad \forall p\in F$.

**Corollary 3.3.** *Let* $H$ *be a real Hilbert space, let* $C$ *be a nonempty closed convex subset of* $H$ *and let* $A: C\to H$ *be an u-inverse-strongly monotone mapping. Let* $f: C\to C$ *be a contraction with the coefficient* $\alpha$ $(0<\alpha<1)$ *and* $\{T_i\}_{i=1}^{\infty}$ *be an infinite nonexpansive mappings from* $C$ *into itself generated by (2.2) such that* $F=\bigcap_{i=1}^{\infty}F(T_i)\cap VI(C,A)\neq\emptyset$*. Assume that* $x_1\in C$ *and* $\{x_n\}$ *is generated by*

$$\begin{cases} x_1\in C \\ y_n = \beta_n f(x_n) + (1-\beta_n)W_nP_C(I-r_nA)x_n\big), \\ x_{n+1} = P_C\big(\alpha_n x_n + (1-\alpha)y_n + e_n\big), \quad n\ge 1, \end{cases}$$

*where* $e_n$ *is a bounded sequence in* $H$*,* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in* $(0,1)$*. If* $\{\alpha_n\}$*,* $\{\beta_n\}$ *and* $\{r_n\}$ *are chosen such that*

(a) $0<\liminf_{n\to\infty}\alpha_n\le\limsup_{n\to\infty}\alpha_n<1$;

(b) $\lim_{n\to\infty}\beta_n = 0,\ \sum_{n=1}^{\infty}\beta_n = \infty$;

(c) $\lim_{n\to\infty}|r_{n+1}-r_n| = 0, \sum_{n=1}^{\infty}\|e_n\|<\infty$;

(d) $\{r_n\}\subset[a,b]$ *for some* $a,b$ *with* $0<a<b<2u$.

*Then* $\{x_n\}$ *converges strongly to* $q\in F$*, where* $q=P_Ff(q)$*, which solves the variation inequality* $\langle f(q)-q, p-q\rangle\le 0, \quad \forall p\in F$.

## Conflict of Interests

The author declares that there is no conflict of interests.

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